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VARIETIES OF METABELIAN JORDAN ALGEBRAS

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In this paper we describe the lattice of all subvarieties of the metabelian variety of Jordan algebras over a field of characteristic 0. As a consequence we establish the asymptotic behaviour of the codimension sequence and estimate the topological rank of the proper subvarieties of the metabelian variety.

Introduction. In this paper we study varieties of metabelian Jordan algebras over a field of characteristic 0. S. Pchelintsev [1] has shown that the variety $\mathfrak{M}$ of all metabelian (i. e. solvable of class 2) algebras defined by the identity $(x_1x_2)(x_3x_4)=0$ is quite complicated. Our main purpose is to describe the lattice $\Lambda(\mathfrak{M})$ of all subvarieties of $\mathfrak{M}$. The description is given in the language of graph theory. It is proved that this lattice is distributive and the proper subvarieties of $\mathfrak{M}$ are simpler than $\mathfrak{M}$ itself. Let $\mathfrak{U}$ be a proper subvariety of $\mathfrak{M}$ and let $f=0$ be a polynomial identity for $\mathfrak{U}$ of degree d such that f does not vanish on $\mathfrak{M}$. We compute explicitly the sequence and the series of codimensions of $\mathfrak{M}$ and estimate asymptotically the codimensions of $\mathfrak{U}$. In particular,

$$c(\mathfrak{M},\ t)=(2t^3+t+1)/2t^2+(2t^3-4t^2-t+1)/2t^2(2t-1)(1-4t^2)^{1/2}$$

and $c_n(\mathfrak{M})\approx 2^n(n/2\pi)^{1/2}$ for n large enough. On the other hand side, $c_n(\mathfrak{U})=\mathcal{O}(n^{d+1})$ and $c(\mathfrak{U},\ t)=f(t)/(1-t)^{d+2}$, where $f(t)\in \mathrm{Z}[t]$. The topological rank of $\mathfrak{M}$ is infinite [1] and it turns out that for $\mathfrak{U}$ this rank is finite and is bounded by $d+2$. Another consequence of the description of the subvarieties of $\mathfrak{M}$ is that a subvariety of $\mathfrak{M}$ is nilpotent if and only if it satisfies a Jordan standard identity. The proofs of the main results of the paper are based on the representation theory of the symmetric and general linear groups.

1. Preliminaries. We consider only Jordan algebras without 1 over a fixed field K of characteristic 0. Let $J=J(X)=J(x_1,\ x_2,\dots)$ be the free Jordan algebra over K and let J_m be the subalgebra of J generated by $x_1,\dots,x_m$. Sometimes we denote the free generators of J by other letters, e. g. y, y_i, u_i etc. We denote by P_n the vector subspace of J_n of all multilinear elements of degree n.

For a variety $\mathfrak{U}$ of Jordan algebras we denote by $T(\mathfrak{U})$ the T-ideal of J of the polynomial identities for $\mathfrak{U}$; $F(\mathfrak{U})=J/T(\mathfrak{U})$ is the relatively free algebra of $\mathfrak{U}$, $F_m(\mathfrak{U})$ is the subalgebra of rank m in $F(\mathfrak{U})$; $P_n(\mathfrak{U})$ and $F_m^{(n)}(\mathfrak{U})$ are the spaces of the multilinear elements of degree n and of the homogeneous elements of degree n in $F_m(\mathfrak{U})$, respectively. Especially, we denote by $\mathfrak{M}$ the metabelian variety (which coincides with the class of all solvable of class ≤ 2 algebras) defined by the identity

$$(1)\qquad (x_1x_2)(x_3x_4)=0.$$

We use left-normed products only, i. e. $x_1x_2x_3=(x_1x_2)x_3$.

Let S_n be the symmetric group with its left action on the set of symbols $\{1,\dots,n\}$, and let GL_m be the general linear group canonically acting on the vector space span-

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ned by $x_1, \ldots, x_m$. The representations of S_n and GL_m are described by partitions and Young diagrams [2]. For a partition $\lambda=(\lambda_1, \ldots, \lambda_r)$ of n, $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$, $\lambda_1+\cdots+\lambda_r=n$, we denote by $[\lambda]$, $M(\lambda)$ and $N_m(\lambda)$, respectively the Young diagram, and the irreducible S_n and GL_m modules related to λ.

The symmetric group S_n acts on P_n by the rule

$$\sigma(\Sigma a_i(x_{i_1}\ldots)(\ldots x_{i_n}))=\Sigma a_i(x_{\sigma(i_1)}\ldots)(\ldots x_{\sigma(i_n)}), \quad \sigma \in S_n,$$

(with an arbitrary distribution of the brackets in $(x_{i_1}\ldots)(\ldots x_{i_n})$) and P_n is a left S_n-module. Similarly, GL_m acts on J_m by

$$g(\Sigma a_i(x_{i_1}\ldots)(\ldots x_{i_n}))=\Sigma a_i(g(x_{i_1})\ldots)(\ldots g(x_{i_n})), \quad g \in GL_m.$$

Let $\mathfrak{U}$ be a variety of algebras. The actions of S_n and GL_m on P_n and J_m are inherited respectively by $P_n(\mathfrak{U})$ and $F_m(\mathfrak{U})$. It is known [3] that $P_n(\mathfrak{U})$ and $F_m^{(n)}(\mathfrak{U})$ have the same module structure: If $P_n(\mathfrak{U})=\Sigma k(\lambda)M(\lambda)$, then $F_m^{(n)}(\mathfrak{U})=\Sigma k(\lambda)N_m(\lambda)$. In the module $N_m(\lambda)=N_m(\lambda_1, \ldots, \lambda_r)\subset F_m(\mathfrak{U})$ there exists a multihomogeneous element $f_\lambda(x_1, \ldots, x_r)$ of degree λ_i in x_i which is uniquely determined up to a multiplicative constant. We call f_λ the standard generator of $N_m(\lambda)$.

