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ON ORTHOGONAL POLYNOMIALS ASSOCIATED WITH DIFFERENTIAL EQUATIONS OF LAGUERRE TYPE

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This paper deals with a study of certain class of orthogonal polynomials, obtained as solutions of differential equations of Laguerre type. Such differential equations are derived through the application of the properties of fractional (generalized) calculus, after establishing the equivalence properties between the operator equations and the corresponding differential equations. The sclutions of the equivalent operator equations and consequently those of the corresponding differential equations have been obtained by using properties of the integro-differential operator of generalized order. Among the solutions is the Rodrigues formula by means of which the orthogonal polynomials have been defined.

1. Introduction. Fractional (generalized) calculus deals with the derivatives and integrals of arbitrary orders, called the differintegrals or the integro-differentials [1, 2, 10, 13, 14, 15, 17]. The concept of differintegral of complex order ν, which is a generalization of the ordinary n-th derivative and n-times integral, can be introduced in several ways. One of the simple definitions of an integral of arbitrary order is based on an integral transform, called the Riemann—Liouville operator of fractional integration [14, 15, 17] $R^{\alpha}f$:

$$R^{\alpha}f : I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \mathrm{Re}(\alpha)>0 \tag{1.1a}$$

$$= \frac{d^n}{dx^n} R_x^{\alpha+n} f \tag{1.1b}$$

for $\mathrm{Re}(\alpha) \leq 0$

Another fundamental definition of a differintegral of order ν, due to Grunwald [15, 17] is as follows:

$$\frac{d^{\nu}f}{[d(x-a)]^{\nu}} = \lim_{N\to\infty} \left\{ \frac{\left[\frac{x-a}{N}\right]^{-\nu}}{\Gamma(-\nu)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\nu)}{\Gamma(j+1)} f\left(x-j\left[\frac{x-a}{N}\right]\right) \right\}, \tag{1.2}$$

where ν is arbitrary. It is interesting to observe that here no explicit use is made of classical definitions of derivatives or integrals of f.

K. Nishimoto [14] considers the differintegral using Cauchy's formula for analytic functions and *integration in a complex domain* Al-Bassam [1, 2—10] and [15, p. 12] has proved the equivalence between Holmgren and M. Riesz definitions, thus combining the two definitions in the form:

If $f(x)$ is a real valued function $\in C^{(n)}$, $a \leq x \leq b$ and $\mathrm{Re}(\alpha+n)>0$, then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha+n)} D_x^n \int_a^x (x-t)^{\alpha+n-1} f(t)\, dt, \quad n=0, 1, 2, \dots \tag{1.3a}$$

which can also be written as

$$(1.3b)\qquad \overset{x}{\underset{a}{I}}{}^{\alpha} f = \sum_{i=0}^{n-1} \frac{(x-a)^{\alpha+i}}{\Gamma(\alpha+i+1)} f^{(i)}(a) + \frac{1}{\Gamma(\alpha+n)} \int_a^x (x-t)^{\alpha+n-1} f^{(n)}(t)dt$$

($n=1, 2, \ldots$), where D_x^n is the n-th derivative operator with respect to x, $f^{(n)}=D_x^n f$, $C^{(n)}$ is the set of functions with contionuous n-th derivatives on $[a, b]$.

The result (1.3a) can be written as $\overset{x}{\underset{a}{I}}{}^{\alpha} f = D_x^n \overset{x}{\underset{a}{I}}{}^{\alpha+n} f$ and using this Al-Bassám [1, 2] has established many properties of this operator. For example,

(i) If $f \in C^{(m+n)}[a, b]$ and $\mathrm{Re}(\alpha+m)>0$, then

$$(1.4)\qquad D_x^n \overset{x}{\underset{a}{I}}{}^{\alpha} f = \overset{x}{\underset{a}{I}}{}^{\alpha-n} f.$$

(ii) If $f \in C^0[a, b]$, then

$$(1.5)\qquad \lim_{\alpha\to 0} \overset{x}{\underset{a}{I}}{}^{\alpha} f = \overset{x}{\underset{a}{I}}{}^{0} f = f(x).$$

From (1.4) and (1.5), it follows that $\overset{x}{\underset{a}{I}}{}^{-n} f = D_x^n f = f^{(n)}(x)$.

(iii) If $f(x)$ and $g(x)$ are differintegrable and K is a constant, then $\overset{x}{\underset{a}{I}}{}^{\alpha} K(f+g)$ $= K \overset{x}{\underset{a}{I}}{}^{\alpha} f + K \overset{x}{\underset{a}{I}}{}^{\alpha} g$.

