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A REMARK ON THE PROBABILITY OF DEGENERATION OF A MULTIDIMENSIONAL BRANCHING GALTON — WATSON PROCESS

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This paper is aimed at obtaining the sufficient conditions for the probability of degeneration of an n-dimensional Galton—Watson process to be less than 1 in the case when only the first terms of the distribution law have been made use of.

Introduction. Let us have a time homogeneous Galton — Watson process $\{z^k\}_{k=0}^{\infty}$ where z^k is a random n-dimensional vector; the i-th component z_i^k of the vector z^k can be interpreted as a number of the i-th type of particles in the k-th generation. Let us denote by e^i the n-dimensional vector where the i-th component is equal to 1 and the remaining ones are equal to 0. For $\alpha=(\alpha_1, \dots, \alpha_n)$ let us denote by p_α^i the probability which the i-th type particle will have α_1 1-type descendants, $\cdots$, α_n n-type descendants if $z^\circ=e^i$. Obviously, $\Sigma_\alpha p_\alpha^i=1$ $(i=1, \dots, n)$. The functions $p^i(x)=\Sigma_\alpha p_\alpha^i x^\alpha$ $(i=1, \dots, n)$ are the generating functions of the process. Each of them is analytic in the polycylinder $U=\{z \in C^n : |z_i|<1\}$ and continuous in $\bar{U}$ with the mapping $p=(p^1, \dots, p^n) : \bar{U} \to \bar{U}$.

Let q_i be the probability for some k $z^k=0$ if $z^\circ=e^i$. The vector $q=(q_1, \dots, q_n)$ is called a probability of degeneration of an n-dimensional Galton — Watson process. It is known [1], that q is the limit of the sequence $\{p^{(m)}(0)\}_{m=0}^{\infty}$, where $p^{(m)}$ are the iterations of the mapping p, $p^{(0)}=p$. Hence q is the minimal (for each component) fixed point of the mapping $p: \Pi \to \Pi$, where $\Pi=[0, 1] \times \cdots \times [0, 1]$ (n times). Note that always $p(\bar{1})=\bar{1}$, where $\bar{1}=(1, \dots, 1)$. One of the substantial problems in the theory of branching processes is the case when $q<\bar{1}$. The sufficient condition formulated by Everett-Ulam's theorem is well known [1].

Theorem 1. *Let us assume that at least one of the functions $p^i(x)$ is non-linear, the spectral radius of $p'(\bar{1})$ (Freshe's derivative) is more than* 1 *and a certain power $(p'(\bar{1}))^N$ has positive elements, then $q<\bar{1}$.*

However it will be difficult to apply the theorem to practice, because the functions $p^i(x)$ are not given a priori. We have only their Taylor's coefficients $\{p_\alpha^i\}_{\alpha \in N^n}$. Except that these coefficients are known approximately only for some concrete problems.

The main aim of our paper is to formulate the sufficient conditions for the probability of degeneration to be less that 1 when only the first terms of the distribution law have been taken into consideration.

1. The probability of degeneration for one-dimensional branching process. Let $\{z^k\}_{k=0}^{\infty}$ be a one-dimensional Galton—Watson process, which begins with a single particle and has an offspring distribution $\{p_k\}_{k \in N}$, where $0 \leq p_k \leq 1$ and $\Sigma_{k=0}^{\infty} p_k=1$. Let $p(x)=\Sigma_{k=0}^{\infty} \cdot p_k x^k$ be the generating function of the process for $x \in [0, 1]$.

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In that case theorem 1 is obvious: $q<1$, if $p'(1)>1$. Note, that since all the derivatives $p^{(m)}(x)\geq 0$ in $[0, 1)$, there exists a limit $p'(1)=\lim_{x\to 1-0} p'(x)$. Evidently the condition $p'(1)<1$ may be replaced by the stronger condition $\Sigma_{k=0}^{N} k\, p^k>1$ for the first N coefficients of the function $p(x)$, but it will not be very precise. The next theorem shows what the condition for $p_0, \ldots, p_N$ should be so that $p>1$.

Theorem 2. *If* $-p_0+\Sigma_{k=1}^{N}(1-p_0-\cdots-p_k)>0$, *then* $q<1$. *This condition is exact. That means that if for the non-negative constants* $p_0, \ldots p_N$ *we have* $\Sigma_{k=0}^{N} p_k\leq 1$, $p_0\neq 0$ *and* $-p_0+\Sigma_{k=0}^{N}(1-p_0-\cdots-p_k)\leq 0$, *then there exists a process with the following distribution* $\{p_0, \ldots, p_N, p_{N+1}, 0, \cdots\}$, *where the probability of degeneration will be* $q=1$.

Proof. Since all the derivatives $p^{(m)}(x)$ are non-negative on $[0, 1)$, then neither $p(x)=x$, nor $p(x)$ will have more than one fixed point in $[0, 1)$.

