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THE MULTIPLICATIVITY PROPERTY OF THE FIXED POINT INDEX FOR MULTIVALUED MAPS

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The fixed point index for multivalued maps has been defined by many authors [1, 3, 6-13, 16-18]. The index defined in [16-18, 3, 6, 8] satisfies the commutativity and the mod-p property but it remains unknown whether the multiplicativity property holds. An alternative approach to the fixed point index for Z-acyclic maps of ENR-s is given in [6]. The definition is based on the fact that a Z-acyclic map is homotopic to a single valued map after an appropriate embedding of the space into a sphere. In this paper the uniqueness of the fixed point index for Z-acyclic maps on ENR-s is proved. From the uniqueness it follows that the fixed point index has all the properties of the index defined in [16] and, moreover, the multiplicativity one.

In the present paper we give an affirmative answer to the question about the multiplicativity property of the fixed point index of F-acyclic (and some classes of nonacyclic) maps on ANR-s. This result is obtained by some generalization of the chain approximations and A-systems introduced in [3].

I. A generalization of A-system of multivalued map

1. Block complexes [14, 134], [4, p. 126].

Definition. Let K be a finite simplicial complex with a fixed triangulation. An n-block in K is a pair of subcomplexes $(e, \dot{e})$ such that $\dim e = n$ and $H_r(e, \dot{e}, \mathrm{F}) = H_r(B^n, \dot{B}^n, \mathrm{F})$ (here B^n is the n-dimensional ball and $\dot{B}^n$ its boundary, F is a given field).

The subcomplex $\dot{e}$ is called a boundary of e and the interior of e is the set $e \setminus \dot{e}$.

Definition [14, p. 134]. A block dissection τ of a simplicial complex K with a given triangulation τ' is a set of i-blocks of τ' such that:

(a) each simplex of the triangulation τ' lies in the interior of exactly one block of τ.

(b) the boundary of each n-block is a union of m-blocks for $m<n$.

If in the simplicial complex K a block dissection τ is fixed, we call K a block complex with a block structure τ.

Definition. The mesh of the block complex is $\operatorname{mesh}(K, \tau) = \max\{\operatorname{diam} e : e \in \tau\}$ (in K we consider the metric defined by barycentric coordinates).

Definition. Let (K, τ) be a block complex on the simplicial complex (K, τ') and let τ_1 be a block structure of a barycentric subdivision τ_1' of the triangulation τ'. The block structure τ_1 is called a subdivision of the block structure τ if every block of τ_1 is in the interior of exactly one block of τ.

Definition. Let τ_1 be a block dissection of the triangulation τ_1'. The sequence $\{\tau_i\}$ of block subdivisions of τ_1 is called a fundamental sequence of block subdivisions if the following conditions hold: a) τ_{i+1} is a subdivision of τ_i, $i = 1, 2, \dots$, b) $\lim \operatorname{mesh}(K, \tau_i) = 0$.

Let τ be a block dissection of the triangulation τ'. We denote by $C_*(K, \tau)$ the chain complex of the block complex τ, with the coefficients in the field F, [14, p. 136].

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Let M be a subset of K. We denote by $\mathrm{St}(M,\tau)=\bigcup\{e\in\tau: e\cap M\neq\emptyset\}$ the star of the set M with respect to the block structure τ. By induction $\mathrm{St}^{k+1}(M,\tau)=\mathrm{St}(\mathrm{St}^k(M,\tau),\tau)$.

For a given block dissection τ of the triangulation τ' there is a unique chain map $b(\tau,\tau'):C_*(K,\tau)\to C_*(K,\tau')$ with the following properties:

(a) $b(\tau,\tau')(a)=a$ for every vertex a of τ.

(b) $\mathrm{carr}\, b(\tau,\tau')(e)=e$ for every block $e\in\tau$ [14, p. 136—137].

Lemma 1. *Let (K,τ), (L,μ) be block complexes on the triangulations τ' and μ'. Let*

$$\varphi: C_*(K,\tau')\to C_*(L,\mu')$$

be a chain map with a Kronecker index $KI\varphi=m$, $m\in F$. Then there is a chain map

$$\theta: C_*(K,\tau)\to C_*(L,\mu)$$

and a homotopy D connecting the chain maps θ and $\varphi b(\tau,\tau')$ such that:

(a) $$\mathrm{carr}\,\theta(e)\subset\mathrm{carr}(\varphi b(\tau,\tau')(e),\mu),$$

(b) $$\mathrm{carr}(D(e),\mu)\subset\mathrm{carr}(\varphi b(\tau,\tau')(e),\mu),$$

(c) $$KI\theta=KI\varphi.$$

Proof. This lemma is obtained by using the method of acyclic carriers (in the case $\varphi=$ identity it is proved in [4, p. 127-129], where the properties (a) and (b) are not stated explicitely but they follow from the construction). The chain map θ and the homotopy D are defined by induction on the dimension of the block e. We confine ourselves only to the case $\dim e=0$.

Let a be a point of the block dissection τ. Then a is also a vertex of the triangulation τ'. Consider the chain $\varphi(a)=\Sigma\lambda_i b_i$ (b_i are the vertices of the triangulation μ'), $KI\varphi(a)=\Sigma\lambda_i=m$. The vertex b_i has a carrier — the block $\bar{e}_i$ in the block structure μ. Let c_i be a simplicial path in $\bar{e}_i$ (in the triangulation μ') connecting the point b_i with a vertex $\hat{b}_i$ of the block $\bar{e}_i$. Define θ_0 and D_0 by $\theta_0(a)=\Sigma\lambda_i\hat{b}_i$ and $D_0(a)=\Sigma\lambda_i c_i$. The maps θ_0 and D_0 satisfy the conditions (a)-(c). The maps θ and D are defined by induction following the construction in [4, p. 127-129].