(iv) For differintegrable functions f and g,

$$\overset{x}{\underset{a}{I}}{}^{-\alpha} fg = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\,\Gamma(n+1)} \overset{x}{\underset{a}{I}}{}^{-n} f \overset{x}{\underset{a}{I}}{}^{\alpha+n} g$$

and if $f(x)$ is a polynomial of degree k and $Re(\alpha)>0$, then

$$\overset{x}{\underset{a}{I}}{}^{\pm\alpha} fg = \sum_{n=0}^{k} \binom{\mp\alpha}{n} f^{(n)}(x) \overset{x}{\underset{n}{I}}{}^{\pm\alpha+n} g.$$

(v) If $x>a$, a real number, and $\beta \neq -m$ (a negative integer), then

$$\overset{x}{\underset{a}{I}}{}^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}.$$

In this paper, we introduce a class of modified *special* functions and polynomials and study some of their properties. The class has been obtained through a study of some types of operator equations of Laguerre type and their corresponding equivalent differential equations by an appeal to the properties of the differintegral or the integro-differential operators [1]. This approach is similar to that of [3–9], where Al-Bassam has studied equations related to other classes of special functions.

Differential equations of Laguerre type have been developed through their equivalent operator equations, which are more general than Laguerre equations [11]. Solutions of these equations have been obtained by solving the equivalent operator equations. One of these solutions is expressed by a generalized Rodrigues formula of arbitrary order (that is the derivatives involved are of arbitrary order instead of derivatives of integral order). The main study is confined to this *class of solutions* which is proved to be orthogonal.

Due to the obvious advantages of the notation (1.3) for differintegral, we follow the notations and symbols used by Al-Bassam [1—10] in his previous works.

2. Some equivalence properties. The operator equation

$$\overset{x}{\underset{a}{I}}{}^{-\alpha-1}\, e^{-\mu x}(x-1)^{\alpha+\beta+1}\overset{x}{\underset{a}{I}}{}^{-1}\, e^{\mu x}(x-1)^{-\alpha-\beta}\overset{x}{\underset{a}{I}}{}^{\alpha}\, y=0, \tag{2.1a}$$

where α, δ, μ are numbers such that $\operatorname{Re}\alpha>0$, and

$$\overset{x}{\underset{a}{I}}{}^{\gamma} F=\frac{1}{\Gamma(\gamma+n)}\, D_x^n \int_a^x (x-t)^{\gamma+n-1} F(t)dt, \quad (n=0,\ 1,\ 2,\dots),\quad a\leqq x\leqq b,$$

$F\in C^{(n)}$ on $a\leqq x\leqq b$, $\operatorname{Re}\gamma+n>0$, n is a positive integer, is equivalent to the second order differential equation

$$(x-1)y''+[1-\beta+\mu(x-1)]y'+\mu(\alpha+1)y=0, \tag{2.1b}$$

where $y'=dy/dx=Dy$, etc.

This can be easily shown by expanding the left hand side of (2.1a) and using the Leibnitz rule of generalized differentiation with properties of the integro-differential operator of generalized order [1, p. 9].

By applying the transformation $y=e^{-\mu x}(x-1)^{\beta}Y$ equation (2.1) can be written as

$$\overset{x}{\underset{a}{I}}{}^{-\alpha-1}e^{-\mu x}(x-1)^{\alpha+\beta+1}\ \overset{x}{\underset{a}{I}}{}^{-1}\, e^{\mu x}(x-1)^{-\alpha-\beta}\ \overset{x}{\underset{a}{I}}{}^{\alpha}\, e^{-\mu x}(x-1)^{\beta}Y=0 \tag{2.1c}$$

and $(x-1)Y''+[\beta+1-\mu(x-1)]Y'+\mu\alpha Y=0$.

