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DIFFEOMORPHISMS OF PSEUDO-RIEMANNIAN MANIFOLDS AND THE VALUES OF THE CURVATURE TENSOR ON DEGENERATE PLANES

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1. Introduction. Examining the inverse problem of the so-called "theorem egregium" of Gauss, R. S. Kulkarni [7] has proved that if f is a sectional-curvature-preserving diffeomorphism of a Riemannian manifold of nowhere constant sectional curvature and of dimension ≥ 4 onto another Riemannian manifold, then f is an isometry. For other results in this direction, see [8, 9, 14].

In contrast to the definite case, when the metric is indefinite a sectional-curvature-preserving diffeomorphism is not necessarily an isometry. This occurs when the manifolds are conformal flat and of recurrent curvature in the sense of A. Walker [12].

We note, that for pseudo-Riemannian manifolds the degenerate planes have an important role. Although the sectional curvature is defined only for nondegenerate planes, the values of the curvature tensor on degenerate planes give a good information about the manifold, as it is shown e. g. in [3, 4].

In this paper it is introduced a condition, which is more general than the condition f being a sectional-curvature-preserving diffeomorphism and which is in connection with the values of the curvature tensor on degenerate planes. It is proved, that if this condition is fulfilled for weakly degenerate planes, then f is sectional-curvature-preserving or the manifolds are of quasi-constant curvature. The corresponding condition for strongly degenerate planes implies that the diffeomorphism is necessarily an isometry.

2. Notations and preliminary results. Let M be a (pseudo-) Riemannian manifold with metric tensor g . By a plane we mean a 2-dimensional subspace of a tangent space. A plane α is said to be nondegenerate, weakly degenerate or strongly degenerate, if the restriction of g on α is of rank 2, 1 or 0, respectively. Of course, if M is Riemannian, i. e. if the metric is definite, all planes are nondegenerate. The sectional curvature $K(\alpha)$ of a nondegenerate plane α is defined by

$$K(\alpha) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)},$$

where $\{x, y\}$ is a basis of α , R denotes the curvature tensor and π_1 is defined by

$$\pi_1(z, u, v, w) = g(z, w)g(u, v) - g(z, v)g(u, w).$$

A vector ξ in a tangent space is called isotropic if $g(\xi, \xi) = 0$ and $\xi \neq 0$. It is easy to see, that if a plane α is weakly (resp. strongly) degenerate, then there exists just one isotropic direction in α and it is orthogonal to α (resp. each vector in α is isotropic and orthogonal to α). On the other hand if a plane α admits two orthogonal noncolinear vectors, one of which is nonisotropic (resp. isotropic) and the other—isotropic, then α is weakly (resp. strongly) degenerate.

As usual by a curvature-like tensor on a vector space V we mean a tensor T of type (0,4) on V with the properties:

- 1) $T(x, y, z, u) = -T(y, x, z, u)$;
- 2) $T(x, y, z, u) + T(y, z, x, u) + T(z, x, y, u) = 0$;
- 3) $T(x, y, z, u) = -T(x, y, u, z)$.

The following lemma will be useful:

Lemma A [3, 4]. *Let T be a curvature-like tensor in a point p of a pseudo-Riemannian manifold M of dimension ≥ 3 . If T vanishes identically on each weakly degenerate plane in p , then it has the form $T = c\pi_1$ with a constant c . In particular, if T is the curvature tensor of M , then M is of constant sectional curvature in p .*

According to the F. Schur's theorem, if the conditions of Lemma A are fulfilled in a connected open set U with T — the curvature tensor, then U is of constant sectional curvature.

Let us recall that the Weil conformal curvature tensor C for M is defined by

$$C = R - \frac{1}{n-2} \varphi(S) + \frac{\tau}{(n-1)(n-2)} \pi_1,$$

where $n = \dim M$, S is the Ricci tensor, τ is the scalar curvature and φ is defined by

$$\varphi(Q)(x, y, z, u) = g(x, u)Q(y, z) - g(x, z)Q(y, u) + g(y, z)Q(x, u) - g(y, u)Q(x, z)$$

for any symmetric tensor Q of type (0,2). It is well known [5] that if $n \geq 4$ then M is conformally flat if and only if C vanishes identically. The following two lemmas give criteria for vanishing of the Weil conformal curvature tensor.