An important numerical invariant of $\mathfrak{U}$ is its codimension sequence $c_n(\mathfrak{U})=\dim P_n(\mathfrak{U})$, $n=1, 2, \ldots$. The generating function of this sequence $c(\mathfrak{U}, t)=\Sigma c_n(\mathfrak{U})t^n$ is called the codimension series of $\mathfrak{U}$.

In the sequel we shall use a particular case of the Littlewood — Richardson rule for the tensor product of GL_m-modules [4]:

$$(2) \qquad N_m(2, 1^p)\otimes_K N_m(1^p)\simeq \sum_{k=0}^{p-1} N_m(3, 2^k, 1^{2p-2k-1})+\sum_{k=1}^{p+1} N_m(2^k, 1^{2p-2k+2}),$$

$$(2') \qquad N_m(2, 1^p)\otimes_K N_m(1^{p+1})\simeq \sum_{k=0}^{p} N_m(3, 2^k, 1^{2p-2k})+\sum_{k=1}^{p+1} N_m(2^k, 1^{2p-2k+3}).$$

Besides, we need the following rule for describing the consequences of a multilinear identity.

Proposition 1.1. [5, Lemma 2.5]. *Let M be an S_n-submodule of P_n and let Q be the set of the multilinear consequences of degree $n+1$ of the polynomial identities of M. Then Q is an S_{n+1}-submodule of P_{n+1} which is a homomorphic image of the S_{n+1}-module*

$$((M\downarrow S_{n-1})\otimes_K M(2))\uparrow S_{n+1}+(M\otimes_K M(1))\uparrow S_{n+1}.$$

In the first summand, S_{n-1} acts on the set $\{1, \ldots, n-1\}$ fixing n and S_2 acts on $\{n, n+1\}$, the tensor product is an $S_{n-1}\times S_2$-module, where the direct product $S_{n-1}\times S_2$ is canonically embedded in S_{n+1}, similarly for the second summand. For a subgroup H of the group G and R and S being respectively H- and G-modules, $R\uparrow G$ and $S\downarrow H$ denote respectively the G-module induced by R and the module S considered as an H-module.

Corollary 1.2. [5, Lemma 2.6]. *Let λ be a partition of n and let $M(\lambda)\subset P_n$. Then the S_{n+1}-module $M'(\lambda)$ of all multilinear consequences of $M(\lambda)$ in P_{n+1} equals $\Sigma a_\mu M(\mu)$, where the non-negative integers a_μ are bounded by the number of diagrams $[\mu]$ obtained by the following devices:*

(i) *We remove a box from $[\lambda]$ and obtain a diagram $[\nu]$. Then we add two new boxes to $[\nu]$ and produce a diagram $[\mu]$ such that these two new boxes do not belong to the same column of $[\mu]$.*

(ii) *We add a new box to $[\lambda]$ and obtain $[\mu]$.*

2. The metabelian variety. We work in the relatively free algebra $F(\mathfrak{M})$ of the metabelian variety $\mathfrak{M}$ defined by the identity (1). The identities $xy=yx$, $yx(xx)=y(xx)x$ which hold for all Jordan algebras give that $F(\mathfrak{M})$ is a commutative algebra satisfying

(3) $$xxyx=0.$$

Linearizing (3), we obtain

(4) $$x_3x_2\,yx_1=-x_2x_1\,yx_3-x_3x_1\,yx_2,$$

(4′) $$x_2x_1\,yx_1=-x_1x_1\,yx_2/2.$$

In (4) we replace x_3 by u_1u_2 and obtain

(5) $$u_1u_2x_2\,yx_1=-u_1u_2x_1\,yx_2.$$

As a consequence of (5) we establish the identity

(6) $$u_1u_2x_{\sigma(1)}\,y_1x_{\sigma(2)}\,y_2\dots x_{\sigma(n-1)}y_{n-1}\,x_{\sigma(n)}=(\operatorname{sign}\sigma)\,u_1u_2x_1\,y_1x_2\,y_2\dots x_{n-1}\,y_{n-1}\,x_n,\quad \sigma\in S_n.$$

In virtue of (4), (4′) and (6), $F(\mathfrak{M})$ is spanned by

(7) $$x_k\,x_{i_1}\,x_{j_1}\,x_{i_2}\,x_{j_2}\dots x_{i_p}\,x^{\varepsilon}_{j_p},$$

$$k\geq i_1<i_2<\dots<i_p,\ j_1<j_2<\dots<j_p,\ p\geq 0,\ \varepsilon=0,\ 1.$$

We recall the construction of the algebra introduced by S. Pchelintsev [1] Let V be a vector space with a basis $v_1, v_2, \dots,$ and let E be the unitary exterior (or Grassmann) algebra over V, $E=\Sigma E^{(s)}$, $s\geq 0$, where $E^{(s)}$ is the homogeneous component of degree s and $E^{(-1)}=\varnothing$. Then E has a basis $\{v_{i_1}v_{i_2}\dots v_{i_s}\mid i_1<i_2<\dots<i_s\}$. We denote by W the vector space with a basis $\{(b,\ e_1,\ e_2)\}$, where b is a formal symbol and $e_1\in E^{(s)}$, $e_2\in E^{(s)}\cup E^{(s-1)}$, $s\geq 0$, are monomials from the basis of E. We define a multiplication in the vector space $V+W$ by $v_iv_j=0$, $(b,\ e_1,\ e_2)(b,\ e_1',\ e_2')=0$,

$$(b,\ e_1,\ e_2)\,v_i=v_i\,(b,\ e_1,\ e_2)=\begin{cases}(b,\ e_1v_i,\ e_2), & \text{when } e_1,\ e_2\in E^{(s)},\\ (b,\ e_1,\ e_2v_i), & \text{when } e_1\in E^{(s)},\ e_2\in E^{(s-1)}.\end{cases}$$

