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ON JACOBSON RADICAL RINGS

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It is well known that Amitsur—Kurosh radical classes $\boldsymbol{R}$ of rings determine in each ring A a maximal radical ideal $\boldsymbol{R}(A)$, called the $\boldsymbol{R}$-radical of A; and conversely, the class of all rings A such that $\boldsymbol{R}(A)=A$ holds, forms a radical class $\boldsymbol{R}$. (See [2], [4]). Among all concrete radicals of rings the Jacobson radical seems to be the most useful. (See [3], [4], [6] and [7].)

The purpose of this note is to give six further characterizations for the Jacobson radical rings. The word "ring" here always means an associative ring, and $\boldsymbol{J}$ always denotes the Jacobson radical. Moreover, $(x)_r$ denotes the principal right ideal, generated by the element x of A. Obviously $(x_1)_r=(x_2)_r$ determines an equivalence relation $x_1\approx x_2$, being also a left congruence of the multiplicative semigroup of the ring A, which was before used by [6], for another characterization of $\boldsymbol{J}$. Let us remark that $\approx$ is an one-sided analogon of another equivalence relation $*$, defined by $(x_3)=(x_4)$, where (y) denotes the principal two sided ideal, generated by the element y of A. That is $x_3 * x_4$ holds if and only if $(x_3)=(x_4)$ is valide. This $*$ was used by [7] to give a new characterization for the antisimple radical (see [1] and yet [5]).

Since, by Chapter I. of [3], $\boldsymbol{J}$ is left-right dual, therefore conditions (II*), (III*) and (IV*), obtained by always taking "left", instead of "right" in the conditions (II), (III) and (IV) of our Theorem, are also equivalent to (I), (II), (III) and (IV). Thus we obtain six different, but equivalent condition for the case $I=A$.

In the sequel we shall use the following:

Lemma. *Assume $\boldsymbol{J}\neq A$ for the Jacobson radical $\boldsymbol{J}$ of a ring A. Then there exists a modular maximal right ideal R of A such that $I\nsubseteq R$ implies $I^k\nsubseteq R$ for arbitrary right ideal I of A and for arbitrary exponent $k\geq 2$.*

Remark. The condition $\boldsymbol{J}\neq A$ in this Lemma is important, since e. g. in the case $A^2=0$, if we put $I=A$, then $I^2=0\subseteq R$, but $I\subseteq R$ holds for any proper additive subgroup R of A^+.

Proof. By $\boldsymbol{J}\neq A$ there exists a modular maximal right ideal R of A such that $R\neq A$, since by Chapter I. of [3] we have $\boldsymbol{J}=\bigcap_\alpha R_\alpha$, where R_α runs over all modular maximal right ideals of A. Now assume $I\subseteq R$ for some right ideal I of A. Then the maximality of R in A yields

$$(\triangle)\qquad I+R=A.$$

Then, obviously, $P=\{z;\ A\cdot z\subseteq R\}=(R:A)=(0:A/R)$ is a primitive ideal of A such that $P\subseteq R$ holds; see [3].

Therefore $I\nsubseteq P$, being $I\nsubseteq R$ and $P\subseteq R$. Thus, by the above, $AI\nsubseteq R$ and

$$(\triangle\triangle)\qquad AI+R=A$$

are valide. Now multiplying $(\triangle)$ from right by I, we have $I^2+R\cdot I=A\cdot I$, which yields by $(\triangle\triangle)$ and $(\triangle)$ at once

$$I^2+R=I^2+RI+R=(I+R)\cdot I+R=A\cdot I+R=A.$$

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Hence $I^2 \nsubseteq R$, and by induction on k, also $I^k \nsubseteq R$.

Theorem. *For an arbitrary ring A the following conditions are equivalent:*

(I.) *A is a Jacobson radical ring that is* $J = A$;

(II.) *For arbitrary element x of A, and for arbitrary element* $y \in (x)_r^2$ *holds* $(x)_r = (x+y)_r$ *that is* $x \approx x+y$;

(III.) *There exists a fixed exponent* k_0 *for all elements* $x \in A$ *such that for any* $y \in (x)_r^{k_0}$ *holds* $x \approx x+y$;

(IV.) *For arbitrary element x of A there exists an exponent* $k = k(x)$, *generally depending on x, such that for any* $x \in A$ *and any* $y \in (x)_r^k$, *holds* $x \approx x+y$.

Proof will be cyclic. (I.) implies (II.). Assume $J = A$. If (II.) does not hold for A, then there exists an element $x \in A$ and an element $y \in (x)_r^2$ such that $(x)_r \neq (x+y)_r$. Thus we have $x \notin (x+y)_r$. Consider the set $\mathfrak{M}$ of those right ideals N of A, which satisfy $x \notin N$ and $(x+y)_r \subseteq N$. This set $\mathfrak{M}$ is not empty by $(x+y)_r \in \mathfrak{M}$. By the Zorn lemma there exists a maximal right ideal M in $\mathfrak{M}$, such that $x \notin M$ and $(x+y)_r \subseteq M$. Then A/M, as an A-right module, is subdirectly irreducible with heart $(\bar{x})_r = (x+M)_r$. Thus by $J = A$ we obtain $(\bar{x})_r \cdot A = M/M = \bar{0}$, which yields $(x)^2 \subseteq M$. Therefore $y \in (x)_r^2 \subseteq M$ and $x+y \in M$ imply $x = y + y - y \in M$, which contradicts to $x \notin M$. Thus (I.) implies (II.), indeed.

(II.) implies (III.). This is evident.

(III.) implies (IV.). This is also clear.

(IV.) implies (I.). Namely assume (IV.), and $J \neq A$. We shall deduce a contradiction. By Chapter I. of [3] there exists in the case $J \neq A$ an element $x \in A$ such that $x \notin R$ for a suitable modular maximal right ideal R of A. By (IV.) there exists an exponent $k = k(x)$ such that $x \approx x+y$ for arbitrary $y \in (x)_r^k$. Now $x \notin R$ and $(x)_r \nsubseteq R$ imply, by our Lemma, $(x)_r^k \nsubseteq R$. Then $(x)_r^k + R = A$ implies the existence of elements $y_0 \in (x)_r^k$ and $^0 \in R$ such that

$$x = y_0 + r_0$$

rolds. Furthermore $r_0 = x - y_0 \in R$, $(x)_r = (x - y_0)_r \subseteq R$ and $y_0 \in (x_r)k$ yield, by (IV.), evidently $x \in R$, which is a contradiction $x \notin R$.

Thus the proof is completed.

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