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A TYPE OF STABLE STOCHASTIC PROCESSES USING LEVY-KHINTCHINE FORMULAE

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The method of the "point sources of influence" (p.s.i.) is mathematically described in the book [1] of V. Zolotorev. In fact this method had been used in many different problems in physics, astronomy, biology, radiotechnics (see Chapter 1, [1]). Its attractivity lies in the simple construction. Poisson summability gives us the possibility of analytical evaluation of all probabilistic characteristics of the output random variable. In general, this random variable is a sum of influences of different sources distributed in the space randomly following Poisson law. For example, the classical result of J. Holtzmark (1919, [7], [41]) describes the distribution of random gravitation field of a system of stars, which turns out to be stable.

In previous work [12] we exploited p.s.i. method to construct an example of random magnetic field using two different kinds of influences: the first one corresponding to electrically active particals represented by microcurrents and the second one corresponding to macroobjects — stars, planets etc. represented by magnetic dipols. Both models produced stable distributions with shape parameters 1. and 1.5 respectively.

In this work we try to extend our results to a dynamic situation, when the output depends somehow of the time and has a kind of memory. In fact this corresponds to the stochastic process of "moving average" type or "linear filter" of Poisson white noise. We are going to construct this 3-dimensional process as a stochastic integral and to prove necessary and sufficient conditions for its existence.

Basic assumptions. Let $X=\mathrm{R}^{n_1}$ and $T=\mathrm{R}^{n_2}$ (we call them "space" and "time" respectively). Let μ and ρ be σ-finite Borel measures on X and T respectively. Ω is the standard probability space and Z the linear space of all random variables with values in R^3. Now we shall recall some properties of Poisson point process (p.p.p.) in $X\times T$ with intensity $\mu\otimes\rho$, defined on the σ-field of all Borel sets $\mathscr{B}(X\times T)$, ([9]).

C1. If $A\in\mathscr{B}(X\times T)$ and $(\mu\otimes\rho)(A)<\infty$, then the number of points $\{x_i\}$ of p.p.p. such that $x_i\in A$ is random variable with Poisson distribution and $E\#(A)=(\mu\otimes\rho)(A)$.

C2. If A, $B\in\mathscr{B}(X\times T)$ and $A\cap B=\emptyset$, then $\#(A)$ and $\#(B)$ are independent.

The p.p.p. can be easily equipped with "labels" so that any point of the trajectory carries some additional information. Such point processes are called labelled. In the simplest case the labels are independent from the basic point process. So let $\{M_i\}$ be a sequence of i.i.d. random variables independent of the p.p.p. Define the mapping $W_\omega:\mathscr{B}(T)\to\mathrm{Z}$ in the following way:

$$W_\omega(J)=\sum_{(x_i,\,t_i)\in p.p.p.,\ t_i\in J} u(x_i,\,t_i,\,M_i),\quad J\in\mathscr{B}(T),\quad \rho(J)<\infty. \tag{1}$$

The function $u(x,t,M)$ describes the perturbation caused by the point x at the moment t with the label M (usually we denote by M only the norm of corresponding spherically symmetric r.v.). This function is called influence function. We consider the simplest case when u does not depend on t, i.e. $u(x,t,M)=u(x,M)$. Then (1) may be rewritten in the form

$$W_\omega(J)=\sum_{(x_i,\,t_i)\in p.p.p.,\ t_i\in J} u(x_i,\,M_i).\quad J\in\mathscr{B}(T),\quad \rho(J)<\infty. \tag{2}$$

In the previous paper we studied the r. v. W_ω in the case $T^{n_2}=T^1$ and showed under some week assumptions on u, that it has symmetric stable distribution with shape parameter α. Without any troubles this result can be extended to the present case:

Theorem 1. *The mapping W_ω (defined in (2)) is σ-finite random vector measure on T and for fixed $A\in\mathcal{B}(T)$, $W_\omega(A)$ has a symmetric stable distribution ($S\alpha S$) with shape parameter α (not depending on A).*

Proof: A) With probability 1 the sum (2) has countably many terms. Suppose the opposite. Then there exists a compact set K such that in $K\times J$ there are infinitely many points (x_i, t_i) and $\mu\otimes\rho(K\times J)<\infty$. But this means that (x_i,t_i) is not found from p.p.p. what contradicts to the supposition. So we can rearrange the points (x_i, t_i) so that $\ldots\leq|x_i|\leq|x_{i+1}|\leq\ldots$