Equations (2.1) are particular cases of the operator equation

$$\overset{x}{\underset{a}{I}}{}^{-\alpha-1}\, e^{-\mu x}(x-b)^{\gamma+1}\ \overset{x}{\underset{a}{I}}{}^{-1}\, e^{\mu x}(x-b)^{-\gamma}\overset{x}{\underset{a}{I}}{}^{\alpha}y=0 \tag{2.2a}$$

and of its equivalent differential equation

$$(x-b)y''+[\alpha-\gamma+1+\mu(x-b)]y'+\mu(\alpha+1)y=0. \tag{2.2b}$$

If the transformation $y=(x-b)^{\lambda}e^{-\mu x}Y$ is applied to equations (2.2), then we would have

$$\overset{x}{\underset{a}{I}}{}^{-\alpha-1}\, e^{-\mu x}(x-b)^{\gamma+1}\ \overset{x}{\underset{a}{I}}{}^{-1}\, e^{\mu x}(x-b)^{-\gamma}\overset{x}{\underset{a}{I}}{}^{\alpha}(x-b)^{\lambda}e^{-\mu x}Y=0 \tag{2.2c}$$

and its equivalent differential equation

$$(x-b)Y''+[(\alpha+2\lambda-\gamma+1)-\mu(x-b)]Y'+[\mu(\gamma-\lambda)+\lambda(\lambda+\alpha-\gamma)\ (x-b)^{-1}]Y+0. \tag{2.2d}$$

Equations (2.2) are reduced to equations (2.1) if $\lambda=\beta$, $\gamma=\alpha+\beta$, $b=1$. Throughout this work there will be no loss of generality in using and dealing with equations (2.1) instead of equations (2.2).

It is to be noted that if in (2.1) $\mu=1$, we would have the Laguerre differential equation satisfied by the Laguerre function $L_{\alpha}^{(\beta)}(x-1)$, [11, p. 182—183].

3. Solutions and orthogonality. By using the operational properties of the integro-differential operator of generalized order [1] solutions of (2.1b) and (2.1d) can be obtained by solving the equivalent operator equations (2.1a) and (2.1c) respectively. Thus from these operator equations of the forms

(3.1a) $$y_1 = K_1 \overset{x}{\underset{a}{I}}{}^{-\alpha} e^{-\mu x}(x-1)^{\alpha+\beta},$$

(3.1b) $$y_2 = \overset{x}{\underset{a}{I}}{}^{-\alpha} e^{-\mu x}(x-1)^{\alpha+\beta} \overset{x}{\underset{a}{I}} e^{\mu x}(x-1)^{-\alpha-\beta-1}(\overset{x}{\underset{a}{I}}{}^{\alpha+1}\, 0)$$

and

(3.1c) $$Y_1 = K_1 e^{\mu x}(x-1)^{-\beta} \overset{x}{\underset{a}{I}}{}^{-\alpha} e^{-\mu x}(x-1)^{\alpha+\beta},$$

(3.1d) $$Y_2 = e^{\mu x}(x-1)^{-\beta} \overset{x}{\underset{a}{I}}{}^{-\alpha} e^{-\mu x}(x-1)^{\alpha+\beta} \overset{x}{\underset{a}{I}} e^{\mu x}(x-1)^{\alpha-\beta-1}(\overset{x}{\underset{a}{I}}{}^{\alpha+1}\, 0),$$

where

$$\overset{x}{\underset{a}{I}}{}^{\alpha+1}\, 0 = \sum_{i-0}^{n-1} C_{i+1} \frac{(x-)^{\alpha-i}}{\Gamma(\alpha-i+1)}, \quad [1, \text{ p. } 20]$$

K_1, C_j ($j=0, 1, \ldots, n-1$) are arbitrary constants, $\operatorname{Re}\alpha > n-1$ ($n=1, 2, \ldots$) and a can be assumed to be equal to one.

Solutions y_1 in (3.1a) and Y_1 in (3.1c) represent the generalized Rodrigues formulae for solutions of equations (2.1a) and (2.1d) respectively, i. e. the Rodrigues formulae of any order α.

The expanded form of (3.1c). Let $Y_1 = A_\alpha^{(\beta, \mu)}(x)$. Then (3.1c) *takes* the expanded form

$$A_\alpha^{(\beta,\mu)}(x) = K_1(x-1)^{-\beta} e^{\mu x} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-\mu)^k [\overset{x}{\underset{a}{I}}{}^{k-\alpha}(x-1)^{\alpha+\beta}] e^{-\mu x}$$

$$= K_1 \sum_{k=0}^{\infty} \frac{(-\mu)^k \Gamma(\alpha+1)\Gamma(\alpha+\beta+1)\,(x-1)^k}{\Gamma(k+1)\,\Gamma(\alpha-k+1)\Gamma(\beta+k+1)} = K_1 \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)\Gamma(\alpha+\beta+1)\,[\mu(x-1)]^k}{\Gamma(\alpha-k+1)\Gamma(\beta+k+1)\Gamma(k+1)}.$$

If $K_1 = \frac{1}{\Gamma(\alpha+1)}$, then

(3.2) $$A_\alpha^{(\beta,\mu)}(x) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \sum_{k=1}^{\infty} \frac{(-\alpha)_k [\mu(x-1)]^k}{(\beta+1)_k \Gamma(k+1)} = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\, {}_1F_1(-\alpha;\ \beta+1;\ \mu(x-1)),$$

where $(\alpha)_r = \alpha(\alpha+1)\ldots(\alpha+r-1)$, $(\alpha)_0 = 1$.