Lemma B [3, 11]. *Let M be a (pseudo-) Riemannian manifold of dimension ≥ 4 . Assume that for every orthogonal quadruple $\{x, y, z, u\}$ of vectors in a point $p \in M$*

$$R(x, y, z, u) = 0$$

holds good. Then the Weil conformal curvature tensor of M vanishes in p . In particular, if this holds in every point of an open set U , then U is conformally flat.

Lemma C [3]. *Let M be an n -dimensional pseudo-Riemannian manifold of signature $(s, n-s)$, where $s \geq 2, n-s \geq 2$. If the curvature tensor of M vanishes identically on each strongly degenerate plane in a point p , then the Weil conformal curvature tensor of M vanishes in p . In particular, if this holds in every point of an open set U , then U is conformally flat.*

If \bar{M} is another (pseudo-) Riemannian manifold, we denote the corresponding objects for \bar{M} by a bar overhead. A diffeomorphism f of M onto \bar{M} is said to be sectional-curvature-preserving [7], if

$$\bar{K}(f*\alpha) = K(\alpha)$$

for each nondegenerate plane α in M , whose image is also nondegenerate. The corresponding condition for degenerate planes is

$$(2.1) \quad \lim_{\alpha \rightarrow \alpha_0} \frac{\bar{K}(f*\alpha)}{K(\alpha)} = 1$$

where the degenerate plane α_0 is approximated by nondegenerate planes whose images are also nondegenerate. Note, that if f is conformal, then (2.1) holds when and only when f preserves the null sectional curvature, defined in [6].

It is easy to prove the following:

Lemma 1. *Let α_0 be a degenerate plane in M , such that (2.1) holds. If R doesn't vanish identically on α_0 , then π_1 vanishes identically on $f*\alpha_0$.*

In what follows let M and \bar{M} be pseudo-Rimannian manifolds of dimension n and let f be a diffeomorphism of M onto \bar{M} . Then f_*x will be denoted by \bar{x} .

Lemma 2. Let $n \geq 3$ and M be of nonconstant sectional curvature in a point p . If (2.1) is satisfied for each weakly degenerate plane α_0 in the tangent space T_pM , then there exists an isotropic vector ξ in p , such that each isotropic vector, which is sufficiently close to ξ is mapped by f_* in an isotropic vector in $f(p)$.

Proof. According to Lemma A there exists a pair $\{x, \xi\}$ of vectors in T_pM such that x is a unit vector, ξ is isotropic and orthogonal to x and $R(x, \xi, \xi, x) \neq 0$. Let us assume that there exists a sequence of isotropic vectors ξ_i converging to ξ , such that $\bar{\xi}_i$ are all nonisotropic. Then we can find easily a pair $\{x_i, \xi_i\}$ of orthogonal vectors in T_pM , such that x_i is a unit vector, ξ_i is isotropic, $\bar{\xi}_i$ is nonisotropic and $R(x_i, \xi_i, \xi_i, x_i) \neq 0$. So, without loss of generality we assume that $\bar{\xi}$ is nonisotropic. We may suppose also that \bar{x} is orthogonal to $\bar{\xi}$. Hence, according to Lemma 1 $\text{span}\{\bar{x}, \bar{\xi}\}$ is degenerate, i. e. \bar{x} is isotropic. Let $\bar{\xi} = y + a$, where the vectors x, y, a are orthogonal and $g(x, x) = g(y, y) = -g(a, a) = 1$ or -1 . Let u be any unit vector, orthogonal to x, y, a if $n \geq 3$ and let u be the zero vector, if $n = 3$. We put $\varepsilon = g(y, y)g(u, u)$ and $y_s = (y + su) / \sqrt{1 + \varepsilon s^2}$, where s is a real number, $|s| < 1$. Since $R(x, \xi, \xi, x) \neq 0$, then by Lemma 1 there exists a real number δ , $0 < \delta < 1$, such that for any real s, t with $|s|, |t| < \delta$ we have