Suppose τ_1 is a block dissection of the triangulation τ_1' and τ_2 is its block subdivision (τ_2 is a block dissection of the triangulation τ_2' which is a barycentric subdivision of the triangulation τ_1'). There is a chain map (subdivision)

$$b(\tau_1,\tau_2): C_*(K,\tau_1)\to C_*(K,\tau_2),$$

defined by $b(\tau_1,\tau_2)(e)=\Sigma e_i$, where $e_i\in\tau_2$, $\dim e_i=\dim e$, $e_i\subset e$ and the orientation of e_i is induced by the orientation of the block e.

The chain map $b(\tau_1,\tau_2)$ is obtained also if applying Lemma 1 to the chain map $b(\tau_1',\tau_2'): C_*(K,\tau_1')\to C_*(K,\tau_2')$, where $b(\tau_1',\tau_2')$ is the barycentric subdivision of simplicial chains, [14, p. 115].

Let us consider the chain map $\chi(\tau_2',\tau_1'): C_*(K,\tau_2')\to C_*(K,\tau_1')$. This is the chain map induced by a simplicial approximation of the identity map $\mathrm{id}:(K,\tau_2')\to(K,\tau_1')$.

Applying Lemma 1 to the chain map $\chi(\tau_2,\tau_1)$, we obtain the following chain map $\chi(\tau_2,\tau_1): C_*(K,\tau_2)\to C_*(K,\tau_1)$ satisfying the conditions of Lemma 1.

2. Chain approximations of upper semicontinuous maps. For upper semicontinuous maps (u.s.c. maps) see [5].

Lemma 2. *Let $\Phi : K \to L$ be an* F*-acyclic map. Let $\{\tau_i\}$, $\{\mu_i\}$ be fundamental systems of block dissections of K and L, respectively. Let $\tau_{k_0} \in \{\tau_i\}$, $\mu_{l_0} \in \{\mu_i\}$, $n \in$* N. *Then there exist k_i, $l_i \in$* N *such that $k_0 < k_1 < \cdots < k_{n+1}$, $l_1 < l_2 < \cdots < l_n$. For every block $e \in \tau_{k_i}$ there are $l_{i-1}(e) \in$* N *and a point $a(l) \in K$ satisfying the following conditions:*

(1) $$l_{i-1} < l_{i-1}(e) < l_i,$$

(2) $$e \subset \mathrm{St}\,(a(e),\ k_{i-1}),$$

(3) $$\Phi(\mathrm{St}\,(e,\ k_i)) \subset \mathrm{St}\,(\Phi(a(e)),\ l_{i-1}(e)),$$

(4) *The homomorphism* $i_* : \widetilde{H}_*(\mathrm{St}\,(\Phi(a(e)),\ l_{i-1}(e)) \to \widetilde{H}_*(\mathrm{St}\,(\Phi(a(e)),\ l_{i-1})$

is zero.

Here i_* is induced by the identity embedding i. ($\mathrm{St}(M,\ i) = \mathrm{St}(M,\ \tau_i)$ for $M \subset K$ and $\mathrm{St}\,(M,\ i) = \mathrm{St}\,(M,\ \mu_i)$ for $M \subset L$). Recall that the map $\Phi : K \to L$ is acyclic if it is u.s.c. and for every point $x \in K$ the compact set $\Phi(x)$ is not empty and acyclic with respect to the homology of Alexandroff — Čech with coefficients in the field F, [12].

Proof. By induction on n. $n = 0$. The Čech homologies with coefficients in the field F are continuous and are finite dimensional vector spaces over the field F for finite polyhedra. The compact set $\Phi(x)$ is acyclic. Then it follows that there exists $l_0(x) \in$ N, $l_0(x) > l_0$ such that the identity embedding

$$i(x) : \mathrm{St}\,(\Phi(x),\ l_0(x)) \subset \mathrm{St}\,(\Phi(x),\ l_0)$$

induces the zero homomorphism in the reduced homology with coefficients in the field F.

Since the map Φ is u.s.c., then for every point x in K there is a neighbourhood Ox such that

$$\Phi(Ox) \subset \mathrm{St}\,(\Phi(x),\ l_0(x))$$

and $\mathrm{diam}\ Ox \leq \delta(\tau_{k_0})$ where $\delta(\tau_{k_0})$ is the Lebesgue number of the covering

$$\mathrm{St}\,\tau_{k_0} = \{\mathrm{St}\,(e,\ k_0) : e \in \tau_{k_0},\ \dim e = 0\}.$$

Let $\omega = \{Ox_1, \ldots, Ox_t\}$ be a finite subcovering of the covering $\{Ox : x \in K\}$ and $k_i \in$ N be such that $\mathrm{St}\,\tau_{k_1} > \omega$ and $k_1 > k_0$.

For every block $e \in \tau_{k_1}$ there is $Ox_i \in \omega$ such that $\mathrm{St}\,(e,\ k_1) \subset Ox_i$.

Let us define $a(e) = x_i$ and $l_0(e) = l_0(x_i)$. The conditions (1)-(4) of Lemma 2 are satisfied for $n = 0$.

Let $l_1 \in$ N be such that $l_1 > \max\{l_0(e) : e \in \tau_{k_1}\}$.

