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OPERATIONAL CALCULUS FOR THE GENERALIZED BESSEL OPERATOR

J. RODRÍGUEZ

In this paper, by following a line similar to Mikusinski's, an operational calculus for the Bessel operator $B_{\mu,\nu}=t^{-\nu-\mu-1}Dt^{2\nu+1}Dt^{\mu-\nu}$ is developed. The calculus thereof obtained meets some useful applications with regard to the solutions of certain time varying networks.

1. Introduction. In this article we develop an operational calculus for the operator

$$B_{\mu,\nu}=t^{-\nu-\mu-1}Dt^{2\nu+1}Dt^{\mu-\nu}=t^{\nu-\mu-1}Dt^{1-2\nu}Dt^{\mu+\nu}, \tag{1.1}$$

by following an algebraic procedure analogous to Mikusinki's. Since (1.1) involves several important differential operators for particular values of the parameters μ and ν such as:

$$tB_{0,\nu}=DtD$$

$$tB_{\nu,\nu}=t\Delta_\nu=tD^2+(2\nu+1)D$$

$$B_{-1/2,\nu}=S_\nu=D^2-(4\nu^2-1)/t^2$$

$$B_{0,\nu}=B_\nu=D^2+\frac{1}{t}D-\frac{\nu^2}{t^2},$$

the results that we have achieved turn out to be extensions to the research previously carried out in [2, 4, 11, 12] and give rise to a very general unified theory. By considering the fractional operator I_m^α together with its inverse $I_m^{-\alpha}$ [6] we can define convolutions for the given values of $\nu(-\infty<\nu<\infty)$ and any μ, which enables us to construct the corresponding field extension, in which certain operational rules are analysed and, further, some practical applications of the developed calculus are studied.

2. The operators I_m^α and δ. The operator I_m^α, given for $\alpha>0$ and $m>0$ as

$$I_m^\alpha f(t)=\frac{m}{\Gamma(\alpha)}\int_0^t (t^m-\xi^m)^{\alpha-1}\xi^{m-1}f(\xi)d\xi \quad (0<t<\infty)$$

as well as the operator D_m, defined by

$$D_m f(t)=\frac{d}{dt^m}f(t)=m^{-1}t^{1-m}Df(t)$$

have been treated by A. C. McBride [6]. In the following some properties of these two operators are shown, which will be invoke further.

Proposition 1. *If $f(t)\in C^1(\mathrm{R})$, then the following holds true*

$$\delta t^\alpha f(t)=t^\alpha(\delta+\alpha)f(t),\ \textit{where}\ \delta=t\frac{d}{dt}. \tag{2.1}$$

Proposition 2. *If* $f(t) \in C^1([0, \infty))$ *with* $f(0)=0$ *and* $\alpha>0$, *then the following expression*

$$t^{-2}\delta I_2^\alpha f(t) = I_2^\alpha t^{-2}\delta f(t) \tag{2.2}$$

holds.

Proposition 3. *If* $f(t) \in C^1([0, \infty))$ *and* $\alpha>0$, *then the following*

$$\delta I_2^\alpha f(t) = I_2^\alpha(\delta + 2\alpha) f(t) \tag{2.3}$$

can be established.

Proposition 4. *Let* k *be a real constant and* $\alpha>0$. *If* $f(t) \in C^1([0, \infty))$, *then the following*

$$I_2^\alpha(\delta + k) f(t) = (\delta + k - 2\alpha) I_2^\alpha f(t) \tag{2.4}$$

is true.

For the corresponding proofs of propositions 1, 2, 3 and 4, see [11] and [13].

Proposition 5. *Let* $\nu < -1/2$ *and* $T_{\mu,\nu,1} = t I_2^{\nu+1/2} t^{\mu-\nu}$. *If* $f(t) \in C^2([0, \infty))$ *satisfies the* following *condition* $t^{2\nu+1} D t^{\mu-\nu} f(t)|_{t=0+} = 0$, *then, we can derive that*

$$T_{\mu,\nu,1} B_{\mu,\nu} f(t) = D^2 T_{\mu,\nu,1} f(t). \tag{2.5}$$

Proof. Let start with equality:

$$T_{\mu,\nu,1} B_{\mu,\nu} f(t) = I_2^{\nu+1/2+} t^{\mu-\nu} B_{\mu,\nu} f(t) = D_2^n I_2^{n+\nu+1/2} t^{\mu-\nu} t^{-2} [\delta + (\mu+\nu)] [\delta + (\mu-\nu)] f(t),$$

where n is the least integer which is greater than $-(\nu+1/2)$ and

$$D_2^n = \left(\frac{1}{2t}\frac{d}{dt}\right)^n = 2^{-n} t^{-2n} \prod_{i=0}^{n-1} (\delta - 2i).$$

