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A MINIMUM PROBLEM FOR A CLASS OF POLYNOMIALS

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In this note we solve a certain minimum problem. As an application we give an integral inequality.

1. A new identity for binomial coefficients.

Lemma. Let k and N be nonnegative integers. Then

$$(1) \quad \sum_{p=0}^N (2k+2p+1) \binom{2k+p}{p}^2 = (2k+1) \binom{2k+N+1}{N}^2.$$

Proof. We keep k fixed and proceed by induction on N . For $N=0$ the identity (1) is clearly true. Assuming the validity of (1) up to N , we have to show that

$$(2k+1) \binom{2k+N+1}{N}^2 + (2k+2N+3) \binom{2k+N+1}{N+1}^2 = (2k+1) \binom{2k+N+2}{N+1}^2,$$

i. e. that

$$\frac{1}{2k+1} + \frac{2k+2N+3}{(N+1)^2} = \frac{(2k+N+2)^2}{(2k+1)(N+1)^2},$$

which is obvious.

2. Main result. Let n and k be integers with $0 \leq k \leq n$. Now we will determine the minimum value $M_{n,k}$ of

$$\int_0^1 \{P_{n,k}(x)\}^2 dx,$$

where $P_{n,k}$ runs through the set of all polynomials with real coefficients and degree at most n such that the coefficient of x^k is 1.

Theorem.

$$M_{n,k} = \left\{ (2k+1) \binom{n+k+1}{n-k}^2 \binom{2k}{k}^2 \right\}^{-1}.$$

Proof. We start by noting that the normalized Legendre polynomials L_k , $k=0, 1, 2, 3, \dots$, transformed to the interval $[0, 1]$ form an orthonormal basis of all polynomials defined on $[0, 1]$. By Rodrigues' formula we get the representation

$$L_k(x) = \frac{\sqrt{2k+1}}{k!} \left(\frac{d}{dx} \right)^k (x^2-x)^k, \quad k=0, 1, 2, \dots$$

(See for instance [1], p. 183).

Clearly $\deg L_k = k$. We now put $P_{n,k}(x) = \sum_{j=0}^n a_j L_j(x)$, where $a_0, a_1, a_2, \dots, a_n \in \mathbb{R}$.

Then

$$(2) \quad \int_0^1 \{P_{n,k}(x)\}^2 dx = \sum_{j=0}^n a_j^2 = \min!$$

From

$$L_j(x) = \frac{\sqrt{2j+1}}{j!} \left(\frac{d}{dx}\right)^j \sum_{p=0}^j (-1)^p \binom{j}{p} x^{2j-p} = \sqrt{2j+1} \sum_{p=0}^j (-1)^p \frac{(2j-p)!}{p! [(j-p)!]^2} x^{j-p}$$

we get that the coefficient of x^k in $P_{n,k}$ equals

$$(3) \quad \sum_{j=k}^n a_j (-1)^{j-k} \sqrt{2j+1} \frac{(j+k)!}{(j-k)! [k!]^2} = 1.$$

This result shows already (in view of (2)) that

$$a_0 = a_1 = \dots = a_{k-1} = 0.$$

Furthermore, (2) and (3) yield the function

$$F(a_k, \dots, a_n, \lambda) = \sum_{j=k}^n a_j^2 - \lambda \left(\sum_{j=k}^n a_j (-1)^{j-k} \frac{(j+k)!}{(j-k)! [k!]^2} - 1 \right)$$

to be minimized.

But $\frac{\partial F}{\partial a_j} = 0$, $j = k, \dots, n$ imply

$$(4) \quad a_j = \frac{1}{2} \lambda (-1)^{j-k} \sqrt{2j+1} \frac{(j+k)!}{(j-k)! [k!]^2}, \quad j = k, \dots, n.$$

Hence, via (3) we get

$$\frac{1}{2} \lambda \sum_{j=k}^n (2j+1) \frac{[(j+k)!]^2}{[(j-k)!]^2 [k!]^4} = 1,$$

i. e.

$$\lambda \binom{2k}{k}^2 \sum_{j=k}^n (2j+1) \binom{j+k}{j-k}^2 = 2.$$

This and the lemma immediately lead to

$$(5) \quad \lambda = 2 \{(2k+1) \binom{n+k+1}{n-k}^2 \binom{2k}{k}\}^{-1}.$$

As the matrix corresponding to the second derivative of F is positive-definite (subject to condition (3)), F attains minimum at the values given by (4) and (5). Finally, via (2)

$$\begin{aligned} M_{n,k} &= \lambda^2 \frac{1}{4} \sum_{j=k}^n (2j+1) \frac{[(j+k)!]^2}{[(j-k)!]^2 [k!]^4} \\ &= \frac{\lambda^2}{4} (2k+1) \binom{n+k+1}{n-k}^2 \binom{2k}{k}^2 = \{(2k+1) \binom{n+k+1}{n-k}^2 \binom{2k}{k}\}^{-1}. \end{aligned}$$

3. An integral inequality. As an application of the above theorem we prove the following

Corollary (see [2]). Let m, n be nonnegative integers. Furthermore, let $f: [0, 1] \rightarrow \mathbf{R}$ be n times continuously differentiable, $f^{(k)}(0) = f^{(k)}(1)$, $k = 0, 1, \dots, n-1$ and $\int_0^1 x^j f(x) dx = 0$, $j = 1, 2, \dots, m$. Then

$$\left(\int_0^1 f(x) dx \right)^2 \leq (2n+1) \left[\frac{n! m!}{(2n+m+1)!} \right]^2 \int_0^1 [f^{(n)}(x)]^2 dx.$$

Proof. Let Q be a polynomial such that $Q(0) = 1$ and $\deg Q \leq m$. By the assumption on f we get (via n times partial integration)

$$\int_0^1 f(x) dx = \int_0^1 f(x) Q(x) dx = (-1)^n \int_0^1 f^{(n)}(x) Q_n(x) dx,$$

where $Q_n^{(n)}(x) = Q(x)$. By Cauchy–Schwarz' inequality we arrive at

$$(6) \quad \left(\int_0^1 f(x) dx \right)^2 \leq \int_0^1 [Q_n(x)]^2 dx \int_0^1 [f^{(n)}(x)]^2 dx.$$

Now, $Q_n(x)$ is a polynomial such that $\deg Q_n \leq m+n$ and the coefficient of x^n equals 1. By the theorem we conclude that

$$\int_0^1 [Q_n(x)]^2 dx \geq M_{m+n, n} = \left\{ (2n+1) \binom{2n+m+1}{m}^2 \left(\frac{2n}{n} \right)^2 \right\}^{-1} = (2n+1) \left[\frac{n! m!}{(2n+m+1)!} \right]^2.$$

This and (6) yield the desired inequality.

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