$$\bar{\pi}_1(\bar{x} + t\bar{y}_s, \frac{\bar{y}_s - t\bar{x}}{\sqrt{1+t^2}} + \bar{a}, \frac{\bar{y}_s - t\bar{x}}{\sqrt{1+t^2}} + \bar{a}, \bar{x} + t\bar{y}_s) = 0,$$

which implies

$$(2.2) \quad (1+t^2)\bar{\pi}_1(\bar{x}, \bar{y}_s, \bar{y}_s, \bar{x}) + 2\sqrt{1+t^2}\bar{\pi}_1(\bar{x}, \bar{y}_s, \bar{a}, \bar{x}) + \bar{\pi}_1(\bar{x}, \bar{a}, \bar{a}, \bar{x}) + t^2\bar{\pi}_1(\bar{y}_s, \bar{a}, \bar{a}, \bar{y}_s) = 0.$$

For $t=0$ this reduces to

$$(2.3) \quad \bar{\pi}_1(\bar{x}, \bar{y}_s, \bar{y}_s, \bar{x}) + 2\bar{\pi}_1(\bar{x}, \bar{y}_s, \bar{a}, \bar{x}) + \bar{\pi}_1(\bar{x}, \bar{a}, \bar{a}, \bar{x}) = 0.$$

From (2.2) and (2.3) we find

$$t^2\{\bar{\pi}_1(\bar{x}, \bar{y}_s, \bar{y}_s, \bar{x}) + \bar{\pi}_1(\bar{y}_s, \bar{a}, \bar{a}, \bar{y}_s)\} + 2(\sqrt{1+t^2} - 1)\bar{\pi}_1(\bar{x}, \bar{y}_s, \bar{a}, \bar{x}) = 0$$

for any real t, s with $|t|, |s| < \delta$. Hence $\bar{\pi}_1(\bar{x}, \bar{y}_s, \bar{a}, \bar{x}) = 0$. Comparing this with (2.3) we get

$$\bar{\pi}_1(\bar{x}, \bar{y}_s, \bar{y}_s, \bar{x}) + \bar{\pi}_1(\bar{x}, \bar{a}, \bar{a}, \bar{x}) = 0$$

and since \bar{x} is isotropic, it is easy to conclude

$$(2.4) \quad \bar{g}(\bar{x}, \bar{a}) = 0,$$

$$(2.5) \quad \bar{g}(\bar{x}, \bar{y}_s) = 0.$$

From (2.5) it follows immediately

$$(2.6) \quad \bar{g}(\bar{x}, \bar{y}) = \bar{g}(\bar{x}, \bar{u}) = 0.$$

Since \bar{x} is isotropic, (2.4) and (2.6) imply $\bar{g}(\bar{x}, \bar{z}) = 0$ for each vector \bar{z} in $f(p)$ and hence $\bar{x} = 0$, which is a contradiction. This proves the lemma.

The following assertion is an analogue of Lemma 2 for strongly degenerate planes.

Lemma 3. *Let the Weil conformal curvature tensor of M do not vanish identically in p and let M have the signature $(s, n-s)$, where $s \geq 2, n-s \geq 2$. If f satisfies (2.1) for each strongly degenerate plane α_0 in $T_p M$, then there exists an isotropic vector ξ in p , such that each isotropic vector, which is sufficiently close to ξ , is mapped by f_* in an isotropic vector in $f(p)$.*

Proof. According to Lemma C there exists an orthogonal pair $\{\xi, \eta\}$ of isotropic vectors in $T_p M$, such that $R(\xi, \eta, \eta, \xi) \neq 0$. Let us assume that there exists a sequence η_i of isotropic vectors converging to η , such that $\bar{\eta}_i$ are nonisotropic. Then we can find an orthogonal pair $\{\xi_i, \eta_i\}$ of isotropic vectors in $T_p M$, such that $\bar{\eta}_i$ is nonisotropic and $R(\xi_i, \eta_i, \eta_i, \xi_i) \neq 0$. So we may assume that $\bar{\eta}$ is nonisotropic. Also without loss of generality we suppose that $\bar{\xi}$ and $\bar{\eta}$ are orthogonal. Then Lemma 1 implies that $\bar{\xi}$ is isotropic. Let $\xi = x + a, \eta = y + b$, where x, y, a, b are orthogonal and $g(x, x) = g(y, y) = -g(a, a) = -g(b, b) = 1$. For any vector z in $T_p M$ orthogonal to ξ , we put

$$\xi_t = \xi + t(x + z); \quad \eta_t = \eta + ty; \quad \alpha_t = \text{span}\{\xi_t, \eta_t\}.$$