The following steps can be inferred from (2.1), proposition 2 and (2.3) as well.

$$\begin{aligned} D_2^n I_2^{n+\nu+1/2} t^{-2}\delta(\delta+2\nu) t^{\mu-\nu} f(t) &= D_2^n t^{-2}\delta I_2^{n+\nu+1/2}(\delta + 2\nu(t^{\mu-\nu} f(t) \\ &= D_2^n t^{-2}\delta(\delta - 2n - 1) I_2^{n+\nu+1/2} f(t). \end{aligned} \tag{2.6}$$

Now, by taking into account $D_2^n t^{-2}\delta(\delta - 2n - 1) = D^2 D_2^n$, the right member of (2.6) becomes

$$D^2 D_2^n I_2^{n+\nu+1/2} t^{\mu-\nu} f(t) = D^2 I_2^{\nu+1/2} t^{\mu-\nu} f(t),$$

which completes the proof.

Proposition 6. *Let* $\nu = -1/2$ *and* $T_{\mu,\nu,2} = t^{\mu+1/2}$. *If* $f(t) \in C^2([0, \infty))$, *then the following equality holds*

$$T_{\mu,\nu,2} B_{\mu,\nu} f(t) = D^2 T_{\mu,\nu,2} f(t). \tag{2.7}$$

Proof. It suffices to note that $t^{\mu+1/2} B_{\mu,-1/2} f(t) = D^2 t^{\mu+1/2} f(t)$.

Proposition 7. *Let* $\nu > -1/2$ *and* $T_{\mu,\nu,3} = t I_2^{-(\nu+1/2)} t^{\mu+\nu}$. *If* $f(t) \in C^2([0, \infty))$ *and the condition* $t^{2\nu+1} D t^{\mu-\nu} f(t)|_{t=0+} = 0$, *is satisfied, then we have*

$$T_{\mu,\nu,3} B_{\mu,\nu} f(t) = D^2 T_{\mu,\nu,3} f(t). \tag{2.8}$$

Proof is similar to the one to proposition 5.

3. The field extension. Operational calculus. By means of propositions 5, 6 and 7 we can introduce different spaces according to the values taken up by ν.

a) For $\nu>-1/2$, let us form the set

$$C^2_{\nu-\mu}=\{f(t)/f(t)=t^{\nu-\mu}f_1(t);\ f_1(t)\in C^2([0,\infty))\}. \tag{3.1}$$

Now, from (2.8) and the similarity method (N. A. Meller [8]) as well, it follows that the operation $*$ can be defined on $C^2_{\nu-\mu}$ as

$$f(t)*g(t)=\frac{\Gamma(1/2)}{\Gamma(\nu+1)}\,T^{-1}_{\mu,\nu,3}[T_{\mu,\nu,3}f(t)\circ T_{\mu,\nu,3}g(t)], \tag{3.2}$$

for each $f(t)$ and $g(t)$ in $C^2_{\nu-\mu}$. Here notation $(\circ)$ stands for the convolution for the operator D, which was considered in [2]. $T^{-1}_{\mu,\nu,3}f(t)$ is defined to be

$$T^{-1}_{\mu,\nu,3}f(t)=t^{-\mu-\nu}I_2^{(\nu+1/2)}t^{-1}f(t). \tag{3.3}$$

From the definition of the operator $T_{\mu,\nu,3}$ and its inverse $T^{-1}_{\mu,\nu,3}$, and the elements of $C^2_{\nu-\mu}$ as well, we can easily derive that the following equality

$$t^{\nu-\mu+p}*t^{\nu-\mu+q}=\frac{\Gamma(1/2)}{\Gamma(\nu+1)}\cdot\frac{\Gamma(\frac{2\nu+p+q}{2})\,\Gamma(\frac{2\nu+q+2}{2})}{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{q+1}{2})}\cdot\frac{\Gamma(p+1)\Gamma(q+1)\,\Gamma(\frac{p+q+1}{2})}{\Gamma(p+q+1)\Gamma(\frac{2\nu+p+q+2}{2})}\,t^{\nu-\mu+p+q}$$

