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NOTE ON CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS WITH QUASICONFORMAL EXTENSION

SHIGEYOSHI OWA, VLADIMIR MIĆIĆ, MILUTIN OBRADOVIĆ

By use of some results due to H. Silverman [7] and J. Brown [1] some conditions, sufficient or necessary and sufficient, for a function f from certain subclasses of univalent functions, that provide the belonging of f to the class of functions with quasiconformal extension, are obtained. Further on some distortion inequalities are proved.

A fractional calculus was developed by Owa in [5, 6]. In the final chapter of the paper some estimates of the fractional integrals and fractional derivatives of functions having quasiconformal extension, studied in the previous chapters, are obtained.

1. Introduction. Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc U. Further, let S be the subclass of A consisting of functions univalent in U.

A function $f\in A$ is said to be starlike of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha$$

for some α, $0\leq\alpha<1$, and for all $z\in U$. We denote by $S^*(\alpha)$ the subclass of A consisting of functions which are starlike of order α in U.

A function $f\in A$ is said to be convex of order α if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha$$

for some α, $0\leq\alpha<1$, and for all $z\in U$. Also we denote by $K(\alpha)$ the subclass of A consisting of all functions convex of order α in U.

We note that $f(z)\in K(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$ and that $S^*(\alpha)\subseteq S^*(0)=S^*$, $K(\alpha)\subseteq K(0)\equiv K$, for $0\leq\alpha<1$.

Let T be the subclass of A consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \quad (a_n\geq 0). \tag{1.2}$$

We denote by $T^*(\alpha)$ and $C(\alpha)$ the classes $T^*(\alpha)=S^*(\alpha)\cap T$, $C(\alpha)=K(\alpha)\cap T$.

We begin by recalling the following lemmas due to H. Silverman [7].

Lemma 1. *If the function f, defined by* (1.1), *satisfies* $\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leq 1-\alpha$ *for* $0\leq\alpha<1$, *then* $f\in S^*(\alpha)$.

Lemma 2. *If the function f, defined by* (1.1), *satisfies* $\sum_{n=2}^{\infty} n(n-\alpha)|a_n|\leq 1-\alpha$ *for* $0\leq\alpha<1$, *then* $f\in K(\alpha)$.

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Lemma 3. *The function f, defined by* (1.2), *is in the class* $T^*(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n \leq 1-\alpha. \tag{1.3}$$

The result (1.3) *is sharp.*

Lemma 4. *The function f, defined by* (1.2), *is in the class* $C(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha)\, a_n \leq 1-\alpha. \tag{1.4}$$

The result (1.4) *is sharp.*

2. Quasiconformal extension. f is a K-quasiconformal mapping ($1 \leq K < +\infty$) of a domain $D \subset \mathbb{C}$ if it is a sense-preserving homeomorphism of D, absolutely continuous on almost all lines parallel to the coordinate axes and

$$|f_{\bar z}| \leq k\,|f_z| \quad \text{a. e. in } D, \tag{2.1}$$

with $k=(K-1)/K+1)$ (i. e. $K=(1+k)/(1-k)$).

We say that a function f is in the class S_k if it is in S and has a K-quasiconformal extension on $\mathbb{C}$ (cf. [2, 3, 4]).

We need the following lemma, due to J. E. Brown [1].

Lemma 5. *If the function f, defined by* (1.1), *satisfies*

$$\sum_{n=2}^{\infty}\left(n+\frac{k-1}{k+1}\right)|a_n| \leq \frac{2k}{1+k} \tag{2.2}$$

for $0 \leq k < 1$, *then* $f \in S^* \cap S_k$.

Since $f \in K$ if and only if $zf'(z) \in S^*$, replacing $|a_n|$ with $n\,|a_n|$ we obtain the following lemma.

Lemma 6. *If the function f, defined by* (1.1), *satisfies*

$$\sum_{n=2}^{\infty} n\left(n+\frac{k-1}{k+1}\right)|a_n| \leq \frac{2k}{1+k} \tag{2.3}$$

for $0 \leq k < 1$, *then* $f \in K \cap S_k$.

