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NUMERICAL METHOD FOR SOLVING AN ELLIPTIC EQUATION OF FOURTH ORDER

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The paper treats the second boundary value problem for an elliptic equation of fourth order in a bounded domain in R^2. The problem is reduced to a problem of the same type in a rectangular and to a system of two boundary integral equations. The Galerkin and collocation methods are applied to the studied problem. Theoretical estimates for them are obtained.

The paper treats the second boundary value problem for an elliptic equation of fourth order with constant coefficients. The domain is bounded and simply connected in R^2 and has smooth boundary. This problem arises in the analysis of plates resting on an elastic foundation of the Winkler type.

The direct application of finite element or finite differences methods to problems in irregular domains is connected with some specific difficulties. Hence other equivalent formulations of the considered problem have to be investigated. The important works [1, 2, 3] in this area are devoted to the reduction of the problem to a system of integral equations on the boundary. But efficient computation of the solution on detail meshes requires new ways of reduction.

In this paper we reduce the investigated problem to: (i) a differential problem of the same type in a rectangular containing the given domain and (ii) a system of two boundary integral equations. Similar methods have been studied for elliptic equations of second order and biharmonic equations [4, 5].

The paper is organized as follows. Section 1 states the initial problem. In Section 2 we examine the same problem in a rectangular. In Section 3 we give the definition of the Green function used later. In Section 4 we reduce the problem to a system of integral equations. Section 5 deals with the properties of this system. First we investigate the main characteristics of the integral operators. Then we obtain existence and uniqueness theorems for the solution of the considered system. In Sections 6 and 7 we apply the Galerkin and collocation methods for numerical solving of the integral equations. In Section 8 we formulate the final error estimates of the methods and discuss the obtained results.

1. Let us consider the bounded simply connected domain $D\subset R^2$ with boundary Σ. Let $r=r(s)=(r_1(s), r_2(s))$, $0\leq s\leq 2\pi$ be the parametrization of Σ and let $r(s)$ be 4-times continuously differentiable function, $r(s)\in C^4(0, 2\pi)$. We assume that the functions $g_1(s)$, $g_2(s)$ and $f(r)$ are given and belong to the Sobolev spaces $W_2^{3,5}(0, 2\pi)$, $W_2^{1,5}(0, 2\pi)$ and $L_2(D)$ respectively. Let β be a fixed real constant.

Consider the solution $u(r)$ to the second boundary value problem for the fourth order differential equation

$$\begin{aligned} &\Delta^2 u(r)+\beta^2 u(r)=f(r), \quad r\in D, \\ &u(s)=g_1(s),\ \Delta u(s)=g_2(s), \quad 0\leq s\leq 2\pi. \end{aligned} \tag{1}$$

Under the above assumptions problem (1) has a unique solution $u(r)\in W_2^4(D)$.

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Our purpose is to construct an efficient numerical method for computation of $u(r)$.

2. Let Ω be a rectangular containing the closure of the domain D. Let $f_0(r)$ be the L_2-extension of $f(r)$ from D to Ω. Let $u_0(r)$ be the solution of the problem

$$\begin{aligned} &\Delta^2 u_0(r)+\beta^2 u_0(r)=f_0(r), \quad r\in\Omega, \\ &u_0(r)=\Delta u_0(r)=0, \quad r\in\partial\Omega \end{aligned} \tag{2}$$

in the rectangular Ω.

Denote $v(r)=u(r)-u_0(r)$. Then $v(r)$ solves the following problem

$$\begin{aligned} &\Delta^2 v(r)+\beta^2 v(r)=0, \quad r\in D, \\ &v(s)=g_3(s), \ \Delta v(s)=g_4(s), \quad 0\leq s\leq 2\pi, \end{aligned} \tag{3}$$

where $g_3(s)=g_1(s)-u_0(s)$, $g_4(s)=g_2(s)-\Delta u_0(s)$.