Let us have $f(x)=(p(x)-x)/(x-1)$. Since this function is analytical in the unit circle of a complex plane, $f(x)$ can be represented by Taylor's series $f(x)=\Sigma_{k=0}^{\infty} c_k x^k$ in the half-interval $[0, 1)$. The equating of the coefficients of the series for $p(x)-x$ and $(x-1)f(x)$ will result in $c_0=-p_0$ and $c_k=(1-p_0-\cdots-p_k)$, $k\geq 1$. Thus,

$$f(x)=-p_0+\sum_{k=1}^{\infty}(1-p_0-\cdots-p_k)x^k$$

will be a function strictly increasing on $[0, 1)$. We have $f(0)=-p_0\leq 0$. If $-p_0+\Sigma_{k=0}^{N}(1-p_0-\cdots-p_k)>0$, then according to the continuity

$$f(x_0)=-p_0+\sum_{k=1}^{N}(1-p_0-\cdots-p_k)x_0^k>0$$

for a point $x_0\in(0, 1)$. Hence according to the Caushy theorem there is q such that $0\leq q<x_0$ and $f(q)=0$. Evidently $p(q)=q$ and $q<1$.

Now let us have the non-negative quantities $p_0, \ldots, p_N$, $\Sigma_{k=0}^{N} p_k\leq 1$, such that $p_0\neq 0$ and

$$-p_0+\sum_{k=1}^{N}(1-p_0-\cdots-p_k)\leq 0.$$

Let us assume that $p_{N+1}=1-p_0-\cdots-p_N\geq 0$ and let us consider the one-dimensional branching process with an offspring distribution law $\{p_0, \cdots, p_N, p_{N+1}, 0, \ldots\}$. The polynomial $p(x)=\Sigma_{k=0}^{N+1} p_k x^k$ will be a process generating function. As above, $p(x)-x=(x-1)f(x)$, where

$$f(x)=-p_0+\sum_{k=0}^{N}(1-p_0-\ldots-p_k)x^k,$$

so that $f(0)=-p_0<0$ and $f(1)\leq 0$. Hence the only zero of the function $f(x)$ may be the point $x=1$. Thus $q=1$. The theorem is proved.

Remark. If $p_0=0$, then $q=0$.

2. General case. Let us return to the general case. The aim is to give the conditions for p_α^i, $|\alpha|\leq N_i$ $(i=1, \ldots, n)$, that should be sufficient to obtain a degeneration probability less than 1. Naturally, these conditions generalize the sufficient ones of theorem 2, but for $n\geq 2$ they will not be minimal.

Theorem 3. *If* $-p_0^i+\Sigma_{k=1}^{N_i}(1-\Sigma_{|\alpha|\leq k} p_\alpha^i)>0$ *at* $i=1, \ldots, n$, *then* $q<\bar{1}$.

Proof. Let us denote $s_k^i=\Sigma_{|\alpha|=k} p_\alpha^i$, so that $s_k^i\geq 0$ and $\Sigma_{k=0}^{\infty} s_k^i=1$ $(i=1, \ldots, n)$ Consider the functions

$$\varphi^i(\xi)=\sum_{k=0}^{\infty} s^i_k \xi^k \quad (k=1,\ldots,n),$$

given on the segment [0, 1]. That condition would mean

$$-s^i_0+\sum_{k=1}^{N_i}(1-s^i_0-\cdots-s^i_k)>0$$

so with respect to theorem 2 for all $i=1,\ldots,n$ there will be such points $\xi_i\in[0,1)$, for which $\varphi^i(\xi_i)=\xi_i$. Since the functions $\varphi^i(\xi)$ are strictly down-convex, then the unequality $\varphi^i(\xi)<\xi$ $(i=1,\ldots,n)$ will be true for each $\xi\in\xi_i, 1]$. Hence, having chosen r, $\max_{1\leq i\leq n}\xi_i<r<1$, get $\varphi^i(r)<r$ for all $i=1,\ldots,n$. Thus, denoting a polycircular norm in C^n by $\|z\|=\max_{1\leq i\leq n}|z_i|$, we have

$$\sup_{\|z\|\leq r}\|p(z)\|\leq\max_{1\leq i\leq n}\varphi^i(r)<r.$$

Thus according to theorem 2.1 from [2] there will be a s-fixed point q of the mapping p in the polycircle $U_r=\{z\in C^n:\|z\|\leq r\}$. Since the mapping p iterations beginning from zero are non-negative, the components of the point q will be also non-negative too. Hence, $q<\bar{1}$. The theorem is proved.

Corollary. *If* $2p^i_0+\Sigma_{|\alpha|=1}p^i_\alpha<1$ *at* $i=1,\ldots n$, *then* $q<\bar{1}$.

Example. Let us have a Galton—Watson branching process with particles of two types: A and B. Let the probabilities of A and B having no descendants be equal respectively to 0,2 and 0,25; the probabilites of reproducing only one descendant of the same type are 0,44 and 0,12; and the probabilities of reproducing only one descendant of the other type are 0,1 and 0,3 respectively. Then in spite of the remaining distribution (which follows from the probabilities of reproducing two and more descendants and mixed-type descendants) the probability of degeneration will be less than 1. In fact, the corollary conditions: $2\times0{,}2+0{,}4+0{,}1<1$ and $2\times0{,}25+0{,}12+0{,}3<1$ have been observed.

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