We denote by T_k the subclass of S_k with members of the form (1.2). As simple consequences of lemmas 1-6 the following propositions and assertions follow.

Proposition 1. *If the function f, defined by* (1.1), *satisfies the inequality* (2.2) *for* $0 < k < 1$, then $f \in S^*\left(\frac{1-k}{1+k}\right) \cap S_k$.

Proposition 2. *If the function f, defined by* (1.1), *satisfies the inequality* (2.3) *for* $0 < k < 1$, *then* $f \in K\left(\frac{1-k}{1+k}\right) \cap S_k$.

A1. *The function f, defined by* (1.2), *is in the class* $T^*\left(\frac{1-k}{1+k}\right) \cap T_k$ *if and only if*

$$\sum_{n=2}^{\infty}\left(n+\frac{k-1}{k+1}\right)a_n \leq \frac{2k}{1+k}, \tag{2.4}$$

where $0 < k < 1$. *Equality in* (2.4) *is attained for*

$$f(z)=z-\frac{2k}{(n+1)k+(n-1)}\,z^n, \quad (n \geq 2). \tag{2.5}$$

A2. *If the function f, defined by* (1.2), *is in the class* $T^*(\frac{1-k}{1+k})\cap T_k$ *with* $0<k<1$, *then*

$$a_n \leqq \frac{2k}{(n+1)k+(n-1)}, \quad (n\geqq 2). \tag{2.6}$$

Equality in (2.6) *is attained for f given by* (2.5).

A3. *The function f, defined by* (1.2) *is in the class* $C(\frac{1-k}{1+1})\cap T_k$ *if and only if*

$$\sum_{n=2}^{\infty} n(n+\frac{k-1}{k+1})\, a_n \leqq \frac{2k}{1+k}, \tag{2.7}$$

where $0<k<1$. *Equality in* (2.7) *is attained for the function*

$$f(z)=z-\frac{2k}{n((n+1)k+(n-1))}z^n, \quad (n\geqq 2). \tag{2.8}$$

A4. *If the function f, defined by* (1.4), *is in the class* $C(\frac{1-1}{1+k})\cap T_k$ *with* $0<k<1$, *then*

$$a_n \leqq \frac{2k}{n((n+1)k+(n-1))}, \quad (n\geqq 2). \tag{2.9}$$

Equality in (2.9) *is attained for the function f given by* (2.8).

3. Distortion inequalities.

Theorem 1. *If the function f, defined by* (1.2), *is in the class* $T^*(\frac{1-k}{1+k})\cap T_k$ *with* $0<k<1$, *then*

$$|z|-(\frac{2k}{3k+1})|z|^2 \leq |f(z)| \leq |z|+(\frac{2k}{3k+1})|z|^2 \tag{3.1}$$

and

$$1-(\frac{4k}{3k+1})|z| \leq |f'(z)| \leq 1+(\frac{4k}{3k+1})|z| \tag{3.2}$$

for $z\in U$. *Equalities in* (3.1) *and* (3.2) *are attained for the function*

$$f(z)=z-(\frac{2k}{3k+1})z^2. \tag{3.3}$$

Proof. By A1 we obtain

$$(\frac{3k+1}{k+1})\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty}(n+\frac{k-1}{k+1})\, a_n \leq \frac{2k}{1+k} \tag{3.4}$$

or

$$\sum_{n=2}^{\infty} a_n \leq \frac{2k}{3k+1}. \tag{3.5}$$

Therefore we have

$$|f(z)\geq |z|-|z|^2\sum_{n=2}^{\infty} a_n \geq |z|-(\frac{2k}{3k+1})|z|^2$$

and

$$|f(z)|\leq |z|+|z|^2\sum_{n=2}^{\infty} a_n \leq |z|+(\frac{2k}{3k+1})|z|^2.$$

In order to prove the second part of the theorem we need

$$\frac{3k+1}{2(k+1)}\sum_{n=2}^{\infty} na_n \leq \sum_{n=2}^{\infty}\left(n+\frac{k-1}{k+1}\right)a_n \leq \frac{2k}{1+k},$$