Then we find

$$1 = \lim_{t \rightarrow 0} \frac{\bar{K}(\bar{\alpha}_t)}{K(\alpha_t)} = \frac{1}{R(\xi, \eta, \eta, \xi)} \lim_{t \rightarrow 0} \frac{\bar{R}(\bar{\xi}_t, \bar{\eta}_t, \bar{\eta}_t, \bar{\xi}_t)}{t(2+t)\{2+tg(x+z, x+z)\} - t\{g(z, \eta) + tg(z, y)\}^2} \times \frac{1}{\{2g(\bar{\xi}, \bar{x} + \bar{z}) + tg(\bar{x} + \bar{z}, \bar{x} + \bar{z})\}g(\bar{\eta}_t, \bar{\eta}_t) - t\{g(\bar{\xi}, y) + g(\bar{x} + \bar{z}, \bar{\eta}) + tg(\bar{x} + \bar{z}, \bar{y})\}^2}$$

Hence we obtain $\bar{g}(\bar{\xi}, \bar{x} + \bar{z})\bar{g}(\bar{\eta}, \bar{\eta}) = 0$ or

$$(2.7) \quad \bar{g}(\bar{\xi}, \bar{x} + \bar{z}) = 0.$$

In particular, if $z = 0$ this implies $\bar{g}(\bar{\xi}, \bar{x}) = 0$ and hence $\bar{g}(\bar{\xi}, \bar{a}) = 0$. Using again (2.7) we conclude $\bar{g}(\bar{\xi}, \bar{z}) = 0$. Consequently $\bar{\xi}$ is orthogonal to every vector in $f(p)$, which is a contradiction. This proves the lemma.

Lemma 4. *Let $n \geq 3$ and in a point $p \in M$ there exists an isotropic vector ξ , such that each isotropic vector, which is sufficiently close to ξ is mapped by f_* into an isotropic vector. Then f is a homothety in p .*

Proof. Let $\xi = x + a$, where $\{x, a\}$ is an orthonormal pair of vectors. Let y by an arbitrary unit vector, orthogonal to x, a . Then e. g. $g(x, x) = g(y, y) = -g(a, a)$ and for each real t the vector

$$\xi_t = (x + ty)(1 + t^2)^{-1/2} + a$$

is isotropic. By the condition $\bar{\xi}_t$ is also isotropic for each sufficiently small t . This implies

$$\bar{g}(\bar{x}, \bar{x}) + t^2 \bar{g}(\bar{y}, \bar{y}) + 2t \bar{g}(\bar{x}, \bar{y}) + 2\sqrt{1 + t^2} \{ \bar{g}(\bar{x}, \bar{a}) + t \bar{g}(\bar{y}, \bar{a}) \} + (1 + t^2) \bar{g}(\bar{a}, \bar{a}) = 0.$$

Hence we derive

$$\bar{g}(\bar{x}, \bar{y}) = \bar{g}(\bar{y}, \bar{a}) = \bar{g}(\bar{x}, \bar{a}) = 0, \quad \bar{g}(\bar{x}, \bar{x}) = \bar{g}(\bar{y}, \bar{y}) = -\bar{g}(\bar{a}, \bar{a}),$$

from which the assertion follows easily.

An n -dimensional nonflat (pseudo-) Riemannian manifold M is said to be a K_n^* -manifold [12], if it has one of the following properties:

- 1) M is recurrent, i. e. $\nabla R = \alpha \otimes R$, where $\alpha \neq 0$;
 2) M is symmetric ($\nabla R = 0$) and there exists a differential form $\alpha \neq 0$, such that

$$\sum_{\text{cycl } X, Y, Z} \alpha(X)R(Y, Z, U, V) = 0.$$

A. Walker showed in [12] that α is defined by $\alpha(X) = g(\nabla v, X)$, where v is a function (called recurrence-function) and ∇v is the gradient of v .