B) The series (2) converges. Consider the following sequence of i.i.d.r.v.—s.

$$W_1(J)=\sum_{(x_i,t_i)\in p.p.p.,\, t_i\in J} u(x_i, M_i),\quad \rho(J)<\infty,\quad |x|<1.$$

$$W_2(J)=\sum_{(x_i,t_i)\in p.p.p.,\, t_i\in J} u(x_i, M_i),\quad \rho(J)<\infty,\quad 1\leq|x|<2.$$

$$\vdots$$

Then $W_\omega(J)=\Sigma_{n=1}^{\infty} W_n(J)$. Denote by $\Phi_n(\tau)$ the characteristic function (c. f.) of $W_n(J)$ and use the form of $S\alpha S$ c. f. (see [1, 12])

$$\ln\Phi_n(\tau)=(\mu\otimes\rho)(S_n\setminus S_{n-1})\int_{(S_n\setminus S_{n-1})\times M}[e^{i(\tau,u)}-1]\,dx\mathbf{P}(dM),$$

where $S_n=(n.S)\times J$, S being the unit ball in X. Then

$$\ln\prod_{n=1}^{\infty}|\Phi_n(\tau)|=\sum_{n=1}^{\infty}\ln|\Phi_n(\tau)|=\sum_{n=1}^{\infty}(\mu\times\rho)(S_n\setminus S_{n-1})\int[e^{i(\tau,u)}-1]\,dx\mathbf{P}(dM)$$

$$=(\mu\otimes\rho)(X\times J)\int[e^{i(\tau,u)}-1]\,dx\,\mathbf{P}(dM)=-\gamma|\tau|^{\alpha}.$$

Therefore $\Pi_{n=1}^{\infty}|\Phi_n(\tau)|>0$ on a set with positive Lebesque measure and applying Th. 2.7. [5] we conclude that $\Sigma_{n=1}^{\infty}w_n(J)<\infty$ a. s. The σ-finiteness of ρ implies σ-finiteness of W_ω.

C) W_ω is σ-additive. Immediately follows from Lemma 3 in [10].

The measure W_ω is called Poisson $S\alpha S$ measure. Following [10], it could be also called Stable Wiener Process of type α.

From now we shall study the stochastic process

$$\xi(t)=\int_T f(t,\tau)\,dW_\omega(\tau), \tag{3}$$

where $f: T\times T\to\mathbf{R}$ and $\xi: T\to\mathbf{Z}$. The nonrandom function f we called 'memory', i. e. $f(t_1, t_2)=f(t_1-t_2)$. In the case $T=\mathbf{R}$ a simple interpretation of fading memory could be achieved with the additional assumptions:

$f(t)\to0$ when $t\to-\infty$ — fading

$f(t)=0$ when $t>0$ — independence of future.

Stochastic integrals. Following [10], we shall develop a necassary and sufficient condition for the existence of $\xi(t)$. Theorems 2 and 3 are borrowed from this work and given here for completeness only. The multivariateness of our process makes no difference in the proofs.

Suppose W is a σ-finite random measure on T with infinite divisible distribution (i. d. d.). Denote by $\varphi(A, t)$ the logarithm of c. f. of $W(A)$, i. e.

$$\varphi(A, t)) = \ln \mathsf{E}[\exp i(t, W(A))].$$

The Levy — Khintchine formulae ([8], [10]) will then be fulfilled

$$\varphi(A, t) = i(\delta, t) + \int_{\mathsf{R}^n} K(t, x)\lambda(\mathrm{d}x),$$

where $\int_{\|x\|<1} \int \|x\|^2 \lambda(\mathrm{d}x) < \infty$ and λ is a σ-finite Borel measure on R^n. The kernel can be written in the form

$$K(t, x) = \exp(i(t, x)) - 1 - i(t, x)/(1 + \|x\|^2).$$

The following modification of Levy — Khintchine formulae is proved in [6] and [11].

Theorem 2. *If W is a Wiener process with i. d. d., then*

$$\varphi(A, t) = i(\delta(A), t) + \int_{\mathsf{R}^n} K(t, x)\lambda(A, \mathrm{d}x),$$

where $\int_{\|x\|<1} \|x\|^2 \lambda(A, \mathrm{d}x) < \infty$, δ *is σ-finite vector measure on* $T, \lambda(A, \cdot)$ *and* $\lambda(\cdot, x)$ *are σ- finite measures on* R^n *and* T *and* $\lambda(A, \{0\}) = 0$ *for fixed* A.