By [1], $V+W$ is a metabelian Jordan algebra.

Proposition 2.1. *The elements* (7) *form a basis of* $F(\mathfrak{M})$.

Proof. Let

(8) $$f(x_1,\dots,x_m)=\sum_{k>i_1}\alpha_{kij}\,x_k\,x_{i_1}\,x_{j_1}\dots x_{i_p}\,x_{j_p}+\Sigma\,\beta_{ksj}\,x_k\,x_k\,x_{j_1}\,x_{s_2}\,x_{j_2}\dots x_{s_p}\,x_{j_p}=0$$

in $F(\mathfrak{M})$ be a sum of monomials of the form (7) and let a coefficient α_{kij} or β_{ksj} be nonzero in K. Since $F(\mathfrak{M})$ is a graded algebra, without loss of generality we may assume that $f(x_1,\dots,x_m)$ is homogeneous in every variable $x_1,\dots,x_m$. Let k be the largest integer with $\alpha_{kij}\neq 0$ or $\beta_{ksj}\neq 0$. We substitute $x_k=x_k+u_1u_2$ and consider the linear component in u_1:

(8′) $$\sum_{k>i_1}\alpha_{kij}\,u_1\,u_2\,x_{i_1}\,x_{j_1}\dots x_{j_p}\,x^{\varepsilon}_{j_p}+2\Sigma\beta_{ksj}u_1u_2x_kx_{j_1}\,x_{s_2}\dots x_{s_p}\,x^{\varepsilon}_{j_p}=0,$$

in the latter summands $k<s_2$. In (8′) we substitute the following elements from the algebra of Pchelintsev: $u_1=b$, $u_2=v_{m+1}$, $x_l=v_l$ and obtain

$$\Sigma\,(b,\ v_{m+1}\,v_{j_1}\dots v^{\varepsilon}_{j_p},\ \sum_{k>i_1}\alpha_{kij}\,v_{i_1}\dots v_{i_p}+2\Sigma\beta_{ksj}\,v_k\,v_{s_2}\dots v_{s_p})=0.$$

Therefore

(8″) $$\sum_{k>i_1} \alpha_{kij} v_{i_1} \dots v_{i_p} + 2\Sigma\beta_{ksj} v_k v_{s_2} \dots v_{s_p} = 0$$

holds for any fixed sets $\{i_1, \dots, i_p\} = \{k, s_2, \dots, s_p\}$, $i_1 < \dots < i_p$, $k < s_2 < \dots < s_p$. If all β_{ksj} are zeros, (8″) yields $\alpha_{kij} = 0$; analogously $\alpha_{kij} = 0$ gives $\beta_{ksj} = 0$. In the other case (8″) implies $i_1 = k$, $i_2 = s_2, \dots, i_p = s_p$ and this is a contradiction with the assumption $k > i_1$. Therefore, all coefficients in (8) equal zero and the elements (7) are linearly independent. Since $F(\mathfrak{M})$ is spanned by (7), these elements form its basis.

The following theorem gives the description of the S_n-module structure of $P_n(\mathfrak{M})$ and a formula for the codimensions.

Theorem 2.2.

(i) $$c_n(\mathfrak{M}) = \begin{cases} \binom{2p+1}{p} p, & n = 2p+1,\ p = 1, 2, \dots, \\ \binom{2p+2}{p} (p+1), & n = 2p+2,\ p = 0, 1, 2, \dots \end{cases}$$

and for n sufficiently large, $c_n(\mathfrak{M}) \approx 2^n (n/2\pi)^{1/2}$.

(ii) $c(\mathfrak{M}, t) = (2t^3 + t + 1)/2t^2 + (2t^3 - 4t^2 - t + 1)/2t^2(2t-1)(1-4t^2)^{1/2}$.

(iii) *The following GL_m-module isomorphism holds*

$$F_m(\mathfrak{M}) \simeq N_m(1) + N_m(2) + \sum_{p \ge 1} (N_m(2, 1^{p-1}) + N_m(2, 1^p)) \otimes_K N_m(1^p).$$

(iv) *For* $n \ge 3$, $P_n(\mathfrak{M}) = \Sigma M(3, 2^p, 1^q) + \Sigma M(2^r, 1^s)$, *where the summation runs over all partitions* $(3, 2^p, 1^q)$ *and* $(2^r, 1^s)$ *such that* $3 + 2p + q = 2r + s = n$, $r > 0$.