An n -dimensional (pseudo-) Riemannian manifold M is said to be of quasi-constant curvature [1, 2] if it is conformally flat and there exist functions H, N and a unit vector field V , such that the curvature tensor has the form

$$R = (N - H)\varphi(B) + H\pi_1,$$

where $B(X, Y) = g(X, V)g(Y, V)$. Note that for any point p of M we have $K(a) = H(p)$ for any nondegenerate plane a in $T_p M$, perpendicular to V_p and $K(a) = N(p)$ for any nondegenerate plane a in $T_p M$, containing V_p . Such a manifold we shall denote by $M(H, N, V)$. Of course, if $\dim M \geq 4$ and the curvature tensor has the above mentioned form, the manifold is necessarily conformally flat.

Two standart examples of manifolds of quasi-constant curvature are the following (see also [2]).

Example 1. Let $M_1(c)$ be an $(n-1)$ -dimensional (pseudo-) Riemannian manifold of constant curvature and $M = M_1(c) \times \mathbb{R}$. Then M is of quasi-constant curvature.

Example 2. Let \mathbb{R}_s^{n+1} be the pseudo-Euclidean space with an indefinite metric of the signature $(s, n+1-s)$. Let M be the n -dimensional indefinite hypersurface given by the equations

$$\begin{cases} x^i = 2y^i y^n / \Delta & i = 1, \dots, n-1 \\ x^n = y^n (\Delta - 2) / \Delta \\ x^{n+1} = f(y^n) \end{cases}$$

for $\Delta > 0, y^n > 0$, where $\Delta = -\sum_{i=1}^s (y^i)^2 + \sum_{i=s+1}^{n-1} (y^i)^2 + 1$ and f is a smooth function. Then M is a manifold of quasi-constant curvature and

$$H = f'^2 / (y^n)^2 (1 + f'^2), \quad N = f' / f'' / y^n (1 + f'^2).$$

In the following section we shall use the well-known fact [5], that if the metrics g and \bar{g} on M are related by $\bar{g} = e^{2\sigma} g$ (thus (M, g) and (M, \bar{g}) are conformal), then

$$(2.8) \quad \bar{R} = e^{2\sigma} \{R + \varphi(Q)\},$$

where

$$(2.9) \quad Q(X, Y) = X\sigma Y\sigma - g(\nabla_X \nabla \sigma, Y) - \frac{1}{2} g(\nabla \sigma, \nabla \sigma) g(X, Y).$$

3. Main results. We begin this section with a theorem, which follows immediately from Lemmas 2 and 4.

Theorem 1. Let M and \bar{M} be pseudo-Riemannian manifolds of dimension $n \geq 3$ and let M be nowhere of constant sectional curvature. If f is a diffeomorphism of M onto \bar{M} satisfying (2.1) for each weakly degenerate plane a_0 on M , then f is conformal.

If the diffeomorphism f is conformal, we have $f^* \bar{g} = e^{2\sigma} g$ or $f^* \bar{g} = -e^{2\sigma} g$ for a smooth function σ . Then without loss of generality we may identify M with \bar{M} via f and assume $\bar{g} = e^{2\sigma} g$. We state:

Theorem 2. *Under the conditions of Theorem 1 the following propositions hold :*

a) *if $\nabla\sigma$ vanishes identically or $n \geq 4$ and M is nowhere conformally flat, then f is an isometry;*

b) *if $\nabla\sigma$ is isotropic and either $n \geq 4$ or M is conformally flat of dimension $n=3$, then M is a conformal flat K_n^* -space and σ is a function of the recurrence-function;*

c) *if $\|\nabla\sigma\|^2 = g(\nabla\sigma, \nabla\sigma)$ doesn't vanish and either $n \geq 4$ or M is conformally flat of dimension $n=3$, then M is a manifold $M(H, N, V)$ of quasi-constant curvature, where $\Delta H, \nabla N$ and V are proportional to $\nabla\sigma$.*