Suppose that Lemma 2 is proved for every $i \leq n$. For $i = n+1$ the proof is the same as for $n = 0$ but in this case we begin with k_n and l_n instead of k_0 and l_0.

Lemma 3. *Let $\Phi : K \to L$ be an* F*-acyclic map. Let $\{\tau_i\}$, $\{\mu_i\}$ be fundamental systems of block dissections of K and L, respectively. Suppose $\tau_k \in \{\tau_i\}$, $\mu_l \in \{\mu_i\}$. Then there exist $\bar{k} \in$* N, *$\bar{k} > k$ and a chain map*

$$\varphi : C_*(K,\ \bar{k}) \to C_*(L,\ l)$$

satisfying the following conditions: for every block $e \in \tau_{\bar{k}}$ there is a point $s(e)$ such that

(1) $$e \subset \mathrm{St}\,(s(e),\ k),$$

(2) $$\operatorname{carr} \varphi(e) \subset \operatorname{St}(\Phi(s(e)), l).$$

Here $C_*(K, i) = C_*(K, \tau_i)$, $C_*(L, i) = C_*(L, \mu_i)$.

Proof. Let us apply Lemma 2 to $k = k_0$, $l = l_0$, $n = \dim K$. We obtain $k_i, l_i \in \mathrm{N}$, $k_0 < k_1 < \cdots < k_{n+1}$, $l_0 < l_i \cdots < l_n$, $l_{i-1}(e) \in \mathrm{N}$, $a(e) \in K$ for every block $e \in \tau_{k_i}$ satisfying the conditions (1)-(4) of Lemma 2.

We shall construct the chain map $\varphi : C_*(K, k_{n+1}) \to C_*(L, l)$ by induction on the dimension j. We denote by $(K, k_{n+1})^{(j)}$ the j-dimensional skeleton of the block complex $(K, \tau_{k_{n+1}})$.

$j=0$. Let e be a vertex of the block dissection $\tau_{k_{n+1}}$. Let a be a vertex in the carrier of the set $\Phi(a(e))$ in the block dissection μ_{l_n}. By setting $\varphi^0(e) = a$, $s(e) = a(e)$ we obtain a chain map

$$\varphi^0 : C_*((K, k_{n+1})^{(0)}) \to C_*(L, l_n)$$

for which the conditions of Lemma 2 hold.

$j=1$. Let e be an one-dimensional block of the block dissection $\tau_{k_{n+1}}$. The carrier of the block e in τ_{k_n} is denoted by $\widehat{e}$. From Lemma 2 we obtain a point $a(\widehat{e}) \in K$ satisfying the conditions of this lemma. Moreover, since $\partial e = e_1 - e_0 \subset e \subset \widehat{e}$ we have

$$\{a(e_0), a(e_1)\} \subset \operatorname{St}(e, k_n) \subset \operatorname{St}(\widehat{e}, k_n).$$

Therefore $\Phi(a(e_0)) \cup \Phi(a(e_1)) \subset \Phi(\operatorname{St}(\widehat{e}, k_n))$. From Lemma 2, (3) it follows that $\Phi(a(e_0)) \cup \Phi(a(e_1)) \subset \operatorname{St}(\Phi(a(\widehat{e})), l_{n-1}(\widehat{e}))$.

Since $l_n > l_{n-1}(\widehat{e}) > l_{n-1}$, then $\varphi^0(e_0)$ and $\varphi^0(e_1)$ are vertices in the block complex $\operatorname{St}(\Phi(a(\widehat{e})), l_{n-1}(\widehat{e}))$. The chain $\varphi^0(e_1) - \varphi^0(e_0) = \varphi^0 \partial e$ is a cycle in the block complex $\operatorname{St}(\Phi(a(\widehat{e})), l_{n-1}(\widehat{e}))$. From Lemma 2, (4) we obtain that the chain $\varphi^0 \partial e$ is homologous to zero in the block complex $\operatorname{St}(\Phi(a(\widehat{e})), l_{n-1})$. Therefore there is a one dimensional chain c in this complex such that $\partial c = \varphi^0 \partial e$.

We define $\varphi^1(e) = c$, $\varphi^1(b) = \varphi^0(b)$ for every vertex b of $\tau_{k_{n+1}}$ and obtain a chain map $\varphi^1 : C_*((K, k_{n+1})^{(1)}) \to C_*(L, l_{n-1})$ satisfying the following conditions: for every block e in $(K, k_{n+1})^{(1)}$ there is a point $s(e) \in K$ such that

$$e \subset \operatorname{St}(s(e), k_{n-1}),$$

$$\operatorname{carr} \varphi^1(e) \subset \operatorname{St}(\Phi(s(e)), l_{n-1}),$$

$$K I \varphi^1 = 1.$$

Suppose that the chain map $\varphi^i : C_*((K, k_{n+1})^{(i)}) \to C_*(L, l_{n-i})$ has already been defined and satisfies the following conditions: for every block $e \in (K, k_{n+1})^{(i)}$ there is a point $s(e) \in K$ such that

$$e \subset \operatorname{St}(s(e), k_{n-i}),$$

$$\operatorname{carr} \varphi^i(e) \subset \operatorname{St}(\Phi(s(e)), l_{n-i}),$$

$$K I \varphi^i = 1.$$

Let e be $(i+1)$-dimensional block in $\tau_{k_{n+1}}$ and $\widehat{e}$ be its carrier in $\tau_{k_{n-i+1}}$. Let us consider the chain $\partial e = \Sigma e_j$ (here e_j are i-dimensional blocks in $\tau_{k_{n+1}}$). By the induction hypothesis $s(e_j) \in \operatorname{St}(e_j, k_{n-i}) \subset \operatorname{St}(\widehat{e}, k_{n-i})$.