holds true for each $p,q\in\mathrm{N}\cup\{0\}$. Therefore, from the characterization of the elements of $C^2_{\nu-\mu}$ by invoking Weierstrass's approximation theorem, it can be inferred that $*$ is a closed operation on $C^2_{\nu-\mu}$. Moreover, the following properties can be verified:

i $\quad f(t)*g(t)=g(t)*f(t),$

ii $\quad f(t)*(g(t)*h(t))=(f(t)*g(t))*h(t),$

iii $\quad f(t)*(g(t)+h(t))=(f(t)*g(t))+(f(t)*g(t)),$

iv $\quad f(t)*g(t)=0,\ \Leftrightarrow\ f(t)=0$ or $g(t)=0,$

v $\quad t^{\nu-\mu}*f(t)=f(t),$

where $f(t)$, $g(t)$ and $h(t)\in C^2_{\nu-\mu}$.

Proposition 8. *When $C^2_{\nu-\mu}$ is endowed with addition $+$ and multiplication $*$, it becomes a unitary commutative ring without zero divisors.*

As a consequence, $C^2_{\nu-\mu}$ can be extended to the quotient field

$$M=C^2_{\nu-\mu}\times(C_{\nu-\mu}-\{0\})/\sim$$

where the equivalence noted by $\sim$ is defined over $C^2_{\nu-\mu}\times(C^2_{\nu-\mu}-\{0\})$ in the usual way, i. e.,

$$(f(t),g(t))\sim(\tilde f(t),\tilde g(t)),\ \Leftrightarrow\ f(t)*\tilde g(t)=g(t)*\tilde f(t).$$

From now on the element $(f(t),g(t))$ will be noted as $f(t)/g(t)$.

If M is endowed with addition, multiplication and product by scalar, then it becomes an algebra.

Note 1. Note that there exists a subset $M'\subset M$, which is isomorphic to $C^2_{\nu-\mu}$ under the mapping

$$M'\subset M \longrightarrow > C^2_{\nu-\mu}$$

$$\frac{t^{\nu-\mu} * f(t)}{t^{\nu-\mu}} \longrightarrow > f(t).$$

Note 2. On the other hand, since

$$B_{\mu,\nu} t^{\nu-\mu+k} = k(2\nu+k) t^{\nu-\mu+k-2}$$

holds true, the following can be established

$$B_{\mu,\nu}(t^{\nu-\mu+p} * t^{\nu-\mu+q}) = (B_{\mu,\nu} t^{\nu-\mu+p}) * t^{\nu-\mu+q}. \tag{3.4}$$

It suffices to take into account that the equality

$$B_{\mu,\nu}(t^{\nu-\mu+p} * t^{\nu-\mu+q})$$

$$= \frac{\Gamma(1/2)\Gamma(\frac{2\nu+p+2}{2})\Gamma(\frac{2\nu+q+2}{2})\Gamma(p+1)\Gamma(q+1)\Gamma(\frac{p+q+1}{2})}{\Gamma(\nu+1)\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})\Gamma(p+q+1)\Gamma(\frac{2\nu+p+q+2}{2})} (p+q)(2\nu+p+q)t^{\nu-\mu+p+q-2}$$

$$= p(p+2\nu)t^{\nu-\mu+p-2} * t^{\nu-\mu+q} = (B_{\mu,\nu} t^{\nu-\mu+p}) * t^{\nu-\mu+q}$$

holds.

For this reason, we say (by following I. Dimovski [1]) that $*$ is a convolution for the operator $B_{\mu,\nu}$ in $C^2_{\nu-\mu}$.

Now, consider the integral operator given as

$$L_{\mu,\nu} f(t) = t^{\nu-\mu} \int_0^t \xi^{-1-2\nu} d\xi \int_0^\xi \eta^{\mu+\nu+1} f(\eta) d\eta = t^{-\mu-\nu} \int_0^t \xi^{2\nu-1} d\xi \int_0^\xi \eta^{\mu-\nu+1} f(\eta)\, d\eta, \tag{3.5}$$

which turns out to be the right-hand inverse operator for the Bessel operator $B_{\mu,\nu}$ i. e., $B_{\mu,\nu} L_{\mu,\nu} f(t) = f(t)$ holds. Now, if we restrict the domain of $B_{\mu,\nu}$ to be the set

$$\{ f(t) \in C^2_{\nu-\mu} / t^{\mu-\nu} f(t) |_{t=0+} = 0 \},$$

then $L_{\mu,\nu}$ can also be regarded as the left-hand inverse operator for the operator $B_{\mu,\nu}$ i. e., $L_{\mu,\nu} B_{\mu,\nu} f(t) = f(t)$ also holds.