Notice that the solution $u_0(r)$ of problem (2) can be efficiently evaluated because of the domain being a rectangular. Therefore the numerical solution of (1) is reduced to the one of (3).

3. Define Green's function $G(r,\rho)$ with singularities on the curve Σ as the solution of the problem (4)

$$\begin{aligned} &\Delta^2 G(r,\rho)+\beta^2 G(r,\rho)=\delta(r-\rho), \quad r\in\Omega, \quad \rho\in\Sigma, \\ &G(r,\rho)=\Delta G(r,\rho)=0, \quad r\in\partial\Omega, \quad \rho\in\Sigma. \end{aligned} \tag{4}$$

Here $\delta(r)$ is the Dirac function.

The fundamental solution of equation (4) is [2] the real part of the Hankel function $\mathrm{Re}(H_0^{(1)}(\sqrt{i|\beta|}\,|r-\rho|))/|\beta|^2$. We shall suggest an effecient method for the calculation of the Green function $G(r,\rho)$ in a following paper.

4. The solution $v(r)$, $r\in D$ of (3) can be represented as a sum of potentials

$$v(r)=\int_\Sigma \lambda(\rho)\,\Delta G(r,\rho)\,d\rho+\int_\Sigma \mu(p)G(r,\rho)\,d\rho, \tag{5}$$

where $\lambda(\rho)$ and $\mu(\rho)$ are unknown density functions.

Using (5) the boundary conditions of problem (3) yield a system of two integral equations of the first kind with respect to $\mu(\rho)$ and of the second kind with respect to $\lambda(\rho)$

$$\begin{aligned} &\int_\Sigma \lambda(\rho)\,\Delta G(r,\rho)\,d\rho+\int_\Sigma \mu(\rho)\,G(r,\rho)\,d\rho=g_3(r), \quad r\in\Sigma, \\ &\lambda(r)-\int_\Sigma \beta^2\lambda(\rho)\,G(r,\rho)\,d\rho+\int_\Sigma \mu(\rho)\,\Delta G(r,\rho)\,d\rho=g_4(r), \quad r\in\Sigma. \end{aligned} \tag{6}$$

The kernel $\Delta G(r,\rho)$ from (6) has a logarithmic singularity (as the principal part of $G(r,\rho)$ is equal to $|r-\rho|^2\ln|r-\rho|$). Let us study the properties of system (6).

5. Let $W_2^k(\Sigma)$, $k\in R$, be the Sobolev space of all 2π-periodic functions $g(s)=\Sigma_{|l|\geq 0}\,g(l)\exp(ils)$ with scalar product

$$(g_1,g_2)_k=g_1(0)\,g_2(0)+\sum_{|l|>0} g_1(l)\,g_2(l)\,|l|^{2k}.$$

Let $\langle x,y\rangle_k$ be the inner product in the product space $W_2^k(\Sigma)\times W_2^{k-2}(\Sigma)$ of 2-vector valued functions $x=(x_1,x_2)^T$:

$$\langle x,y\rangle_k=\langle (x_1,x_2)^T,(y_1,y_2)^T\rangle_k=(x_1,y_1)_k+(x_2,y_2)_{k-2}$$

with corresponding norm $|\langle x\rangle|_k=(\langle x,x\rangle_k)^{0.5}$.

Define operators A and B by the formulae

$$Av(r)=\int_{\Sigma} v(\rho)\,\Delta G(r,\ \rho)\,d\rho, \quad r\in\Sigma,$$

$$Bv(r)=\int_{\Sigma} v(\rho)\,G(r,\ \rho)\,d\rho, \quad r\in\Sigma.$$

Let A_0 and B_0 be the principal parts of A and B. These principal parts are completely characterized in the case when Σ is the unit circle and $\beta=0$.

Let x and y be the vectors $(\lambda,\ \mu)^T$ and $(g_3,\ g_4)^T$; let C be the matrix $\begin{pmatrix} A & B \\ E-\beta^2 B & A \end{pmatrix}$ and C_0 — the matrix $\begin{pmatrix} A_0 & B_0 \\ 0 & A_0 \end{pmatrix}$.