that implies

$$\sum_{n=2}^{\infty} na_n \leq \frac{4k}{3k+1}. \tag{3.6}$$

From (3.6) the desired estimations

$$|f'(z)| \geq 1-|z|\sum_{n=2}^{\infty} na_n \geq 1-\left(\frac{4k}{3k+1}\right)|z|$$

and

$$|f'(z) \leq 1+|z|\sum_{n=2}^{\infty} na_n \leq 1+\left(\frac{4k}{3k+1}\right)|z|$$

follow.

Corollary 1. *Let f, defined by* (1.2), *be in the class* $T^*(\frac{1-k}{1+k})\cap T_k$ *with* $0<k<1$. *Then the unit disk U is mapped by f onto a domain that contains the disk* $|w|<(k+1)/(3k+1)$.

Theorem 2. *If the function f, defined by* (1.2), *is in the class* $C(\frac{1-k}{1+k})\cap T_k$ *with* $0<k<1$, *then*

$$|z|-\left(\frac{k}{3k+1}\right)|z|^2 \leq |f(z)| \leq |z|+\left(\frac{k}{3k+1}\right)|z|^2 \tag{3.7}$$

and

$$1-\left(\frac{2k}{3k+1}\right)|z| \leq |f'(z)| \leq 1+\left(\frac{2k}{3k+1}\right)|z| \tag{3.8}$$

for $z\in U$. *Equalities in* (3.7) *and* (3.8) *are attained for*

$$f(z)=z-\left(\frac{k}{3k+1}\right)z^2. \tag{3.9}$$

This theorem can be proved by the same manner as Theorem 1.

Corollary 2. *Let f, defined by* (1.2), *be in the class* $C(\frac{1-k}{1+k})\cap T_k$ *with* $0<k<1$. *Then the unit disk U is mapped by f onto a domain that contains the disk* $|w|<(2k+1)/(3k+1)$.

4. Fractional calculus. The following definitions are due to Owa [5, 6].

Definition 1. The fractional integral of order δ of the function f is defined by

$$D_z^{-\delta} f(z)=\frac{1}{\Gamma(\delta)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}}\,d\zeta,$$

where $\delta>0$, f is analytic in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 2. The fractional derivative of order δ of the function f is defined by

$$D_z^{\delta} f(z)=\frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}}\,d\zeta,$$

where $0 \geqq \delta < 1$, f is analytic in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\delta}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 3. Let $n \in N_0 = \{0, 1, 2, \ldots\}$. Under the hypotheses of Definition 2 the fractional derivative of order $(n+\delta)$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^\delta f(z).$$

Theorem 3. *If the function f, defined by* (1.2), *is in the class* $T^*\left(\frac{1-k}{1+k}\right) \cap T_k$ *with* $0<k<1$, *then*

$$|D_z^{-\delta} f(z)| \geq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left(1 - \frac{4k}{(2+\delta)(3k+1)} |z|\right) \tag{4.1}$$

and

$$|D_z^{-\delta} f(z)| \leq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left(1 + \frac{4}{(2+\delta)(3k+1)} |z|\right) \tag{4.2}$$

for $\delta>0$ *and* $z \in U$. *Equalities in* (4.1) *and* (4.2) *are attained for f given by* (3.3).

Proof. Note that

$$\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} a_n z^n.$$

Setting $G(n) = \Gamma(n+1)\Gamma(2+\delta)/\Gamma(n+1+\delta)$ $(n \geq 2)$, we have

$$0 < G(n) \leq G(2) = 2/(2+\delta). \tag{4.3}$$

It follows from (3.5) and (4.3)

$$|\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)| \geq |z| - G(2)|z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{4k}{(2+\delta)(3k+1)} |z|^2,$$

that gives (4.1) and

$$|\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)| \leq |z| + G(2)|z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{4k}{(2+\delta)(3k+1)} |z|^2,$$

that shows (4.2). Finally, equalities in (4.1) and (4.2) are attained for f defined by

$$D_z^{-\delta} f(z) = \frac{z^{1+\delta}}{\Gamma(2+\delta)} \left(1 - \frac{4k}{(2+\delta)(3k+1)} z\right),$$

that is, defined by (3.3).