The Case of Y_1 when $\alpha = n$. The solution when $\alpha = n$ (a positive integer) $Y_1 = A_n^{(\beta,\mu)}(x)$ is represented by the usual Rodrigues formula

(3.3a) $$A_n^{(\beta,\mu)}(x) = \frac{1}{\Gamma(n+1)} e^{\mu x}(x-1)^{-\beta} D^n e^{-\mu x}(x-1)^{\beta+n}.$$

This represents a polynomial of degree n. For ($n=0, 1, 2, 3, 4$) this polynomial takes the following forms:

$A_0^{(\beta,\mu)}(x) = 1,$

$A_1^{(\beta,\mu)}(x) = -\mu(x-1) + (\beta+1),$

$A_2^{(\beta,\mu)}(x) = \frac{1}{2}[\mu^2(x-1)^2 - 2\mu(\beta+2)(x-1) + (\beta+1)(\beta+2)],$

$$A_3^{(\beta,\mu)}(x)=\frac{1}{\Gamma(4)}[-\mu^3(x-1)^3+3\mu^2(\beta+3)(x-1)^2-3\mu(\beta+2)(x-1)+(\beta+1)(\beta+2)(\beta+3)],$$

$$A_4^{(\beta,\mu)}(x)=\frac{1}{\Gamma(5)}[\mu^4(x-1)^4-4(\beta+4)\mu^3(x-1)^3+6(\beta+3)(\beta+4)\mu^2(x-1)^2$$
$$-4\mu(\beta+2)(\beta+3)(\beta+4)(x-1)+(\beta+1)(\beta+2)(\beta+3(\beta+4)].$$

In general the polynomial may take the expanded form:

$$A_n^{(\beta,\mu)}(x)=\frac{1}{\Gamma(n+1)}\sum_{k=0}^{n}\frac{\Gamma(n+1)\Gamma(\beta+n+1)(-\mu)^{n-k}(x-1)^{n-k}}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(\beta+n-k+1)}\tag{3.3b}$$

$$=\frac{(\beta+n+1)}{\Gamma(n+1)}\sum_{r=0}^{n}\frac{(-\mu)^r\Gamma(n+1)(x-1)^r}{\Gamma(r+1)\Gamma(\beta+r-1)\Gamma(n-r+1)}=\frac{(\beta+1)n}{\Gamma(n+1)}\sum_{=0}^{n}\frac{(-n)_r[\mu(x-1)]^r}{\Gamma(r+1)(\beta+1)_r}$$

which is the special case of (3.2) when $\alpha=n$, a positive integer, and it represents a polynomial of degree n. This polynomial of (3.3b) can be written as

$$A_n^{(\beta,\mu)}(x)=\frac{(\beta+1)_n}{\Gamma(n+1)}\,{}_1F_1(-n;\ \beta+1;\ \mu(x-1)).\tag{3.3c}$$

It may be noticed that y_1 and Y_1 represent solutions in the form of Rodrigues formulae. We have been dealing only with the form (3.3c) as it is related to the orthogonal polynomials.

Orthogonality. The differential equation (2.1d) for the solution $A_n^{(\beta,\mu)}(x)$ may be written in the form

$$D[(x-1)^{\beta+1}e^{-\mu x}DA_n^{(\beta,\mu)}(x)]+n\mu(x-1)^{\beta}e^{-\mu x}A_n^{(\beta,\mu)}(x)=0\tag{3.5a}$$

since this substitution results in equation (2.2b). Also we have

$$D[(x-1)^{\beta+1}e^{-\mu x}DA_m^{(\beta,\mu)}(x)]+m\mu(x-1)^{\beta}e^{-\mu x}A_m^{(\beta,\mu)}(x)=0.\tag{3.5b}$$

Equations (3.5) lead to $(m-n)\mu(x-1)^{\beta}e^{-\mu x}A_n^{(\beta,\mu)}(x)A_m^{(\beta,\mu)}(x)=A_m^{(\beta,\mu)}(x)D[(x-1)^{\beta+1}e^{-\mu x}$ $DA_n^{(\beta,\mu)}(x)]-A_n^{(\beta,\mu)}(x)D[(x-1)^{\beta+1}e^{-\mu x}DA_m^{(\beta,\mu)}(x)]=D[(x-1)^{\beta+1}e^{-\mu x}\{A_m^{(\beta,\mu)}(x)DA_n^{(\beta,\mu)}(x)]$ $-A_n^{(\beta,\mu)}(x)DA_m^{(\beta,\mu)}(x)\}]$.