Then the system (6) takes the equivalent form

$$(6') \qquad Cx=g.$$

The essential characteristics of the operators A, B and C are formulated in

Theorem 1. *For any real k the following properties are valid:*

(i) *The mappings*

$$A,\ A_0\colon\ W_2^k(\Sigma)\to W_2^{k+1}(\Sigma),$$

$$B,\ B_0\colon\ W_2^k(\Sigma)\to W_2^{k+3}(\Sigma),$$

$$A-A_0\colon\ W_k^2(\Sigma)\to W_2^{k+3}(\Sigma),$$

$$B-B_0\colon\ W_2^k(\Sigma)\to W_2^{k+5}(\Sigma)$$

are continuous.

(ii) *C is a continuous linear operator from $W_2^k(\Sigma)\times W_2^{k-2}(\Sigma)$ into $W_2^{k+1}(\Sigma)\times W_2^{k-1}(\Sigma)$. $C-C_0$ is a compact operator from $W_2^k(\Sigma)\times W_2^{k-2}(\Sigma)$ into $W_2^{k+1}(\Sigma)\times W_2^{k-1}(\Sigma)$.*

(iii) *There exist positive constants γ_1, γ_2 such that for all $x\in W_2^k(\Sigma)\times W_2^{k-2}(\Sigma)$ the inequalities*

$$\langle -C_0 x,\ x\rangle_k \geqq \gamma_1 |\langle x\rangle|_{k-0.5}^2,$$

$$|\langle (C-C_0)x,\ x\rangle_k| \leqq \gamma_2 |\langle x\rangle|_{k-1}^2$$

are satisfied.

Proof. Theorem 1 follows by standard arguments of the classical theory [6] of pseudodifferential operators on the boundary manifold Σ and the facts that: the principal symbols of A and B are $-4\pi/|l|$, $l\neq 0$ and $\pi/2|l|(l^2-1)$, $|l|>1$ resp.; the kernel of $A-A_0$ is a smooth function belonging to the space $C^3(0,\ 2\pi)$ under the assumption $\Sigma\in C^4(0,\ 2\pi)$.

A direct consequence of Theorem 1 is

Corollary 1. *The operator $-C$ is k-coercive: $-C$ satisfies the Gårding inequality*

$$\langle -Cx,\ x\rangle_k \geqq \gamma_1 |\langle x\rangle|_{k-0.5}^2 - \gamma_2 |\langle x\rangle|_{k-1}^2.$$

Now we use the uniqueness of the weak solution of the differential problem (3) in the space $W_2^2(D)$ together with Theorem 1 and the Fredholm theorems to prove the following

Theorem 2. *Let $k\geqq 0.5$ and g be a function belonging to $W_2^{k+1}(\Sigma)\times W_2^{k-1}(\Sigma)$. Then*

Theorem 5. *Assume that* $0.5 \le k-0.5 \le m$, $m \le \min(2l_1, 2l_2+2)$, $k < \min(2l_1, 2l_2+2)$, $l_1 = 1, 2, \ldots$, $l_2 = 1, 2, \ldots$ *If* $x \in W_2^m(\Sigma) \times W_2^{m-2}(\Sigma)$ *then there exist positive constants* γ_7 *and* h_0 *such that for any partition* Δ *with* $h < h_0$ *there is a unique solution* $x_h \in \tilde{H}_h^{l_1, l_2}(\Delta)$ *of the Galerkin equations* (7) *and the error estimate*

$$|\langle x - x_h \rangle|_{k-0.5} \le \gamma_7 h^{m-k+0.5} |\langle x \rangle|_m$$

is valid.