From Lemma 2, (3) we obtain: $\Phi(s(e_j))\subset\Phi(\mathrm{St}(\widehat{e},\ k_{n-i}))\subset \mathrm{St}(\Phi(a(\widehat{e})), l_{n-i-1}(\widehat{e}))$. Therefore

$$\mathrm{carr}\,\varphi^i\partial e\subset \mathrm{St}(\Phi(a(\widehat{e})),\ l_{n-i-1}(\widehat{e})).$$

Since $l_{n-i}>l_{n-i-1}(\widehat{e})>l_{n-i-1}$, then the block chain $\chi(l_{n-i},\ l_{n-i-1}(\widehat{e}))\,\varphi^i\,\partial e$ is a chain of the block complex $\mathrm{St}(\Phi(a(\widehat{e})),\ l_{n-i-1}(\widehat{e}))$ and it is a cycle. It follows from Lemma 2, (4) that the chain $\chi(l_{n-i},\ l_{n-i-1},\ (\widehat{e}))\,\varphi^i\,\partial e$ is homologous to zero in the block complex $\mathrm{St}(\Phi(a(\widehat{e})),\ l_{n-i-1})$ i. e. there is a chain c_{i+1} in this complex such that

$$\chi(l_{n-i},\ l_{n-i-1}(\widehat{e}))\,\varphi^i\partial e=\partial_{c_{i+1}}.$$

We define $\varphi^{i+1}(e)=c_{i+1}$, $s(e)=a(\widehat{e})$ for an $(i+1)$-dimensional block e. If $\dim e\leqq i$ we define

$$\varphi^{i+1}(e)=\chi(l_{n-i},\ l_{n-i-1})\,\varphi^i(e).$$

By definition $\varphi^{i+1}: C_*((K,\ k_{n+1})^{(i+1)})\to C_*(L,\ l_{n-i-1})$ is a chain map satisfying the conditions

$$e\subset \mathrm{St}(s(e),\ k_{n-i-1}).$$

$$\mathrm{carr}\ \varphi^{i+1}(e)\subset \mathrm{St}(\Phi(a(\widehat{e})),\ l_{n-i-1}),$$

$$K\,\mathrm{I}\ \varphi^{i+1}=1.$$

Thus we construct the chain map φ^n. Finally we put $\varphi=\varphi^n$, $\bar{k}=k_{n+1}$. Lemma **3** is proved.

D e f i n i t i o n. Let $\Phi: K\to L$ be a u.s.c. map. Let $\{\tau_i\}$, $\{\mu_i\}$ be fundamental systems of block dissections of the simplicial complexes K and L, respectively. The chain map

$$\psi: C_*(K,\ \widetilde{k})\to C_*(L,\ l)$$

is called $(n,\ \tau_k,\ \mu_l)$ or $(n,\ k,\ l)$ chain approximation of the map Φ if $\widetilde{k}\geqq k$ and if for every block $e\in\tau_{\widetilde{k}}$ there is a point $s(e)\in K$ such that

(1) $e\subset \mathrm{St}^n(s(e),\ k)$,

(2) $\mathrm{carr}\,\psi(e)\subset \mathrm{St}^n(\Phi(s(e)),\ l)$

($(1,\ k,\ l)$ approximations we call $(k,\ l\text{-})$-approximations).

L e m m a 3 shows us that for an F-acyclic map $\Phi: K\to L$ $k,\ l\in\mathrm{N}$ there exists $(k,\ l)$ chain approximation of Φ.

L e m m a 4. *For every* $j\in\mathrm{N}$ *there exists* $\bar{k}\in\mathrm{N}$, $\bar{k}>j$ *such that if* $m, l, m_1, l_1\in\mathrm{N}$ *and if the mappings*

$$\varphi_1: C_*(K,\ l_1)\to C_*(L,\ l),\quad \psi_1: C_*(K,\ m_1)\to C_*(L,\ m)$$

are $(l,\ l)$ *and* $(m,\ m)$ *chain approximations of the map* Φ *respectively and* $m_1\geqq m\geqq l$, $m_1\geqq l_1\geqq l\geqq k\geqq \bar{k}$, then there is a homotopy

$$D: C_*(K,\ m_1)\to C_*(L,\ l)$$

connecting $\varphi_1\chi(m_1,\ l_1)$ *and* $\chi(m,\ l)\,\psi_1$.

The homotopy D *satisfies the following conditions: for every block* $e\in\tau_{m_1}$ *there is a point* $b(e)\in K$ *such that* $e\subset \mathrm{St}(b(e),\ j)$, $\mathrm{carr}\,D(e)\subset St(\Phi(b(e)),\ j)$.

P r o o f. Let us apply Lemma 2 to $k_0=l_0=j$ and $n=\dim K$. Then we obtain the natural numbers $\{k_i\}$, $\{l_i\}$, $\{l_i(e)\}$ and points $a(e)$ satisfying the conditions (1)-(4) of Lemma 2. Define $\bar{k}=k_{n+1}$.

We shall construct the homotopy D by an induction on the dimension t.