In the following it will be shown that the operator $L_{\mu,\nu}$ belongs to M.

Proposition 9. *For each* $f(t) \in C^2_{\nu,\mu}$, *the following holds*

$$\frac{t^{\nu-\mu+2}}{2^2(\nu+1)} * f(t) = L_{\mu,\nu} f(t). \tag{3.6}$$

To prove this it suffices to take into account that $B_{\mu,\nu} t^{\nu-\mu+2} = 2^2(\nu+1)t^{\nu-\mu}$. Then' from (3.4) it follows that

$$B_{\mu,\nu}\left(\frac{t^{\nu-\mu+2}}{2^2(\nu+1)} * f(t)\right) = B_{\mu,\nu} L_{\mu,\nu} f(t),$$

which further leads to $t^{\nu-\mu} * f(t) = f(t)$.

By applying the principle of induction to proposition 9 we can establish the following:

Proposition 10. *For* $k \in \mathrm{N}$ *and* $f(t) \in C^2_{\nu-\mu}$, *we can write that*

$$\frac{\Gamma(\nu+1)}{2^{2k} k!\, \Gamma(\nu+k+1)} t^{\nu-\mu+2k} * f(t) = L^k_{\mu,\nu} f(t). \tag{3.7}$$

Hence, it follows from note 1 that the operator $L^k_{\mu,\nu}$ belongs to M and is represented by

$$L^k_{\mu,\nu}=\frac{\Gamma(\nu+1)}{2^{2k}k!\,\Gamma(\nu+k+1)}\,t^{\nu-\mu+2k}. \tag{3.8}$$

Proposition 11. *For $f(t)\in C^2_{\nu-\mu}$, we have*

$$f(t)=L_{\mu,\nu}B_{\mu,\nu}f(t)+t^{\nu-\mu}f_1(0^+) \tag{3.9}$$

where $f_1(t)=t^{\mu-\nu}f(t)$.

Proof. It suffices to note that

$$L_{\mu,\nu}B_{\mu,\nu}f(t)=t^{\nu-\mu}\int_0^t \xi^{-1-2\nu}d\xi\int_0^{\xi} D\eta^{2\nu+1}D\eta^{\mu-\nu}f(\eta)d\eta,$$

holds true.

After integration, the right member of the preceeding equality becomes $f(t)-t^{\nu-\mu}[t^{\mu-\nu}f(t)]|_{t=0^+}$.

Now, the following can be proved by applying the principle of induction:

Proposition 12. *For $f(t)\in C^{2k}_{\nu-\mu}$, we have*:

$$f(t)=L^k_{\mu,\nu}B^k_{\mu,\nu}f(t)+t^{\nu-\mu}\sum_{j=1}^{k}[t^{\mu-\nu}B^{k-j}_{\mu,\nu}f(t)]|_{t=0^+}. \tag{3.10}$$

Now, let V be the operator defined as $V=2^2(\nu+1)\,t^{\nu-\mu}/t^{\nu-\mu+2}$, where V^k is the k-times application of V. With respect to this operator the following can be established:

Proposition 13. *If $k\in\mathrm{N}$ and $f(t)\in C^{2k}_{\nu-\mu}$, then*

$$V^kf(t)=V^k*f(t)=B^k_{\mu,\nu}f(t)+\sum_{j=1}^{k}[t^{\mu-\nu}B^{k-j}_{\mu,\nu}f(t)]_{t=0^+}V^j, \tag{3.11}$$

holds true.

Proof. By applying V to (3.9) we get:

$$Vf(t)=B_{\mu,\nu}f(t)+[t^{\mu-\nu}f(t)]|_{t=0^+}V,\ \text{for}\ \ k=1.$$

Since the preceeding holds for $k=1$, let us suppose that it is equally certain for $k=m$. Then, the application of V to V^m leads to

$$V(V^mf(t))=V(B^m_{\mu,\nu}f(t))+V\sum_{j=1}^{m}[t^{\mu-\nu}B^{m-j}_{\mu,\nu}f(t)]|_{t=0^+}V^j.$$