Proof. (i) The formula for $c_n(\mathfrak{M})$ follows immediately by counting the multilinear elements of the basis (7) of $F(\mathfrak{M})$. In order to obtain the asymptotic behaviour of $c_n(\mathfrak{M})$ first let $n = 2p$. Then the Stirling formula gives

$$c_{2p}(\mathfrak{M}) = \binom{2p}{p-1} p = (2p)!\, p/(p-1)!\,(p+1)! = (p^2/(p+1))(2p)!/(p!)^2$$
$$\approx (p^2/(p+1))(4\pi p)^{1/2}(2p/e)^{2p}/2\pi p\,(p/e)^{2p} = (p/(p+1))\,2^{2p}(p/\pi)^{1/2} \approx 2^n (n/2\pi)^{1/2}.$$

For $n = 2p+1$ the proof is similar.

(ii) Using the binomial formula the following identities can be verified:

$$\sum_{p \ge 0} (2p+1) \binom{2p}{p} u^p = 1/(1-4u)^{3/2},$$
$$\sum_{p \ge 0} \binom{2p}{p} u^p = 1/(1-4u)^{1/2},$$
$$\sum_{p \ge 0} \binom{2p}{p} u^p/(p+1) = (1-(1-4u)^{1/2})/2u.$$

Hence we obtain for the codimension series of $\mathfrak{M}$

$$c(\mathfrak{M}, t) = t + \sum_{p \ge 0} \binom{2p+2}{p} (p+1) t^{2p+2} + \sum_{p \ge 0} \binom{2p+1}{p} p t^{2p+1}$$
$$= t + \sum_{p \ge 0} ((2p+2)!/p!\,(p+2)!)(p+1)\, t^{2p+2} + \sum_{p \ge 0} ((2p+1)!/p!\,(p+1)!)\, p t^{2p+1}$$

$$=t+\sum_{p\geq 0}\binom{2p}{p}(p^2/(p+1)+(2p+1)pt/(p+1))t^{2p}=t+\sum_{p\geq 0}\binom{2p}{p}((2p+1)(t+1/2)$$

$$-(2t+3/2)+(t+1)/(p+1))t^{2p}=t+(t+1/2)/(1-4t^2)^{3/2}-(2t+3/2)/(1-4t^2)^{1/2}$$

$$+(1+t)(1-(1-4t^2)^{1/2})/2t^2=(2t^3+t+1)/2t^2+(2t^3-4t^2-t+1)/2t^2(2t-1)(1-4t^2)^{1/2}.$$

(iii) Clearly, $F_m^{(1)}(\mathfrak{M})=N_m(1)$, $F_m^{(2)}(\mathfrak{M})=N_m(2)$. Since the Hilbert series $H(N_m, t_1, \ldots, t_m)$ of a GL_m-module N_m completely determines N_m and

$$H(N'_m\otimes_K N''_m,\ t_1,\ldots,t_m)=H(N'_m,\ t_1,\ldots,t_m)H(N''_m,\ t_1,\ldots,t_m),$$

it suffices to prove that

(9) $$H(F_m^{(2p+1)}(\mathfrak{M}),\ t_1,\ldots,t_m)=H(N_m(2,\ 1^{p-1}),\ t_1,\ldots,t_m)H(N_m(1^p),\ t_1,\ldots,t_m),$$

(9′) $$H(F_m^{(2p+2)}(\mathfrak{M}),\ t_1,\ldots,t_m)=H(N_m(2,1^p),\ t_1,\ldots,t_m)H(N_m(1^p),\ t_1,\ldots,t_m),\ p=1,2,\ldots$$

It is known that $H(N_m(\lambda),\ t_1,\ldots,t_m)=\Sigma\, a_\alpha t_1^{\alpha_1}\ldots t_m^{\alpha_m}$, where a_α equals the number of semistandard λ-tableaux of content $\alpha=(\alpha_1,\ldots,\alpha_m)$. For the vector space V_m spanned by $x_1,\ldots,x_m$, let us consider the vector spaces $W_1\subset V_m^{\otimes p+1}$ and $W_2\subset V_m^{\otimes p}$ spanned by $\{x_k\otimes x_{i_1}\otimes\cdots\otimes x_{i_p}\mid k\geq i_1<\cdots<i_p\}$ and $\{x_{j_1}\otimes\cdots\otimes x_{j_p}\mid j_1<\cdots<j_p\}$, respectively. It is easy to see that

$$H(W_1,\ t_1,\ldots,t_m)=H(N_m(2,\ 1^{p-1}),\ t_1,\ldots,t_m),$$
$$H(W_2,\ t_1,\ldots,t_m)=H(N_m(1^p),\ t_1,\ldots,t_m).$$

Since the monomials (7) which are in $F_m^{(2p+1)}(\mathfrak{M})$ form a basis of $F_m^{(2p+1)}(\mathfrak{M})$, the mapping

$$x_k x_{i_1} x_{j_1}\ldots x_{i_p} x_{j_p}\to(x_k\otimes x_{i_1}\otimes\cdots\otimes x_{i_p})\otimes(x_{j_1}\otimes\cdots\otimes x_{j_p})$$

can be extended to an isomorphism of graded vector spaces $F_m^{(2p+1)}(\mathfrak{M})\simeq W_1\otimes_K W_2$. Therefore

$$H(F_m^{(2p+1)}(\mathfrak{M}),\ t_1,\ldots,t_m)=H(W_1\otimes_K W_2,\ t_1,\ldots,t_m)$$

$$=H(W_1,\ t_1,\ldots,t_m)H(W_2,\ t_1,\ldots,t_m)=H(N_m(2,\ 1^{p-1}),\ t_1,\ldots,t_m)H(N_m(1^p),\ t_1,\ldots,t_m)$$

and this gives (9). The proof of (9′) is similar.