Proof. Since $\bar{g} = e^{2\sigma}g$, (2.1) reduces to

$$\bar{R}(x, \xi, \xi, x) = e^{4\sigma}R(x, \xi, \xi, x)$$

for arbitrary p in M and x, ξ in T_pM with $g(x, x) \neq 0, g(\xi, \xi) = g(x, \xi) = 0$. Hence, using Lemma A we obtain

$$\bar{R} = e^{4\sigma}\{R + c\pi_1\}$$

for a function c , from which it follows

$$(3.1) \quad \bar{R} = e^{4\sigma}\{R + (\bar{\tau} - \tau)\pi_1/n(n-1)\}.$$

Let $\nabla\sigma = 0$, i. e. σ is a constant. Then (2.8), (2.9) and (3.1) imply $(e^{-2\sigma} - 1)R = (\bar{\tau} - \tau)\pi_1/n(n-1)$.

Since M cannot be of constant sectional curvature in an open set, this yields $\sigma = 0$, i. e. f is an isometry.

On the other hand, if x, y, z, u are arbitrary orthogonal vectors at a point p of M , from (2.8) and (3.1) we find

$$(3.2) \quad (e^{2\sigma(p)} - 1)R(x, y, z, u) = 0.$$

If $n \geq 4$ and M is nowhere conformally flat, Lemma B and (3.2) imply $\sigma = 0$, i. e. f is an isometry, proving a).

Let us assume that $\nabla\sigma$ is isotropic or $\|\nabla\sigma\|^2$ doesn't vanish. In both cases if p is an arbitrary point, σ cannot vanish in a neighborhood of p , i. e. there exists a sequence p_i converging to p , such that $\sigma(p_i) \neq 0$ for each i . Then (3.2) and Lemma B imply that the Weil conformal curvature tensor of M vanishes in p_i . By continuity it vanishes in p and hence M is conformally flat, if $n \geq 4$. Consequently in cases b) and c) M is conformally flat. Using (2.8) and (3.1) it is not difficult to get

$$(3.3) \quad \bar{P} = e^{2\sigma}P,$$

$$(3.4) \quad Q = (e^{2\sigma} - 1)P/(n-2) + (\bar{\tau}e^{2\sigma} - \tau)g/2n(n-1),$$

where $P = S - \tau/ng$. Since (M, g) and (M, \bar{g}) are conformally flat, we have [13]

$$(3.5) \quad (\nabla_x(S - \frac{\tau}{2(n-1)}g))(Y, Z) - (\nabla_x(S - \frac{\tau}{2(n-1)}g))(X, Z) = 0,$$

$$(\bar{\nabla}_x(\bar{S} - \frac{\bar{\tau}}{2(n-1)}\bar{g}))(Y, Z) - (\bar{\nabla}_x(\bar{S} - \frac{\bar{\tau}}{2(n-1)}\bar{g}))(X, Z) = 0.$$

Hence, using (3.4) and $\bar{\nabla}_x Y = \nabla_x Y + X\sigma Y + Y\sigma X - g(X, Y)\nabla\sigma$ we obtain

$$X\sigma P(Y, Z) - Y\sigma P(X, Z) + \frac{n-2}{2n(n-1)} \{X(\bar{\tau} - \tau)g(Y, Z) - Y(\bar{\tau} - \tau)g(X, Z)\}$$

$$+ g(X, Z)P(Y, \nabla\sigma) - g(Y, Z)P(X, \nabla\sigma) = 0,$$

which implies immediately

$$(3.6) \quad P(Y, \nabla\sigma) = \frac{n-2}{2n^2} Y(\bar{\tau}-\tau),$$

$$(3.7) \quad X\sigma P(Y, Z) - Y\sigma P(X, Z) + \frac{n-2}{2n^2(n-1)} \{X(\bar{\tau}-\tau)g(Y, Z) - Y(\bar{\tau}-\tau)g(X, Z)\} = 0.$$

