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A MINIMUM PROBLEM FOR A CLASS OF POLYNOMIALS

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In this note we solve a certain minimum problem. As an application we give an integral inequality.

1. A new identity for binomial coefficients.

Lemma. *Let k and N be nonnegative integers. Then*

$$\sum_{p=0}^{N}(2k+2p+1)\binom{2k+p}{p}^2=(2k+1)\binom{2k+N+1}{N}^2. \tag{1}$$

Proof. We keep k fixed and proceed by induction on N. For $N=0$ the identity (1) is crearly true. Assuming the validity of (1) up to N, we have to show that

$$(2k+1)\binom{2k+N+1}{N}^2+(2k+2N+3)\binom{2k+N+1}{N+1}^2=(2k+1)\binom{2k+N+2}{N+1},$$

i. e. that

$$\frac{1}{2k+1}+\frac{2k+2N+3}{(N+1)^2}=\frac{(2k+N+2)^2}{(2k+1)\ (N+1)^2},$$

which is obvious.

2. Main result. Let n and k be integers with $0\leq k\leq n$. Now we will determine the minimum value $M_{n\cdot k}$ of

$$\int_0^1 \{P_{n\cdot k}(x)\}^2\,dx,$$

where $P_{n\cdot k}$ runs through the set of all polynomials with real coefficients and degree at most n such that the coefficient of x^k is 1.

Theorem.

$$M_{n\cdot k}=\left\{(2k+1)\binom{n+k+1}{n-k}^2\binom{2k}{k}^2\right\}^{-1}.$$

Proof. We start by noting that the normalized Legendre polynomials L_k, $k=0$, 1, 2, 3, ..., transformed to the interval [0, 1] form an orthonormal basis of all polynomials defined on [0, 1]. By Rodrigues' formula we get the representation

$$L_k(x)=\frac{\sqrt{2k+1}}{k!}\left(\frac{d}{dx}\right)^k(x^2-x)^k,\quad k=0,\ 1,\ 2,\dots$$

(See for instance [1], p. 183).

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Clearly $\deg L_k = k$. We now put $P_{n\cdot k}(x) = \Sigma_{j=0}^{n} a_j L_j(x)$, where $a_0, a_1, a_2, \dots a_n \in \mathbf{R}$. Then

$$\int_0^1 \{P_{n\cdot k}(x)\}^2 dx = \sum_{j=0}^{n} a_j^2 = \min! \tag{2}$$

From

$$L_j(x) = \frac{\sqrt{2j+1}}{j!}\left(\frac{d}{dx}\right)^j \sum_{p=0}^{j} (-1)^p \binom{j}{p} x^{2j-p} = \sqrt{2j+1} \sum_{p=0}^{j} (-1)^p \frac{(2j-p)!}{p!\,[(j-p)!]^2} x^{j-p}$$

we get that the coefficient of x^k in $P_{n\cdot k}$ equals

$$\sum_{j=k}^{n} a_j (-1)^{j-k} \sqrt{2j+1}\; \frac{(j+k)!}{(j-k)!\,[k!]^2} = 1. \tag{3}$$

This result shows already (in view of (2)) that

$$a_0 = a_1 = \dots a_{k-1} = 0.$$

Furthermore, (2) and (3) yield the function

$$F(a_k, \dots, a_n, \lambda) = \sum_{j=k}^{n} a_j^2 - \lambda \Big(\sum_{j=k}^{n} a_j (-1)^{j-k} \frac{(j+k)!}{(j-k)!\,[k!]^2} - 1 \Big)$$

to be minimized.

But $\dfrac{\partial F}{\partial a_j} = 0$, $j = k, \dots, n$ imply

$$a_j = \frac{1}{2} \lambda (-1)^{j-k} \sqrt{2j+1} \frac{(j+k)!}{(j-k)!\,[k!]^2}, \quad j = k, \dots, n. \tag{4}$$

Hence, via (3) we get

$$\frac{1}{2} \lambda \sum_{j=k}^{n} (2j+1) \frac{[(j+k)!]^2}{[(j-k)!]^2 [k!]^4} = 1,$$

i. e.

$$\lambda \binom{2k}{k}^2 \sum_{j=k}^{n} (2j+1) \binom{j+k}{j-k}^2 = 2.$$

This and the lemma immediately lead to

$$\lambda = 2\Big\{(2k+1) \binom{n+k+1}{n-k}^2 \binom{2k}{k}^2\Big\}^{-1}. \tag{5}$$

As the matrix corresponding to the second derivative of F is positive-definite (subject to condition (3)), F attains minimum at the values given by (4) and (5). Finally, via (2)

$$M_{n\cdot k} = \lambda^2 \frac{1}{4} \sum_{j=k}^{n} (2j+1) \frac{[(j+k)!]^2}{[(j-k)!]^2 [k!]^4}$$

$$= \frac{\lambda^2}{4} (2k+1) \binom{n+k+1}{n-k}^2 \binom{2k}{k}^2 = \Big\{(2k+1) \binom{n+k+1}{n-k}^2 \binom{2k}{k}^2\Big\}^{-1}.$$

3. An integral inequality. As an application of the above theorem we prove the following

Corollary (see [2]). *Let m, n be nonnegative integers. Furthermore, let* $f:[0, 1] \to \mathbb{R}$ *be n times continuously differentiable,* $f^{(k)}(0)=f^{(k)}(1)$, $k=0, 1, \dots, n-1$ *and* $\int_0^1 x^j f(x)dx=0$, $j=1, 2, \dots, m$. *Then*

$$\left(\int_0^1 f(x)dx\right)^2 \leq (2n+1)\left[\frac{n!\,m!}{(2n+m+1)!}\right]^2 \int_0^1 [f^{(n)}(x)]^2 dx.$$

Proof. Let Q be a polynomial such that $Q(0)=1$ and $\deg Q \leq m$. By the assumption on f we get (via n times partial integration)

$$\int_0^1 f(x)dx = \int_0^1 f(x)Q(x)dx = (-1)^n \int_0^1 f^{(n)}(x)Q_n(x)dx,$$

where $Q_n^{(n)}(x)=Q(x)$. By Cauchy—Schwarz' inequality we arrive at

$$(6) \qquad \left(\int_0^1 f(x)\,dx\right)^2 \leq \int_0^1 [Q_n(x)]^2\,dx \int_0^1 [f^{(n)}(x)]^2 dx.$$

Now, $Q_n(x)$ is a polynomial such that $\deg Q_n \leq m+n$ and the coefficient of x^n equals 1. By the theorem we conclude that

$$\int_0^1 [Q_n(x)]^2\,dx \geq M_{m+n,n} = \left\{(2n+1)\binom{2n+m+1}{m}^2\binom{2n}{n}^2\right\}^{-1} = (2n+1)\left[\frac{n!\,m!}{(2n+m+1)!}\right]^2.$$

This and (6) yield the desired inequality.

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