But, on the other hand we can write that

$$B^m_{\mu,\nu}f(t)=L_{\mu,\nu}B^{m+1}_{\mu,\nu}f(t)+t^{\nu-\mu}[t^{\mu-\nu}B^m_{\mu,\nu}f(t)]|_{t=0^+},$$

and therefore it can be inferred that

$$V^{m+1}f(t)=B^{m+1}_{\mu,\nu}f(t)+V[t^{\mu-\nu}B^m_{\mu,\nu}f(t)]|_{t=0^+}+\sum_{j=1}^{m}[t^{\mu-\nu}B^{m-j}_{\mu,\nu}]|_{t=0^+}V^{j+1}$$

b) if $\nu=-1/2$, let us consider the set

$$C^2_{-1/2-\mu}=\{f(t)/f(t)=t^{-1/2-\mu}f_1(t);\ f_1(t)\in C^2([0,\infty))\}.$$

From (2.7) it follows that the operation given as:

$$f(t)*g(t)=T^{-1}_{\mu,\nu,2}[(T_{\mu,\nu,2}f(t))\circ(T_{\mu,\nu,2}g(t))], \tag{3.12}$$

can be defined on $C^2_{-1/2-\mu}$ for each $f(t)$, $g(t)$ belonging to $C^2_{-1/2-\mu}$. Let us denote by $(\circ) \longrightarrow$ the convolution for the operator D, which has been already mentioned. $T^{-1}_{\mu,\nu,2}$ is defined to be

$$T^{-1}_{\mu,\nu,2} = t^{-\mu-1/2}.$$

Hence, by introducing the operation $+$ and $*$ given in (3.12), the set $C^2_{-1/2-\mu}$ becomes a unitary commutative ring without zero divisors. Therefore, $C^2_{-1/2-\mu}$ can be extended to a quotient field. As a consequence, the same results obtained for $\nu > -1/2$ hold here true for the particular value $\nu = -1/2$.

c) If $\nu < -1/2$, let us form the set of function given as

$$C^2_{-\nu-\mu-1} = \{ f(t)/f(t) = t^{-\nu-\mu-1} f_1(t)\,;\; f_1(t) \in C^2([0, \infty))\}.$$

By invoking (2.5), the following operation

$$f(t) * g(t) = 1/\Gamma\left(\frac{1-2\nu}{2}\right) T^{-1}_{\mu,\nu,1} f(t)\, [(T_{\mu,\nu,1} f(t) \circ (T_{\mu,\nu,1} g(t))] \tag{3.13}$$

can be defined on $C^2_{-\nu-\mu-1}$, for each $f(t), g(t) \in C^2_{-\nu-\mu-1}$. Here notation $(\circ)$ is used to design the convolution for the operator D. That is:

$$f(t) \circ g(t) = D \int_0^t f(t-\xi)\, g(\xi) d\xi$$

with

$$T^{-1}_{\mu,\nu,1} = t^{\nu-\mu} I_2^{-(\nu+1/2)}.$$

Under the operations of addition $+$ and multiplication $*$ given in (3.13) the set $C^2_{-\nu-\mu-1}$ becomes a unitary commutative ring without zero divisors. Further, the corresponding extension to a quotient field can be carried out.

Now, by following a similar line to the one viewed in case a), the propositions therein contained can be extended to the case under consideration. Here the operators $L_{\mu,\nu}$ and V are given as:

$$L_{\mu,k} = \frac{t^{-\nu-\mu+1}}{(1-2\nu)}, \quad V = (1-2\nu)\, \frac{t^{-\nu-\mu-1}}{t^{-\nu-\mu+1}}.$$

4. Applications. In the following some operational rules may be derived, which will turn out useful to tackle certain practical applications.

The differential equations

$$(B_{\mu,\nu} \pm a) y = 0, \quad a > 0 \tag{4.1}$$

or else

$$y'' + \frac{2\mu+1}{t} y' + \left(\pm a - \frac{\nu^2-\mu^2}{t^2}\right) y = 0, \quad a > 0$$

admit as solution the following function [15]

$$y_1(t) = (a^{1/2}t)^{-\mu} J_\nu(a^{1/2}t) = J_{\mu,\nu}(a^{1/2}t)$$

for the sign $+$, and

$$y_2(t) = (a^{1/2}t)^{-\mu} I_\mu(a^{1/2}t) = I_{\mu,\nu}(a^{1/2}t)$$

for the sign —. Here, $J_\nu(t)$ and $I_\nu(t)$ stand, respectively, for the Bessel function of the first kind and the modified Bessel function of the first kind and order ν. Since