7. With the same notations as in the previous sections we determine the collocation approximation $x_h \in \tilde{H}_h^{l, l-2}(\Delta) = S_h^{2l-1}(\Delta) \times S_h^{2l-5}(\Delta)$ of x from the collocation equations

$$(Cx)(r_j) = g(r_j), \quad j = 1, 2, \ldots, N. \tag{9}$$

Here $\{r_j\}_{j=1}^N$ are collocation points.

Let $l > 2$, then from the Sobolev embedding theorem the functions in the space $\tilde{H}_h^{l, l-2}(\Delta)$ are continuous and we may collocate. In view of Theorem 1 existence of a solution and convergence of the collocation method follow from the standard theory [8] and we establish the next result.

Theorem 6. *Assume that* $l > 2$. *Then for sufficiently small* h *the collocation equations* (9) *have a unique solution* $x_h \in \tilde{H}_h^{l, l-2}(\Delta)$ *and the error estimate*

$$|\langle x - x_h \rangle|_{l-0.5} \le \gamma_8 h^{m-l+0.5} |\langle x \rangle|_m$$

is valid with some constant $\gamma_8 > 0$ *and* $l - 0.5 \le m \le 2l - 2$.

8.1. The error estimates obtained in Theorems 3, 4, 5, 6 can be used to compare the Galerkin method with the collocation method.
(i) Let us first consider the case we use the same degree splines $S_h^{2k-1}(\Delta) \times S_h^{2k-5}(\Delta)$, $k > 2$, for both methods. Then the highest rate of convergence achieved by both methods is the same, $O(h^{k-1.5})$ in the norm $W_2^{k-0.5}(\Sigma) \times W_2^{k-2.5}(\Sigma)$. On the other hand, the construction of the stiffness matrix for the Galerkin method requires the evaluation of double integrals whilst the collocation method requires single integrals only.
(ii) Splines of different degrees $2l_1 - 1$, $2l_2 - 1$ with respect to the density functions λ and μ may be used in the Galerkin method. In the collocation method degrees of the related splines are $2l-1$ and $2l-5$. Therefore the Galerkin method allows to apply splines of lower degrees for numerical evaluation of the solutions of (6′).
(iii) In both methods with splines the order of convergence cannot exceed some constant (for the collocation method $l - 1.5$). On the contrary, the Galerkin method with finite sections of Fourier series converges with any rate $m - k + 0.5$, $m > k - 0.5$.

8.2. In Sections 6 and 7 we have investigated two numerical methods for evaluation of the approximation $x_h = (\lambda_h, \mu_h)^T$ of the density function $x = (\lambda, \mu)^T$. Denote by $v_h(r)$ the corresponding potential function

$$\int_\Sigma \lambda_h(\rho) \Delta G(r, \rho) d\rho + \int_\Sigma \mu_h(\rho) G(r, \rho) d\rho.$$

Since the operators A: $W_2^{k-0.5}(\Sigma) \to W_2^{k+1}(D)$, B: $W_2^{k-2.5}(\Sigma) \to W_2^{k+1}(D)$ are continuous mappings ([6], § 8), the inequality

$$\| v - v_h \|_{W_2^{k+1}(D)} \le \gamma_9 |\langle x - x_h \rangle|_{k-0.5} \tag{10}$$

holds for some constant $\gamma_9 > 0$. The right-hand term in (10) can be estimated by Theorem 3. Thus we get the following

Corollary 2. *Let* $x_h = (\lambda_h, \mu_h)^T \in \tilde{H}_N$ *be the appoximation of* $x = (\lambda, \mu)^T$ *by the Galerkin method* (7) *with basic functions from* $\tilde{H}_N$. *Then the error estimate*

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$$\| v - v_h \|_{W_2^{k+1}(D)} \leq \gamma_{10} N^{k-m-0.5} | \langle g \rangle |_{m+1}$$

holds for $m > k - 0.5$, $k \geq 1$ *and some constant* $\gamma_{10} > 0$.

Consequences of Theorems 4, 5, 6 can be obtained analogously. In this way, we have completed the presentation of our method and its error estimates.

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