To prove b) we assume that $\nabla\sigma$ is isotropic and we put in (3.7) $X=Z=\nabla\sigma$, $Y=\nabla(\bar{\tau}-\tau)$. Using (3.6) we find $(\nabla\sigma)(\bar{\tau}-\tau)=0$. Hence, substituting in (3.7) X by $\nabla\sigma$ we obtain easily $Y(\bar{\tau}-\tau)Z\sigma=0$ for arbitrary vector fields Y, Z on M . Since $\nabla\sigma$ can not vanish this shows that $\bar{\tau}-\tau$ is a constant. Since $\nabla\sigma$ is isotropic, (2.9) implies $Q(X, \nabla\sigma)=0$. This, using (3.4) and (3.6) we derive

$$(3.8) \quad \bar{\tau}e^{2\sigma} - \tau = 0.$$

On the other hand, since $\bar{\tau}-\tau$ is a constant, (3.7) implies $X\sigma P(Y, Z) - Y\sigma P(X, Z) = 0$ and hence $P(Y, Z) = hY\sigma Z\sigma$ for a smooth function h . Then (2.9), (3.4) and (3.8) yield $g(\nabla_X \nabla\sigma, Y) = (1 + (1 - e^{2\sigma})h/(n-2))X\sigma Y\sigma$. Hence we obtain easily

$$(\nabla_X P)(Y, Z) = \{Xh + 2h(1 + (1 - e^{2\sigma})h/n - 2)X\sigma\}Y\sigma Z\sigma.$$

We compare this with (3.5) to find

$$\{XhY\sigma - YhX\sigma\}Z\sigma + \frac{n-2}{2n(n-1)} \{X\tau g(Y, Z) - Y\tau g(X, Z)\} = 0.$$

Here we assume, that X is any vector field on M , such that Z is orthogonal to $\nabla\sigma$ and that Y is not orthogonal to Z . It follows that $X\tau=0$, i. e. τ is a constant. Thus, differentiating (3.8) and using $\bar{\tau}-\tau=\text{const}$ we obtain $\bar{\tau}X\sigma=0$ for any vector field X on M . Since $\nabla\sigma$ can not vanish this implies $\bar{\tau}=0$. According to (3.8) τ vanishes too. Then (3.1) shows that f is sectional-curvature-preserving and b) follows from [10].

Finally, we assume that $\|\nabla\sigma\|^2$ doesn't vanish. Let in (3.7) $X=Z=\nabla\sigma$ and Y be orthogonal to $\nabla\sigma$. Using (3.6) we obtain $Y(\bar{\tau}-\tau)=0$. Hence $\mu\nabla\sigma = \nabla(\bar{\tau}-\tau)$, where μ is a smooth function. Substituting X by $\nabla\sigma$ in (3.7) and making use of (3.6) we find

$$(3.9) \quad P(Y, Z) = \mu \left\{ \lambda Y\sigma Z\sigma - \frac{n-2}{2n^2(n-1)} g(Y, Z) \right\},$$

where $\lambda = (n-2)/2n(n-1)\|\nabla\sigma\|^2$. From (2.9), (3.4) and (3.9) we derive

$$(3.10) \quad g(\nabla_X \nabla\sigma, Y) = (1 + \lambda\mu \frac{1-e^{2\sigma}}{n-2})X\sigma Y\sigma + \nu g(X, Y),$$

where

$$\nu = \mu \frac{e^{2\sigma}-1}{2n^2(n-1)} - \frac{1}{2} \|\nabla\sigma\|^2 - \frac{\bar{\tau}e^{2\sigma}-\tau}{2n(n-1)}.$$

In particular, this implies $X\|\nabla\sigma\|^2=0$ for any X orthogonal to $\nabla\sigma$. From (3.9) and (3.10) one gets

$$\begin{aligned}
 (\nabla_X P)(Y, Z) = & \frac{X\mu}{\mu} P(Y, Z) + \mu X\lambda Y\sigma Z\sigma + 2\lambda\mu(1 + \lambda\mu \frac{1-e^{2\sigma}}{n-2}) X\sigma Y\sigma Z\sigma \\
 & + \lambda\mu\nu\{g(X, Y)Z\sigma + g(X, Z)Y\sigma\}.
 \end{aligned}$$