$t=0$. Suppose σ is a vertex of the block dissection τ_{m_1} and $\bar{\sigma}=\chi(m_1, l_1)(\sigma)$ is a vertex of τ_{l_1}. Since φ_1 is a (l, l) chain approximation of the map Φ we have: there exists a point $s(\bar{\sigma})$ such that

$$(1)\qquad \begin{gathered} s(\bar{\sigma}) \in \mathrm{St}\,(\bar{\sigma}, l), \\ \operatorname{carr} \varphi_1(\bar{\sigma}) \subset \mathrm{St}\,(\Phi(s(\bar{\sigma})), l). \end{gathered}$$

For the chain map ψ_1 — there exists a point $s(\sigma)$ such that

$$(2)\qquad \begin{gathered} s(\sigma) \in \mathrm{St}\,(\sigma, m), \\ \operatorname{carr} \psi_1(\sigma) \subset \mathrm{St}\,(\Phi(s(\sigma)), m) \end{gathered}$$

Obviously $\{s(\sigma), s(\bar{\sigma})\} \subset \mathrm{St}\,(\bar{\sigma}, l)$. Therefore

$$(3)\qquad \Phi(s(\sigma)) \cup \Phi(s(\bar{\sigma})) \subset \Phi(\mathrm{St}\,(\bar{\sigma}, l)).$$

Let $\hat{\sigma}$ be the carrier of $\bar{\sigma}$ in the block dissection $\tau_{k_{n+1}}$. Since $l \geq k_{n+1}$, then from (3) and Lemma 2, (3) it follows

$$(4)\qquad \Phi(s(\sigma)) \cup \Phi(s(\bar{\sigma})) \subset \Phi(\mathrm{St}\,(\hat{\sigma}, k_{n+1})) \subset \mathrm{St}\,(\Phi(a(\hat{\sigma})), l_n(\hat{\sigma})).$$

From (1), (2), (4) and Lemma 2, (1) we obtain

$$\operatorname{carr} \chi(m, l)\psi_1(\sigma) \subset \mathrm{St}^2(\Phi(s(\sigma)), l) \subset \mathrm{St}\,(\Phi(s(\sigma)), l_{n-1}) \subset \mathrm{St}^2(\Phi(a(\hat{\sigma})), l_n(\hat{\sigma})),$$

$$\operatorname{carr} \varphi_1(\bar{\sigma}) \subset \mathrm{St}^2(\Phi(a(\hat{\sigma})), l_n(\hat{\sigma})).$$

Therefore the chain $\chi(l, l_n(\hat{\sigma}))\,[\varphi_1(\bar{\sigma})-\chi(m, l)\psi_1(\sigma)]$ is a cycle in the block complex $C_*(\mathrm{St}^2(\Phi(a(\hat{\sigma})), l_n(\hat{\sigma})), l_n(\hat{\sigma}))$.

From Lemma 2, (4) it follows that there exists an one-dimensional chain $c_1 \in C_*$ $(\mathrm{St}\,(\Phi(a(\hat{\sigma})), l_n), l_n)$ such that $\partial c_1=\chi(l_n(\hat{\sigma}), l_n)\,[\varphi_1(\bar{\sigma})-\chi(m, l)\psi_1(\sigma)]$.

Define $D^0(\sigma)=c_1$, $b(\sigma)=a(\hat{\sigma})$. Obviously

$$(5)\qquad \begin{gathered} \sigma \subset \mathrm{St}\,(\hat{\sigma}, l) \subset \mathrm{St}^2(b(\sigma), l_n), \\ \operatorname{carr} D^0(\sigma) \subset \mathrm{St}(\Phi(b(\sigma)), l_n). \end{gathered}$$

$t=1$. Let σ be an one-dimensional block in τ_{m_1} and $\tilde{\sigma}$ be its carrier in τ_{l_1}. Then it follows

$$(6)\qquad \operatorname{carr} \chi(m_1, l_1)(\sigma) \subset \tilde{\sigma}.$$

We denote by $\hat{\sigma}$ the carrier of $\tilde{\sigma}$ in τ_{k_n}. Since φ_1 and ψ_1 are (l, l) and (m, m) chain approximation of the map Φ, then there exist points $s(\sigma)$ and $s(\tilde{\sigma})$ such that

$$s(\sigma) \in \mathrm{St}\,(\sigma, l),\ s(\tilde{\sigma}) \in \mathrm{St}\,(\tilde{\sigma}, m).$$

Then $\{s(\sigma), s(\tilde{\sigma})\} \subset \mathrm{St}\,(\hat{\sigma}, k_n)$.

If $\chi(m_1, l_1)(\sigma)=\Sigma\lambda_i\sigma_i$, then $\sigma_i \subset \tilde{\sigma}$. For every σ_i there is a point $s(\sigma_i)$ such that

(7) $$s(\sigma_i)\in \mathrm{St}(\sigma_i,\ l)\subset \mathrm{St}(\widehat{\sigma},\ k_n),$$
$$\mathrm{carr}\ \varphi_1(\sigma_i)\subset \mathrm{St}(\Phi(s(\sigma_i)),\ l),$$
$$\mathrm{carr}\ \varphi_1\chi(m_1,\ l_1)\ (\sigma)\subset \mathrm{St}(\Phi(\mathrm{St}(\widehat{\sigma},\ k_n)),\ l)$$
$$\subset \mathrm{St}(\mathrm{St}(\Phi(a(\widehat{\sigma})),\ l_{n-1}(\widehat{\sigma})),\ l)\subset \mathrm{St}^2(\Phi(a(\sigma)),\ l_{n-1}(\widehat{\sigma})).$$