(iv) The proof follows immediately from (iii) in virtue of (2) and (2′).

3. Subvarieties of the metabelian variety. It is known that the lattice $\Lambda(\mathfrak{M})$ of the subvarieties of a variety $\mathfrak{U}$ is distributive if and only if $P_n(\mathfrak{U})$, $n=1,2,\ldots$, is a sum of pairwise non-isomorphic irreducible S_n-submodules. Therefore Theorem 2.2 (iv) gives immediately

Theorem 3.1. *The lattice* $\Lambda(\mathfrak{M})$ *is distributive.*

In order to describe all the subvarieties of $\mathfrak{M}$ we use the following method which has already been applied for several varieties (see e. g. [5]). We associate with $\mathfrak{M}$ an oriented graph $\mathrm{gr}(\mathfrak{M})$. The vertices of $\mathrm{gr}(\mathfrak{M})$ are the irreducible submodules $M(\lambda)$ of $\bigcup_{n\geq 1}P_n(\mathfrak{M})$ and the set of edges consists of all $(M(\lambda),\ M(\mu))$ such that $M(\lambda)\subset P_n(\mathfrak{M})$, $M(\mu)\subset P_{n+1}(\mathfrak{M})$ for some n and the elements of $M(\mu)$ are consequences of these of $M(\lambda)$. For any $\mathfrak{U}\subset\mathfrak{M}$ with a T-ideal $U\subset F(\mathfrak{M})$ we associate a subgraph $\psi(\mathfrak{U})$ of $\mathrm{gr}(\mathfrak{M})$ in the following way. The vertices of $\psi(\mathfrak{U})$ are all $M(\lambda)\subset P_n(\mathfrak{M})\cap U$, $n=1,2,\ldots,$

(i. e. $M(\lambda)$ are identities for $\mathfrak{U}$) and the set of edges for $\psi(\mathfrak{U})$ consists of all edges from $\mathrm{gr}(\mathfrak{M})$ which connect vertices from $\psi(\mathfrak{U})$. Obviously,

$$\psi(\mathfrak{U}_1\cap\mathfrak{U}_2)=\psi(\mathfrak{U}_1)\cup\psi(\mathfrak{U}_2),\quad \psi(\mathfrak{U}_1\cup\mathfrak{U}_2)=\psi(\mathfrak{U}_1)\cap\psi(\mathfrak{U}_2)$$

for $\mathfrak{U}_1, \mathfrak{U}_2\subset\mathfrak{M}$. Hence ψ is a dual isomorphism of $\Lambda(\mathfrak{M})$ onto the lattice of all subgraphs G of $\mathrm{gr}(\mathfrak{M})$ satisfying the property: If v is a vertex of G, then all edges beginning from v belong to G as well. Therefore we shall describe the subvarieties of $\mathfrak{M}$ if we obtain all consequences of degree $n+1$ of $M(\lambda)\subset P_n(\mathfrak{M})$.

Theorem 3.2. (i) *Let* $M=M(3, 2^p, 1^q)\subset P_n(\mathfrak{M})$. *Then all the consequences of* M *in* $P_{n+1}(\mathfrak{M})$ *are*

$$M(3,\ 2^{p+1},\ 1^{q-1})+M(3,\ 2^p,\ 1^{q+1})+M(2^{p+2},\ 1^q). \tag{10}$$

(ii) *For* $M=M(2^r, 1^s)\subset P_n(\mathfrak{M})$ *the consequences of* M *are*

$$M(3,\ 2^r,\ 1^{s-2})+M(3,\ 2^{r-1},\ 1^s)+M(3,\ 2^{r-2},\ 1^{s+2})+M(2^{r+1},\ 1^{s-1})+M(2^r,\ 1^{s+1}). \tag{11}$$

(By convention, $M(3^a, 2^b, 1^c)=0$ if some of the integers a, b, c is negative.)

We shall prove the theorem in several steps.

Lemma 3.3. (i) *The polynomials*

$$f_\lambda(x_1,\ldots,x_{p+q+1})=\Sigma(\mathrm{sign}\,\sigma)(\mathrm{sign}\,\tau)\,x_{\sigma(1)}x_1x_{\tau(1)}x_{\sigma(2)}\ldots x_{\tau(p+1)}x_{\sigma(p+2)}\ldots x_{\sigma(p+q+1)},\quad \sigma\in S_{p+q+1},\ \tau\in S_{p+1}, \tag{12}$$

$$f'_\mu(x_1,\ldots,x_{r+s})=\Sigma(\mathrm{sign}\,\sigma)(\mathrm{sign}\,\tau)\,x_{\sigma(1)}x_{\tau(1)}\ldots x_{\sigma(r)}x_{\tau(r)}x_{\sigma(r+1)}\ldots x_{\sigma(r+s)},\quad \sigma\in S_{r+s},\quad \tau\in S_r, \tag{12'}$$

are nonzero standard generators of $N_m(\lambda)$, $N_m(\mu)\subset F_m(\mathfrak{M})$, *for* $\lambda=(3, 2^p, 1^q)$, $\mu=(2\ 1^s)$, *respectively.*

(ii) *Let*

$$g_{ab}=g_{ab}(u_1, u_2; x_1,\ldots,x_a; y_1,\ldots,y_b)=\Sigma(\mathrm{sign}\,\rho)\,u_1u_2x_1x_1x_2x_2\ldots x_ax_ay_{\rho(1)}\ldots y_{\rho(b)}. \tag{13}$$

Then $g_{p+1,q}$ *is a consequence of* (12) *in* $F(\mathfrak{M})$. *Similarly,* $g_{r,s-1}$ *and* $g_{r-1,s+1}$ *follow from* (12′) *if* $r>0$.