Together with (3.5) and (3.9) this yields

$$\begin{aligned}
 (3.11) \quad & \lambda X\mu Y\sigma Z\sigma + \frac{n-2}{2n(n-1)} (X\tau - \frac{1}{n} X\mu) g(Y, Z) \\
 & - \{ \frac{n-2}{2n(n-1)} Y\tau - \frac{n-2}{2n^2(n-1)} Y\mu - \lambda\mu\nu Y\sigma \} g(X, Z) = 0,
 \end{aligned}$$

when λ is orthogonal to $\nabla\sigma$. Let in (3.11) $g(Y, Y) \neq 0$; Y be orthogonal to $X, \nabla\sigma$; $Z = \nabla\sigma + Y$. Then we obtain $X\tau = 1/n X\mu$. Now we assume in (3.11) $Y = Z = \nabla\sigma$. The result is $X\mu = 0$. Consequently $\nabla\sigma, \nabla\tau$ and $\nabla\mu$ are proportional. Since M is conformally flat (3.9) shows that it is a manifold of quasi-constant curvature $M(H, N, V)$ with V proportional to $\nabla\sigma$. Moreover, we compute

$$H = \frac{1}{n(n-1)} \{ \tau - \frac{1}{n} \mu \} \quad N = \frac{1}{n(n-1)} \{ \tau + \frac{n-2}{2n} \mu \}.$$

Consequently $\nabla H, \nabla N$ and $\nabla\sigma$ are proportional. This proves the theorem.

Remark. Putting in (3.11) $Z = X$ we obtain

$$(3.12) \quad \mu\nu\nabla\sigma = \|\nabla\sigma\|^2 (\nabla\tau - \frac{1}{n} \nabla\mu).$$

If $H - N$ doesn't vanish (3.10) and (3.12) imply

$$g(\nabla_X Y, \nabla H) = \frac{1}{2} \frac{1}{H-N} \|\nabla H\|^2 g(X, Y)$$

for all vector fields X, Y , orthogonal to $\nabla\sigma$. This holds for each manifold of quasi-constant curvature, see [1] (indeed in [1] this is proved in the definite case, but there is no principle difference). Note also that in the case $H = \text{const}$ it follows from (3.10) and (3.12) that M is locally a product $M_1 \times M_2$ in a neighborhood of any point in which M is not of constant sectional curvature, where M_1 is an $(n-1)$ -dimensional manifold of constant sectional curvature H .

The following result shows that there exists no nontrivial diffeomorphisms, satisfying (2.1) for strongly degenerate planes.

Theorem 3. *Let M and \bar{M} be pseudo-Riemannian manifolds, such that M is nowhere conformally flat and of signature $(s, n-s)$ where $s \geq 2, n-s \geq 2$. Let f be a diffeomorphism of M onto \bar{M} , satisfying (2.1) for each strongly degenerate plane α_0 on M . Then f is an isometry.*

Proof. According to Lemmas 3 and 4 f is conformal. So without loss of generality we may identify M with \bar{M} via f and assume $\bar{g} = \varepsilon e^{2\sigma} g$, where $\varepsilon = \pm 1$. Let $\varepsilon = 1$ and let $\{\xi, \eta\}$ be an arbitrary orthogonal pair of isotropic vectors on M . Then (2.1) yields

$$(3.13) \quad \bar{R}(\xi, \eta, \eta, \xi) = e^{4\sigma} R(\xi, \eta, \eta, \xi).$$

On the other hand (2.8) implies

$$(3.14) \quad \bar{R}(\xi, \eta, \eta, \xi) = e^{2\sigma} R(\xi, \eta, \eta, \xi).$$

From (3.13) and (3.14) we obtain

$$(3.15) \quad (e^{2\sigma} - 1)R(\xi, \eta, \eta, \xi) = 0.$$

Since M is nowhere conformally flat, then from (3.15) and by applying Lemma C it follows that $\sigma = 0$. Similar arguments show that the case $\varepsilon = -1$ is not possible. This completes the proof.

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Received 7. 6. 1988.