For the chain map ψ_1 we have

(8) $$\mathrm{carr}\ \psi_1(\sigma)\subset \mathrm{St}(\Phi(s(\sigma)),\ m)\subset \mathrm{St}(\Phi(\mathrm{St}(\widehat{\sigma},\ k_n)),\ m)\subset \mathrm{St}(\mathrm{St}(\Phi(a(\widehat{\sigma})),\ l_{n-1}(\widehat{\sigma})),\ m),$$
$$\mathrm{carr}\ \chi(m,\ l)\psi_1(\sigma)\subset \mathrm{St}(\mathrm{St}(\Phi(a(\widehat{\sigma})),\ l_{n-1}(\widehat{\sigma})),\ l)\subset \mathrm{St}^2(\Phi(a(\widehat{\sigma})),\ l_{n-1}(\widehat{\sigma})).$$

Let us consider the chain $\partial\sigma=\Sigma\lambda_i\bar{\sigma}_i$. Here $\bar{\sigma}_i$ are zero-dimensional blocks in τ_{m_1} and $\bar{\sigma}_i\subset\sigma$. From the first step of the induction we obtain

$$\mathrm{carr}\ D^0(\bar{\sigma}_i)\subset \mathrm{St}(\Phi(a(\delta_i)),\ l_n).$$

Here δ_i is the carrier of $\chi(m_1,\ l_1)\ (\bar{\sigma}_i)$ in $\tau_{k_{n+1}}$ and $\delta_1\subset\widehat{\sigma}$. We also have

$$a(\delta_i)\in \mathrm{St}(\delta_i,\ l)\subset \mathrm{St}(\widehat{\sigma},\ k_n),$$
$$\Phi(a(\delta_i))\subset \Phi(\mathrm{St}(\widehat{\sigma},\ k_n))\subset \mathrm{St}(\Phi(a(\delta_i)),\ l_{n-1}(\widehat{\sigma})),$$
$$\mathrm{carr}\ D^0(\bar{\sigma}_i)\subset \mathrm{St}(\mathrm{St}(\Phi(a(\widehat{\sigma})),\ l_{n-1}(\widehat{\sigma})),\ l_n)\subset \mathrm{St}^2(\Phi(a(\widehat{\sigma})),\ l_{n-1}(\widehat{\sigma})).$$

Therefore

(9) $$\mathrm{carr}\ D^0(\partial\sigma)\subset \mathrm{St}^2(\Phi(a(\widehat{\sigma})),\ l_{n-1}(\widehat{\sigma})).$$

From (7), (8), (9) it follows that the cycle

$$c=\chi(l,\ l_{n-1}(\widehat{\sigma}))\ [\varphi_1\chi(m_1,\ l_1)\ (\sigma)-\chi(m,\ l)\psi_1(\sigma)]-\chi(l_n,\ l_{n-1}(\widehat{\sigma}))D^0(\partial\sigma)$$

belongs to the chain complex $C_*(\mathrm{St}^2(\Phi(a(\widehat{\sigma})),\ l_{n-1}(\widehat{\sigma})),\ l_{n-1}(\widehat{\sigma}))$.

From Lemma 2, (4) we obtain that there exists a chain

$$c_2\in C_*(\mathrm{St}(\Phi(a(\widehat{\sigma})),\ l_{n-1}),\ l_{n-1})$$

such that $\partial c_2=\chi(l_{n-1}(\widehat{\sigma}),\ l_{n-1})\ (c)$.

Define $D_1^1(\sigma)=c_2$, $b(\sigma)=a(\widehat{\sigma})$, $D_0^1=\chi(l_n,\ l_{n-1})D^0$.

Thus the proof of Lemma 4 is complete.

3. *A*-systems — a generalization. Let K, L be finite simplicial complexes. Let $\{\tau_i\}$, $\{\mu_i\}$ be their fundamental systems of block dissections. Let $\Phi: K\to L$ be an u.s.c. map.

Definition. The graduated set $A(\Phi)$ $\{A(\Phi)_i : i\in \mathrm{N}\}$ where

$$A(\Phi)_i\subset \mathrm{hom}\,(C_*(K,\ i),\ C_*(L,\ i))$$

is called an approximation system (generalized A-system) of the map Φ if there exists $n=n(A(\Phi))$ such that:

(1) if $\varphi\in A(\Phi)_i$, then $\varphi=\varphi_1 b(i,\ k)$ and φ_1 is $(n,\ k,\ i)$ chain approximation of the map Φ,

(2) for every $i\in \mathrm{N}$, there is $i_1\geqq i$ such that if

$$m\geqq l\geqq i_1,\ \varphi=\varphi_1 b(l,\ l_1)\in A(\Phi)_l,\ \psi=\psi_1 b(m,\ m_1)\in A(\Phi)_m$$

then the chain maps $\varphi_1\chi(m_1, l_1)$ and $\chi(m, l)\psi_1$ ard homotopic with a chain homotopy D satisfying the following conditions: for every block $\sigma\in\tau_{m_1}$ there is a point $b(\sigma)\in K$ for which

$$\sigma\subset \mathrm{St}^n(b(\sigma), i),$$

$$\mathrm{carr}\, D(\sigma)\subset \mathrm{St}^n(\Phi(b(\sigma)), i).$$

Let U be an open set in the polyhedron K, such that $\bar{U}$ is a subcomplex in some block dissection τ_k of $\{\tau_i\}$. Let $\Phi: K\to K$ be a u.s.c. map. The triple (K, Φ, U) is called admissible if $x\notin\Phi(x)$ for every point $x\in\partial U$.