Proof. (i) It is known (see e. g. [3]) that f_λ, f'_μ are standard generators of GL_m-modules $N_m(\lambda)$ and $N_m(\mu)$, respectively. Therefore, the only problem is to show that (12) and (12′) are nonzero in $F(\mathfrak{M})$. Since $g_{ab}\neq0$ in $F(\mathfrak{M})$ the proof will be completed if we establish (ii).

(ii) In (12) we replace x_1 by $x_1+u_1u_2$ and take the linear component in u_1 (which is a consequence of (12)). In this way we obtain

$$\Sigma(\mathrm{sign}\,\sigma)(\mathrm{sign}\,\tau)u_1u_2x_{\sigma(1)}x_{\tau(1)}\ldots x_{\sigma(p+1)}x_{\tau(p+1)}x_{\sigma(p+2)}\ldots x_{\sigma(p+q+1)}$$

$$+\Sigma(\mathrm{sign}\,\sigma)(\mathrm{sign}\,\tau)\,u_1u_2x_1x_{\tau(1)}x_{\sigma(2)}x_{\tau(2)}\ldots x_{\sigma(p+1)}x_{\tau(p+1)}x_{\sigma(p+2)}\ldots x_{\sigma(p+q+1)},$$

in the latter sum $\sigma(1)=1$. Applying (6), we derive as a consequence of (12)

$$c\,\Sigma(\mathrm{sign}\,\sigma)u_1u_2x_1x_1\ldots x_{p+1}x_{p+1}x_{\sigma(p+2)}\ldots x_{\sigma(p+q+1)},$$

where $0\neq c\in K$ and σ fixes $1,\ldots,p+1$. Therefore $g_{p+1,q}$ follows from (12). Similarly, if $s>0$, replacing in (12′) x_{r+s} by u_1u_2 we obtain

$$c'\,\Sigma(\mathrm{sign}\,\sigma)\,u_1u_2x_1x_1x_2x_2\ldots x_rx_rx_{\sigma(r+1)}\ldots x_{\sigma(r+s+1)},$$

where $0\neq c'\in K$ and $\sigma\in S_{r+s-1}$ fixes $1, 2,\ldots,r$, i. e. $g_{r,s-1}$ follows from (12′). Analogously, the linear component in u_1 of $f'_\mu(x_1,\ldots,x_{r-1}, x_r+u_1u_2, x_{r+1},\ldots,x_{r+s})$ equals

$h(u_1, u_2, x_1, \ldots, x_{r+s}) = kr \Sigma (\operatorname{sign} \sigma) u_1 u_2 x_1 x_1 \ldots x_{r-1} x_{r-1} x_r x_{\sigma(r+1)} \ldots x_{\sigma(r+s)} + k \Sigma (\operatorname{sign} \sigma)$ $. u_1 u_2 x_1 x_1 \ldots x_{r-1} x_{r-1} x_{\sigma(r)} x_{\sigma(r+1)} \ldots x_{\sigma(r+s)}$, $0 \neq k \in K$ and σ fixes $1, 2, \ldots, r$ and $1, 2, \ldots, r-1$ respectively. The alternating summation on $\sigma \in S_{r+s}$, σ fixing $1, 2, \ldots, r-1$, gives the polynomial

$$c'' \Sigma (\operatorname{sign} \sigma) u_1 u_2 x_1 x_1 \ldots x_{r-1} x_{r-1} x_{\sigma(r)} \ldots x_{\sigma(r+s)},$$

where $0 \neq c'' \in K$ and σ fixes $1, \ldots, r-1$, i. e. $g_{r-1, s+1}$ follows from (12′).

Lemma 3.4. *In $F(\mathfrak{M})$ the standard generators f_λ and f'_μ of* (12) *and* (12′) *are consequences of g_{ab} of* (13) *for $\lambda = (3, 2^a, 1^{b-1})$, $(3, 2^{a-1}, 1^{b+1})$ and $\mu = (2^{a+1}, 1^b)$.*

Proof. Applying several times (3), (4), (4′) and (6), we express f_λ as a linear combination of monomials (7) and obtain that for $\lambda = (3, 2^a, 1^{b-1})$, f_λ is proportional to

$$\Sigma (\operatorname{sign} \sigma) x_1 x_1 x_2 x_2 \ldots x_{a+1} x_{a+1} x_1 x_{\sigma(a+2)} \ldots x_{\sigma(a+b)},$$

where $\sigma \in S_{a+b}$, $\sigma(j) = j$ for $j < a+2$. Up to a multiplicative constant this equals

$$\Sigma (\operatorname{sign} \sigma) x_1 x_1 x_2 x_2 \ldots x_{a+1} x_{a+1} x_{\sigma(1)} x_{\sigma(a+2)} \ldots x_{\sigma(a+b)},$$

$\sigma(j) = j$ for $1 < j < a+2$. Obviously, the latter polynomial is a consequence of $g_{ab}(x_1, x_1; x_2, \ldots, x_{a+1}; x_1, x_{a+2}, \ldots, x_{a+b})$. Similarly, for $\lambda = (3, 2^{a-1}, 1^{b+1})$, f_λ equals a linear combination of

$$g_{ab}(x_i, x_1; x_1, \ldots, x_a; x_{a+1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{a+b+1}), \quad i > a.$$