$$\lim_{t\to 0+} t^{\mu-\nu} J_{\mu,\nu}(a^{1/2}t) = \lim_{t\to 0+} t^{\mu-\nu} I_{\mu,\nu}(a^{1/2}t) = a\left(\frac{\nu-\mu}{2}\right)/2^{\nu}\Gamma(\nu+1) \tag{4.2}$$

holds, then by subtituting (4.1) and (4.2) into $Vf(t) = B_{\mu,\nu} f(t) + [t^{\mu-\nu} f(t)]\,|_{t=0} V$ we obtain that

$$Vf(t) = -at^{\nu-\mu} * f(t) + \left(a\left(\frac{\nu-\mu}{2}\right)2^{\nu}\Gamma(\nu+1)\right)V.$$

Therefore, the equality

$$(V + at^{\nu-\mu}) * f(t) = \left(a\left(\frac{\nu-\mu}{2}\right)2^{\nu}\Gamma(\nu+1)\right)V$$

holds, and, as a consequence, the following can be established

$$\frac{V}{V+at^{\nu-\mu}} = \frac{2^{2}\Gamma(\nu+1)}{a^{(\frac{\nu-\mu}{2})}} J_{\mu,\nu}(a^{1/2}t). \tag{4.3}$$

Similarly, the following expression can be derived:

$$\frac{V}{V-at^{\nu-\mu}} = \frac{2^{\nu}\Gamma(\nu+1)}{a^{(\frac{\nu-\mu}{2})}} I_{\mu,\nu}(a^{1/2}t). \tag{4.4}$$

Through a direct calculus the validity of the following formulae can be verified:

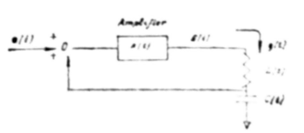
$$\frac{at^{\nu-\mu}}{V+at^{\nu-\mu}} = 1 - \frac{2^{\nu}\Gamma(\nu+1)}{a^{(\frac{\nu-\mu}{2})}} J_{\mu,\nu}(a^{1/2}t) \tag{4.5}$$

$$\frac{-at^{\nu-\mu}}{V-at^{\nu-\mu}} = 1 - \frac{2^{\nu}\Gamma(\nu+1)}{a^{(\frac{\nu-\mu}{2})}} I_{\mu,\nu}(a^{1/2}t) \tag{4.6}$$

$$\frac{V^2}{V^2-a^2t^{\nu-\mu}} = \frac{2^{\nu}\Gamma(\nu+1)}{a^{(\frac{\nu-\mu}{2})}} \left[\frac{I_{\mu,\nu}(a^{1/2}t)+J_{\mu,\nu}(a^{1/2}t)}{2}\right] \tag{4.7}$$

$$\frac{aV}{V^2-a^2t^{\nu-\mu}} = \frac{2^{\nu}\Gamma(\nu+1)}{a^{(\frac{\nu-\mu}{2})}} \left[\frac{I_{\mu,\nu}(a^{1/2}t)-J_{\mu,\nu}(a^{1/2}t)}{2}\right]. \tag{4.8}$$

As an example, consider the positive feedback circuit ([3]; fig 4).



where $q(t)$ satisfies the following equation

$$DL(t)Dq(t) + \frac{q(t)}{C(t)} = k(t)\left(e(t) + \frac{q(t)}{C(t)}\right). \tag{4.9}$$

Assume that $L(t)=at^{1+2\nu}$, $C(t)=bt^{-2\nu-1}$ an $k(t)=1-t^{\nu-\mu}$ with $a, b>0$ and $\nu>-1/2$. Then, equation (4.10) can be re-written as:

$$abB_{\mu,\nu}q_1(t)+(1-k(t))q_1(t)=e_1(t),$$

where $q_1(t)=t^{\nu-\mu}q(t)$ and $e_1(t)=bk(t)e(t)t^{-\nu-\mu-1}$.

Hence, if $q(0)=0$, we have

$$q_1(t)=\frac{t^{\nu-\mu}}{abV+t^{\nu-\mu}}*e_1(t),$$

where, involving (4.6) becomes

$$q_1(t)=t^{\nu-\mu}\left[1-\frac{2^\nu\Gamma(\nu+1)}{(ab)^{-\nu/2}}t^{-\nu}J_\nu((ab)^{-1/2}t)\right]*e_1(t).$$

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Departamento de Análisis Matemático
Universidad de La Laguna
38203 La Laguna Spain

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