The words 'Wiener process' mean in this paper σ-additive random measure with independent increments (see C2).

Let on T a σ-finite measure ρ be defined such that both $\lambda(\cdot, x)$ and $\delta(\cdot)$ are absolutely continuous with respect to ρ (in our case the intensity ρ is such measure). Denote the Radon—Nicodim derivatives of λ and δ with $g(\cdot, x)$ and $\eta(\cdot)$ respectively. Both functions are nonnegative and locally integrable. Additionaly g has natural properties of multidimensional distribution function [2], § 19.

Definition 1. *Denote formally*

$$F(a, t) = i(\eta(a), t) + \int_{\mathsf{R}^n} K(t, x) g(a, \mathrm{d}x)$$

and call it structural function of Wiener process.

Lemma 1.

$$\varphi(A, t) = \int_A F(a, t)\rho(\mathrm{d}a), \ \forall t, \quad A \in \mathscr{B}(T), \quad \rho(A) < \infty.$$

The correctness of Definition 1 and the proof of this Lemma are easily derived from the inequalities

$$|K(t, x)| \leq 2 + \|t\|/2, \quad \|x\| > 1,$$

$$|K(t, x)| \leq \|x\|^2 (\|t\|^2/2) + \|t\|), \quad \|x\| < 1$$

and Theorem 2.

The construction of the stochastic integral follows the usual procedure beginning with simple functions, bounded functions to achieve at the end any measurable function. Let f be a measurable function on T. Then it can be presented as a pointwise limit of bounded functions

$$\psi_n(a) = \begin{cases} f(a), & n-1 \leq |f(a)| < n, \\ 0, & \text{otherwise}, \end{cases}$$

$$f_n(a) = \sum_{i=2}^{n} \psi_i(a), \quad \{f_n\} \to f.$$

Any bounded function itself is a pointwise limit of a sequence of uniform bounded simple functions.

Definition 2. *The function f is integrable (stochastically) with respect to a random measure W if for each $A\in\mathcal{B}(T)$ the sequence*

$$\int_A f_1 \mathrm{d}W, \int_A f_2 \mathrm{d}W, \ldots, \int_A f_n \mathrm{d}W, \ldots$$

converges in probability. The limit is called stochastic integral and denoted by

$$\int_A f \mathrm{d}W. \tag{4}$$

Theorem 3. *Let f be measurable on T. The stochastic integral* (4) *exists (i. e. the Definition 4 is correct) if and only if the structural function F of the Wiener process W is integrable with respect to ρ for each t.*

All considerations above are valid for a Wiener process with i. d. d. Now we apply the result to the random measure defined in (2).

For any $S\alpha S$ Wiener process we have $\varphi(A, t)=-c\,.\,\rho(A)\,\|\,t\,\|^{\alpha}$. So $F(a, t)=-\|\,t\,\|^{\alpha}$.

Then using Theorem 3, we obtain

$$\varphi(f, t)=\int_T F(a, f(a)t)\rho(\mathrm{d}a)=-\|\,t\,\|^{\alpha}\int_T |f(a)|^{\alpha}\rho(\mathrm{d}a),$$

what means that F is integrable with respect to ρ iff $\int_T |f|^{\alpha}\mathrm{d}\rho<\infty$.

Corollary 1. *The stochastic integral* (4) *with respect to $S\alpha S$ Wiener process W exists iff $f\in L_{\alpha}(T, \mathcal{B}(T), \rho)$.*

Finally we shall outline some usefull properties of $\xi(t)$ defined in (3).

1. The r. v. $\xi(t_2)-\xi(t_1)$ has $S\alpha S$ distribution with the same shape parameter α and scale parameter depending on t_2-t_1 only.

2. A correlation function analogue could be defined [3] if $T=\mathbf{R}$

$$R(\tau)=1-[\int_{\mathbf{R}}|f(x+\tau)-f(x)|^{\alpha}\mathrm{d}x/\int_{\mathbf{R}}|f(x)|^{\alpha}\mathrm{d}x].$$

It may be shown that in the case of Gausian law $\alpha=2$ it coincides with the usual definition of correlation. Moreover, under some assumptions on memory f it can be easily derived from Γ what helps the statistical estimation of the process parameters. These results will appear elsewhere in the near future.

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