Now, let us consider the case $\mu = (2^{a+1}, 1^b)$ and let for example τ be the identity substitution of S_{a+1}. Then, applying (6), we express

$$\Sigma (\operatorname{sign} \sigma) x_{\sigma(1)} x_1 x_{\sigma(2)} x_2 \ldots x_{\sigma(a+1)} x_{a+1} x_{\sigma(a+2)} \ldots x_{\sigma(a+b+1)}$$

as a linear combination of

(14) $$\Sigma (\operatorname{sign} \sigma) x_1 x_1 x_2 x_2 \ldots x_{a+1} x_{a+1} x_{\sigma(a+2)} \ldots x_{\sigma(a+b+1)};$$

(14′) $$\Sigma (\operatorname{sing} \sigma) x_i x_1 x_1 x_i x_2 x_2 \ldots x_{i-1} x_{i-1} x_{i+1} x_{i+1} \ldots x_{a+1} x_{a+1} x_{\sigma(a+2)} \ldots x_{\sigma(a+b+1)}, \quad 1 < i < a+2;$$

(14″) $$\Sigma (\operatorname{sign} \sigma) x_{\sigma(a+2)} x_1 x_2 x_2 \ldots x_{a+1} x_{a+1} x_1 x_{\sigma(a+3)} \ldots x_{\sigma(a+b+1)},$$

σ fixes $1, 2, \ldots, a+1$.

Clearly, (14) follows from g_{ab}, the same holds for (14′) applying (4′). In order to handle (14″) it suffices to show that $\Sigma (\operatorname{sign} \rho) y_{\rho(1)} x_1 x_1 y_{\rho(2)} \ldots y_{\rho(b)}$ is a consequence of

(15) $$h(u_1, u_2; y_1, \ldots, y_b) = \Sigma (\operatorname{sign} \rho) u_1 u_2 y_{\rho(1)} \ldots y_{\rho(b)}.$$

In the following calculations we work modulo the polynomial (15).

$$0 = h(y_1, x_1; x_1, y_2, \ldots, y_b) = \Sigma (\operatorname{sign} \rho) ([(b+1)/2] y_1 x_1 x_1 y_{\rho(2)} \ldots y_{\rho(b)} - [b/2] y_1 x_1 y_{\rho(2)} x_1 y_{\rho(3)} \ldots y_{\rho(b)})$$

$$= \Sigma (\operatorname{sign} \rho) ([(b+1)/2] y_1 x_1 x_1 y_{\rho(2)} \ldots y_{\rho(b)} + ([b/2]/2) x_1 x_1 y_{\rho(2)} y_1 y_{(\rho)3} \ldots y_{\rho(b)})$$

and the alternating sum on $y_1, \ldots, y_b$ gives

$$0 = \Sigma (\operatorname{sign} \rho) ([(b+1)/2] y_{\rho(1)} x_1 x_1 y_{\rho(2)} \ldots y_{\rho(b)} - ([b/2]/2) x_1 x_1 y_{\rho(1)} \ldots y_{\rho(b)}) = [(b+1)/2] \Sigma (\operatorname{sign} \rho) y_{\rho(1)} x_1 x_1 y_{\rho(2)} \ldots y_{\rho(b)}.$$

Therefore (14″) follows from (15) and, as a consequence, f'_μ vanishes modulo g_{ab}.

Lemma 3.5. *In the notation of Theorem* 3.2 (i), (ii), *the consequences of M in $P_{n+1}(\mathfrak{M})$ form an S_{n+1}-submodule of the modules* (10) *and* (11), *respectively.*

Proof. Since the consequences of M in $P_{n+1}(\mathfrak{M})$ form an S_{n+1}-submodule of $P_{n+1}(\mathfrak{M})$, the proof follows immediately from Corollary 1.2.

Proof of Theorem 3.2. (i) In virtue of Lemma 3.5 it suffices to show that the polynomials f_ν, $\nu=(3,\ 2^{p+1},\ 1^{q-1})$, $(3,\ 2^p,\ 1^{q+1})$, f'_ρ, $\rho=(2^{p+2},\ 1^q)$ are consequences of (12). By Lemma 3.3 (ii), $g_{p+1,q}$ follows from (12) and Lemma 3.4 gives the desired result.

(ii) Similarly, $g_{r,s-1}$ and $g_{r-1,s+1}$ are consequences of (12′) and hence f_ν, $\nu=(3,\ 2^r,\ 1^{s-2})$, $(3,\ 2^{r-1},\ 1^s)$, $(3,\ 2^{r-2},\ 1^{s+2})$, f'_ρ, $\rho=(2^{r+1},\ 1^{s-1})$, $(2^r,\ 1^{s+1})$, follow from (12′).

Corollary 3.6. *Let $\mathfrak{U}$ be a subvariety of $\mathfrak{M}$. Then $\mathfrak{U}$ is nilpotent if and only if $\mathfrak{U}$ satisfies a Jordan standard identity*

$$\Sigma\,(\operatorname{sign}\sigma)\,x_{n+1}\,x_{\sigma(1)}\ldots x_{\sigma(n)}=0. \tag{16}$$

Proof. Clearly the polynomial from (16) generates in $P_{n+1}(\mathfrak{M})$ the submodule $M(2,\ 1^{n-1})$. Therefore, it suffices to show that $f'_\mu=0$ from (12′), $\mu=(2,\ 1^{n-1})$ implies the nilpotency of $F(\mathfrak{U})$. But it follows immediately from Theorem 3.2 that $M(\lambda)\subset P_N(\mathfrak{M})$ vanishes modulo f'_μ if $\lambda=(3,\ 2^p,\ 1^q)$, $p+q\geq n-1$ or $\lambda=(2^r,\ 1^s)$, $r+s\geq n$. Since this holds for all $M(\lambda)\subset P_N(\mathfrak{M})$ for $N\geq 2n+1$, we obtain that $x_1x_2\ldots x_{2n+1}=0$ is a consequence of (16) in $F(\mathfrak{M})$.