Therefore we find that

$$(m-n)\int_a^b(x-1)^{\beta}e^{-\mu x}A_n^{(\beta,\mu)}(x)A_m^{(\beta,\mu)}(x)dx=[(x-1)^{\beta+1}e^{-\mu x}\{A_m^{(\beta,\mu)}(x)DA_n^{(\beta,\mu)}(x)$$
$$-A_n^{(\beta,\mu)}(x)DA_m^{(\beta,\mu)}(x)\}]_a^b.\tag{3.6}$$

It is clear that for $\operatorname{Re}\mu>0$, the product of $e^{-\mu x}$ and any polynomial in x approaches zero as $x\to\infty$ and $(x-1)^{\beta+1}\to 0$ as $x\to 1$ if $\operatorname{Re}(\beta+1)>0$. Thus equation (3.6) yields the result

$$\int_1^{\infty}(x-1)^{\beta}e^{-\mu x}A_n^{(\beta,\mu)}(x)A_m^{(\beta,\mu)}(x)\,dx=0\tag{3.7}$$

for $m\neq n$ and $\operatorname{Re}\beta>-1$. So (3.7) shows that when $\operatorname{Re}\beta>-1$ the polynomials $A_n^{(\beta,\mu)}(x)$ form an orthogonal set in the invterval $(1,\infty)$ with the weight function $(x-1)^{\beta}e^{-\mu x}$, $\operatorname{Re}\mu>0$.

It may be noted that instead of using (3.5a) for proving orthogonality property we may use the equation $D[(x-1)^{1-\beta}e^{\mu x}De^{\mu x}(x-1)^{\beta}Y_n]+(n+1)(x-1)^{-\beta}e^{\mu x}Y_n=0$. Property

(3.7) can be shown directly from Rodrigues formula (3.1c) as in this approach the integral in (3.7) has to be evaluated. By using the integration by parts we find that

$$(3.8)\qquad I_{m,n}=\int_1^\infty e^{-\mu x}(x-1)A_n^{(\beta,\mu)}(x)A_m^{(\beta,\mu)}(x)dx$$

$$=\frac{1}{\Gamma(n+1)}\int_1^\infty [D^n e^{-\mu x}(x-1)^{\beta+n}]A_m^{(\beta,\mu)}(x)\,dx$$

$$=\frac{1}{\Gamma(n+1)}\int_1^\infty [D^{n-1}e^{-\mu x}(x-1)^{\beta+n}]DA_m^{(\beta,\mu)}(x)dx$$

$$=\frac{1}{\Gamma(n+1)}\int_1^\infty e^{-\mu x}(x-1)^{\beta+n}D^n A_m^{(\beta,\mu)}(x)dx.$$

But $D^n A_m^{(\beta,\mu)}(x)=0$ for $n>m$ and consequently $I_{m,n}=I_{n,m}=0$ for $m\neq n$ which shows the orthogonality property.

From (3.8) if $m=n$ we would have

$$\frac{(-1)^n}{\Gamma(n+1)}\int_1^\infty e^{-\mu x}(x-1)^{\beta+n}D^n A_n^{(\beta,\mu)}(x)\,dx=\frac{\mu n}{\Gamma(n+1)}\int_1^\infty e^{-\mu x}\,(x-1)^{\beta+n}\,dx$$

$$=\frac{\Gamma(\beta+n+1)}{(n+1)}\,\mu^{-\beta-1}\,e^{-\mu}.$$

Therefore $\int_1^\infty e^{-\mu x}(x-1)\,[A_n^{(\beta,\mu)}(x)]^2dx=\frac{\Gamma(\beta+n+1)}{\Gamma(n+1)}\,\mu^{-\beta-1}e^{-\mu}$.

As to other properties of these functions and polynomials it may be noticed that they follow a similar pattern as those possessed by Laguerre polynomials [12, 16].

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