Lemma 5. *Let (K, Φ, U) be an admissible triple and $A(\Phi)$ — an A-system of the map Φ corresponding to a given fundamental system of block dissections of K. The element*

$$I(U, A(\Phi))=\Lambda(\pi(\bar{U}, i)\varphi)=\Sigma(-1)^j \mathrm{tr}\,(\pi(\bar{U}, i)\varphi)_j$$

of the field F *does not depend on the chain map φ for i sufficiently large i (here $\varphi\in A(\Phi)_i$, and $\pi(\bar{U}, i): C_*(K, i)\to C_*(\bar{U}, i)$ is the natural projection, $i\geq k$.*

This lemma is proved in [3, Lemma 2.3., p. 196-197] in the case when τ_i are triangulations. The proof in our more general case is the same.

The element $I(U, A(\Phi))$ is called an index of the approximation system $A(\Phi)$ on the set U. Obviously the requirement of $\bar{U}$ to be subcomplex in some block dissection τ_k is not essential.

4. Induced A-systems. Let $\Phi: K\to L$ be an acyclic map, $\{\tau_i\}$, $\{\mu_i\}$-fundamental systems of block dissections of K and L, respectively. Let us consider

$$A^*(\Phi)_i=\{\varphi=\varphi_1 b(i, j):\quad C_*(K, i)\to C_*(L, i),$$

where φ_1 is an (i, i) chain approximation of the map $\Phi\}$.

It follows from Lemma 4

Lemma 6. *$A^*(\Phi)=\{A^*(\Phi)_i: i\in \mathrm{N}\}$ is an A-system of the map Φ.*

This A-system is called an induced A-system of the map Φ corresponding to the fundamental systems $\{\tau_i\}$, $\{\mu_i\}$.

Corollary 7. *Let $\Phi: K\to K$ be an acyclic map. For every admissible triple* (K, Φ, U) it holds

$$i(\Phi, U)=I(U, A^*(\Phi)).$$

Here $i(\Phi, U)$ is the fixed point index defined in [3] and $A^*(\Phi)$ is the induced A-system of the map corresponding to a given fundamental system of block dissections on K.

Proof. Let us consider Fig. 1 and the diagrams I, II, III. The diagrams I and III are commutative. The diagram II is a homotopy commuting with a homotopy satisfying the conditions of Lemma 4. It follows from [3, Lemma 2.3, p. 196-197] that

$$\Lambda(\pi(\bar{U}, i)\varphi_1 b(\tau_i, \tau_k))=\Lambda(\pi(\bar{U}, i)\bar{\varphi_1}\; b(\tau_i', \tau_k')).$$

The right side is $I(U, A^*(\Phi))$ for i sufficiently large. The left side is $i(\Phi, U)$.

II. A-system for the product of two acyclic maps. Let $\Phi_i: K_i\to L_i$ be F-acyclic maps, $i=1, 2$. Let $\{\tau_k^i\}$, $\{\mu_k^i\}$ be fundamental systems of block dissections of K_i and L_i, res-

$$\begin{array}{ccccccc}
C_*(K,\ \tau_i) & \xrightarrow{b(\tau_i,\ \tau_k)} & C_*(K,\ \tau_k) & \xrightarrow{\varphi_1} & C_*(K,\ \tau_i) & \xrightarrow{\pi(\bar{U},\ i)} & C_*(\bar{U},\ \tau_i) \\
\downarrow{\scriptstyle b(\tau_i,\ \tau_i')} & & \downarrow{\scriptstyle b(\tau_k,\ \tau_k')} & & \downarrow{\scriptstyle b(\tau_i,\ \tau_i')} & & \downarrow{\scriptstyle b(\tau_i,\ \tau_i')} \\
C_*(K,\ \tau_i') & \xrightarrow{b(\tau_i',\ \tau_k')} & C_*(K,\ \tau_k') & \xrightarrow{\bar{\varphi}_1} & C_*(K,\ \tau_i') & \longrightarrow & C_*(\bar{U},\ \tau_i')
\end{array}$$

Fig. 1

pectively, $i=1,\ 2$. Let $A^*(\Phi_i)$ be the induced A-system of the acyclic map Φ_i, $i=1,\ 2$ corresponding to the fundamental systems of block dissections.

Let us consider the map $\psi=\Phi_1\times\Phi_2 : K_1\times K_2\to L_1\times L_2$ defined by $\psi(x_1,\ x_2)=\Phi_1(x_1)\times\Phi_2(x_2)$. The map ψ is F-acyclic.

Let us consider two block dissections $\tau_k^i\in\{\tau_j^i\}$, $i=1,\ 2$. We shall denote by $\theta_k=\tau_k^1\times\tau_k^2$ the product block structure of $K_1\times K_2$ induced by τ_k^1 and τ_k^2.

It follows that $\{\tau_j^1\times\tau_j^2\}$ and $\{\mu_j^1\times\mu_j^2\}$ are fundamental systems of block dissections of $K_1\times K_2$ and $L_1\times L_2$, respectively. We have

$$C_*(K_1\times K_2,\ \tau_j^1\times\tau_j^2)=C_*(K_1,\ \tau_j^1)\otimes C_*(K_2,\ \tau_j^2)$$

and the same formula for $L_1\times L_2$.

Let $A^*(\Phi_1)\times A^*(\Phi_2)=\{A^*(\Phi_1)_i\times A^*(\Phi_2)_i\}$, where for $(\varphi,\ \psi)\in A^*(\Phi_1)_i\times A^*(\Phi_2)_i$, $(\varphi,\ \psi) : C_*(K_1\times K_2,\ \tau_i^1\times\tau_i^2)\to C_*(L_1\times L_2,\ \mu_i^1\times\mu_i^2)$ is defined by $(\varphi,\ \psi)\ (e_1\times e_2)=\varphi(e_1)\otimes\psi(e_2)$ for every two blocks $e_j\in\tau_i^j$, $j=1,\ 2$.