Using (3.9), we can prove

Theorem 4. *If the function f, defined by* (1.2), *is in the class* $C\left(\frac{1-k}{1+k}\right) \cap T_k$ *with* $0<k<1$, *then*

$$|D_z^{-\delta} f(z)| \geq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left(1 - \frac{2k}{(2+\delta)(3k+1)} |z|\right) \tag{4.4}$$

and

$$|D_z^{-\delta} f(z)| \leq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left(1 + \frac{2k}{(2+\delta)(3k+1)} |z|\right) \tag{4.5}$$

for $\delta>0$ *and* $z\in U$. *Equalities in* (4.4) *and* (4.5) *are attained for* f *given by* (3.9).

Theorem 5. *If the function* f, *defined by* (1.2), *is in the class* $T^*(\frac{1-k}{1+k})\cap T_k$ *with* $0<k<1$, *then*

$$|D_z^\delta f(z)|\geq\frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left(1-\frac{4k}{(2-\delta)(3k+1)}|z|\right) \tag{4.6}$$

and

$$|D_z^\delta f(z)|\leq\frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left(1+\frac{4k}{(2-\delta)(3k+1)}|z|\right) \tag{4.7}$$

for $0\leq\delta<1$ *and* $z\in U$. *Equalities in* (4.6) *and* (4.7) *are attained for* f *given by* (3.3).

Proof. It follows from Definition 2 that

$$\Gamma(2-\delta)\,z^\delta D_z^\delta f(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}a_n z^n=z-\sum_{n=2}^{\infty}H(n)na_n z^n,$$

where $H(n)=\Gamma(n)\Gamma(2-\delta)/\Gamma(n+1-\delta)$. Since $0\leq H(n)\leq H(2)=1/(2-\delta)$, by using (3.6) we have

$$|\Gamma(2-\delta)\,z^\delta D_z^\delta f(z)|\geq|z|-H(2)|z|^2\sum_{n=2}^{\infty}na_n\geq|z|-\frac{4k}{(2-\delta)(3k+1)}|z|^2,$$

that proves (4.6), and

$$|\Gamma(2-\delta)\,z^\delta D_z^\delta f(z)|\leq|z|+H(2)|z|^2\sum_{n=2}^{\infty}na_n\leq|z|+\frac{4k}{(2-\delta)(3k+1)}|z|^2,$$

that proves (4.7). Further, it is clear that the equalities in (4.6) and (4.7) are attained for f defined by

$$D_z^\delta f(z)=\frac{z^{1-\delta}}{\Gamma(2-\delta)}\left(1-\frac{4k}{(2-\delta)(3k+1)}z\right),$$

that is, defined by (3.3).

Using (3.9), we can prove

Theorem 6. *If the function* f, *defined by* (1.2), *is in the class* $C(\frac{1-k}{1+k})\cap T_k$ *with* $0<k<1$, *then*

$$|D_z^\delta f(z)|\geq\frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left(1-\frac{2k}{(2-\delta)(3k+1)}|z|\right) \tag{4.8}$$

and

$$|D_z^\delta f(z)|\leq\frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left(1+\frac{2k}{(2-\delta)(3k+1)}|z|\right) \tag{4.9}$$

for $0\leq\delta<1$ *and* $z\in U$. *Equalities in* (4.8) *and* (4.9) *are attained for* f *given by* (3.9).

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S. Owa
Department of Mathematics
Kinki University
Higashi-Osaka,
Osaka 577, Japan

V. Mićić
Department of Mathematics
Faculty of
Civil Engineering
Bulevar revolucije 73
11000 Beograd, Yugoslavia

M. Obradović
Department of Mathematics
Faculty of Technology
and Metallurgy
Karnegijeva 4
11000 Beograd, Yugoslavia

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