Corollary 3.7. *Let $f=0$ be a polynomial identity of degree d for the subvariety $\mathfrak{U}$ of $\mathfrak{M}$ and let $f\neq 0$ for $\mathfrak{M}$. Then*

(i) $c_n(\mathfrak{U})\leq cn^{d+1}$, *where* $c>0$.

(ii) $c(\mathfrak{U},\ t)=f(t)/(1-t)^{d+2}$, *where* $f(t)\in\mathbf{Z}[t]$.

(iii) *The topological rank of $\mathfrak{U}$ is finite and is bounded by $d+2$.*

Proof. Let $\mathfrak{U}$ be a proper subvariety of $\mathfrak{M}$ and let $f=0$ be a polynomial identity of degree d for $\mathfrak{U}$ such that $f\neq 0$ for $\mathfrak{M}$. Then some of the polynomials (12) or (12′) vanishes on $\mathfrak{U}$ and, as a consequence of Theorem 3.2,

$$P_n(\mathfrak{U})\subset\Sigma\,M(3,\ 2^p,\ 1^q)+\Sigma\,M(2^r,\ 1^s), \tag{17}$$

where the summation runs over all partitions of n with $p,\ r<d$.

(i) The hook formula for the dimensions of the irreducible S_n-modules gives that for the partition $(3,\ 2^p,\ 1^q)$ of n, p being fixed, is a polynomial of degree $p+2$ in n; similarly for fixed r, $\dim M(2^r,\ 1^s)$ is a polynomial of degree r. Therefore, by (17)

$$c_n(\mathfrak{U})\leq\Sigma\dim M(3,\ 2^p,\ 1^q)+\Sigma\dim M(2^r,\ 1^s),\quad p,\ r<d.$$

Hence there exists $2d-1$ polynomials $f_i(n)$ of degree at most $d+1$ such that $c_n(\mathfrak{U})\leq\Sigma f_i(n)$ and $c_n(\mathfrak{U})$ is bounded by a polynomial in n of degree $d+1$, i. e. $c_n(\mathfrak{U})\leq cn^{d+1}$ for a suitable $c>0$.

(ii) As a consequence of Theorem 3.2 and (17) we obtain that for n large enough there exists an integer $k\leq d+1$ such that

$$P_n(\mathfrak{U})=\Sigma\,M(3,\ 2^p,\ 1^q)+\Sigma\,M(2^r,\ 1^s),$$

where the summation is over all partitions of n with $p\leq k-2$, $r\leq k$.

Therefore, for n sufficiently large, e. g. $n>N$, $c_n(\mathfrak{U})=g(n)$, where $g(n)$ is a polynomial of degree k and $g(n)\in\mathbf{Z}$. Hence

$$g(n)=\Sigma\,a_i\binom{n+i}{i},\quad a_i\in\mathbf{Z},\quad i=0,\ 1,\ldots,\ k,$$

$$c(\mathfrak{U},\ t)=f_0(t)+\Sigma\,a_i\binom{n+i}{i}t^n=f_0(t)+\Sigma\,a_i/(1-t)^{i+1}=f(t)/(1-t)^{d+1}.$$

Here $f_0(t)=\Sigma \dim M(\lambda)t^n$, $n\leq N$, where the summation is over all partitions λ such that $M(\lambda)\subset P_n(\mathfrak{U})$, $\lambda=(3, 2^p, 1^q)$ and $\lambda=(2^r, 1^s)$ with $p\geqq k-1$ and $r>k$. Hence $f_0(t)\notin \mathbf{Z}[t]$ and, therefore, $f(t)\notin \mathbf{Z}[t]$ as well.

(iii) The definition of the topological rank of a variety $\mathfrak{U}$ satisfying the Specht property is given in [1]; in [5] it is restated in the language of the graph ot the variety $\mathfrak{U}$. We follow the exposition in [5, Theorem 3.2]. It is easy to check that the set of isolated points $\Lambda(\mathfrak{M})'$ of the topological space $\Lambda(\mathfrak{M})$ consists of all subvarieties $\mathfrak{B}$ of $\mathfrak{M}$ such that $\psi(\mathfrak{B})\subset \mathrm{gr}(\mathfrak{M})$ does not contain $M(2, 1^r)$, $r=0, 1, 2, \ldots$. Analogously, for $p\geqq 2$ $\Lambda(\mathfrak{M})^{(p)}=\{\mathfrak{B}\subset\mathfrak{M} \mid \psi(\mathfrak{B})$ does not contain $M(3, 2^{p-2}, 1^q)$ and $M(2^p, 1^q)\}$. On the other hand, every isolated point of $\Lambda(\mathfrak{U})$ is isolated also in $\Lambda(\mathfrak{M})$. Therefore, (17) gives that $\Lambda(\mathfrak{U})^{(d+2)}=\emptyset$ and the topological rank of $\mathfrak{U}$ is bounded by $d+2$.

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