Lemma 8. $A^*(\Phi_1)_i\times A^*(\Phi_2)_i\subset A^*(\Phi_1\times\Phi_2)_i$.

Here $A^*(\Phi_1\times\Phi_2)$ is the induced A-system of the acyclic map $\Phi_1\times\Phi_2$ corresponding to the fundamental systems of block dissections $\{\tau_i^1\times\tau_i^2\}$ and $\{\mu_i^1\times\mu_i^2\}$, $A^*(\Phi_j)$ is the induced A-system corresponding to $\{\tau_i^j\}$, $\{\mu^j\}$ $j=1,\ 2$.

Remark. The chain map subdivision in $A^*(\Phi_1\times\Phi_2)$ is

$$b_1(i,\ j) : C_*(K_1\times K_2,\ \tau_i^1\times\tau_i^2)\to C_*(K_1\times K_2,\ \tau_j^1\times\tau_j^2)$$

and is defined by $b_1(i,\ j)=b(\tau_i^1,\ \tau_j^1)\otimes b(\tau_i^2,\ \tau_j^2)$. While the chain map $\chi_1(i,\ j)=\chi(\tau_j^1\times\tau_j^2,\ \tau_i^1\times\tau_i^2)$ is defined by $\chi_1(i,\ j)=\chi(\tau_j^1,\ \tau_i^1)\otimes\chi(\tau_j^2,\ \tau_i^1)$, $i\leq j$.

III. The multiplicativity property of the fixed point index for multivalued maps on finite polyhedra

Theorem 9. *Let $\Phi_i : K_i\to K_i$ be F-acyclic maps, $i=1,2$. If $(K_i,\ \Phi_i,\ U_i)$ are admissible triples, $i=1,\ 2$, then $(K_1\times K_2,\ \Phi_1\times\Phi_2,\ U_1\times U_2)$ is an admissible triple and*

$$i(\Phi_1\times\Phi_2,\ U_1\times U_2)=i(\Phi_1,\ U_1)\,i(\Phi_2,\ U_2).$$

Proof. Let $\varphi\in A^*(\Phi_1)_k$, $\psi\in A^*(\Phi_2)_k$ (k — sufficiently large). It follows from Lemma 7 that

$$i(\Phi_1\times\Phi_2,\ U_1\times U_2)=\Lambda(\pi(\bar{U}_1\times\bar{U}_2,\ k)\ (\varphi\otimes\psi)).$$

Using $\pi(\bar{U}_1\times\bar{U}_2,\ k)\,(\varphi\otimes\psi)=(\pi(\bar{U}_1,\ k)\varphi)\otimes(\pi(\bar{U}_2,\ k)\psi)$ and the multiplicativity property of the Lefschetz number, we obtain

$$i(\Phi_1\times\Phi_2,\ U_1\times U_2)=\Lambda(\pi(\bar{U}_1,\ k)\varphi)\Lambda(\pi(\bar{U}_2,\ k)\psi)=i(\Phi_1,\ U_1)i(\Phi_2,\ U_2).$$

Remark. 1. With the same arguments we obtain

Theorem 10. *Let* $\Phi_1 : K_1\to K_1$, $\Phi_1\in\mathscr{A}_n(K_1,\ K_1)$ ([8, p. 13]). *Let* $\Phi_2: K_2\to K_2$ *be an acyclic map. If* $(K_1,\ \Phi_1,\ U_1)$ *and* $(K_2,\ \Phi_2,\ U_2)$ *are admissible triples, then*

$$i(\Phi_1\times\Phi_2,\ U_1\times U_2)=i(\Phi_1,\ U_1)\,i(\Phi_2,\ U_2).$$

Remark. 2. If $\Phi_2\in\mathscr{A}_m(K_2,\ K_2)$ we obtain in theorem 10

$$I(U_1\times U_2,\ A^*(\Phi_1\times\Phi_2))=i(\Phi_1,\ U_1)\,i(\Phi_2,\ U_2).$$

IV. The multiplicativity property of the fixed point index for multivalued maps on ANR-s. The fixed point index for acyclic (or for more general of the class (1, *n*)) maps is constructed in [3] ([8]). The construction uses appropriate chain approximations of the maps and reduces this case to that of multivalued maps on a finite polyhedron. This construction (in [3]) follows the same way if we use A-systems corresponding to fundamental systems of block dissections instead of A-systems induced by barycentric subdivisions of a given triangulation. Therefore from theorem 9 we obtain

Theorem 11. *Let* $\Phi_i: X_i\to X_i$ *be* F*-acyclic maps of complete metric spaces* X_i, $i=1,\ 2$. *If* $(X_i,\ \Phi_i,\ U_i)$ *are admissible triples,* $i=1,\ 2$, *then* $(X_1\times X_2,\ \Phi_1\times\Phi_2,\ U_1\times U_2)$ *is admissible triple and*

$$i(\Phi_1\times\Phi_2,\ U_1\times U_2)=i(\Phi_1,\ U_1)\,i(\Phi_2,\ U_2).$$

Remark 1. In [8] it is proved that the condition of completeness of the space X_i is not necessary to define the index. Therefore Theorem 11 remains true even, if we omit this condition.

Remark 2. In Theorem 11 the map Φ_i might be of the class

$$\mathscr{A}_n(X_i,\ X_i)\ ([8]).$$

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