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ONE THEOREM OF PHRAGMEN — LINDELÖF TYPE FOR PARABOLIC EQUATION OF SECOND ORDER

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A modified version of the famous three balls theorem is derived. Then a new theorem of Phragmen — Lindelöf type for the linear parabolic differential equation of second order is proved.

Nowadays we consider as theorems of Phragmen — Lindelöf type a certain amount of results from the qualitative theory of partial differential equations. Usually they concern the behaviour of the solutions with respect to their behaviour on the boundary of the region. Some results concerning certain rates of decrease or growth of the solutions also belong to this kind of theorems.

To prove such qualitative results we need a certain kind of theorems concerning some forms of continuous dependence on the extensions of the solutions on larger domains.

Perhaps, the most suitable one is the so-called "three balls theorem". Its prototype is the classical Hadamard theorem for the three circles for an analytic function in plane. In the case of partial differential equations instead of an analytic function we consider a solution of the equation. There are some variants of the three balls theorem.

In this paper is given a modified version of the generalized theorem of the three balls (Theorem of Nadirishvili). Then a theorem of Phragmen — Lindelöf type for parabolic linear equations will be proved.

Let $C^{x_0}_{T,r}$ be a cylinder in $R^{n+1}=R^n_x\times R^1_t$:

$$C^{x_0}_{T,r}=\{(x,\ t)\in R^{n+1}/|x-x_0|<r;\quad 0<t\leqq T\}.$$

In $C^{x_0}_{T,r}$ we will consider a parabolic equation:

$$\text{(H.1)}\qquad \frac{\partial u}{\partial t}=\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i(x,\ t)\frac{\partial u}{\partial x_i}+c(x,\ t)\,u,$$

$$\text{(H.2)}\qquad \lambda^{-1}|\zeta|^2\leqq\sum a_{ij}(x,\ t)\zeta_i\zeta_j\leqq\lambda|\zeta|^2,\quad \lambda\geqq 1.$$

We assume that the coefficients $a_{ij}(x,\ t)$, $b_i(x,\ t)$, $c(x,\ t)$ are analytic functions. Let them be continual in the complex domain $|\mathrm{Jm}\,x_i|<\delta$, $i=1,\ 2,\ldots n$; $|\mathrm{Jm}\,t|<\delta$ and their moduli be bounded there by a constant M. Let $Q^{x_0}_r$ be the upper base of $C^{x_0}_{T,r}$ and $Q^{x_0}_\rho\subset Q^{x_0}_r$ be a concentric to $Q^{x_0}_r$ ball with radius ρ: $0<\rho<r$.

Theorem 1. *Let $E\subset Q^{x_0}_{r/4}$ be a closed set and*

$$\text{(H.3)}\qquad \operatorname{mes} E>a>0.$$

There exist constants $\sigma>1$ and $\varepsilon_0>0$ which depend on λ, M, a, δ, T, r, such that if

$$\text{(H.4)}\qquad |u(x,\ t)||_{C^{x_0}_{T,r}}<1$$

and

(H.5) $$|u(x, t)||_E < \varepsilon < \varepsilon_0,$$

then $|U(x, t)||_{Q^{x_0}_{r/2}} < \varepsilon^\sigma$.

Proof of theorem 1. The following natural estimates for the derivatives of $u(x, t)$ with respect to x are valid:

(H.6) $$|D^\alpha_x u(x, t)| < c_0 c^k k!,$$

where $k = |\alpha|$ and constants $c_0 > 0$ and $c > 0$ depend on λ, M, δ, T, a, r. Because of these estimates the proof of the inequality coincides with the proof of a similar inequality in [1].

Now let us formulate and prove the following theorem:

Let Ω be the following domain:

$$\Omega = \{(x, t) \in R^{n+1} | 0 < t \leqq T;\ 0 < x_1 < \infty;\ \sum_{i=2}^{n} x_i^2 < r^2\}.$$

We will consider equation (H.1)-(H.2) with analytic coefficients in Ω, which are as above continual in the complex domain: $|\mathrm{Jm}\, x_i| < \delta$, $i = 1, 2, \dots n$, $|\mathrm{Jm}\, t| < \delta$, and bounded by a constant M. Function $u(x, t)$ is a solution of (H.1)-(H.2) in Ω. Let us denote by G_T and x_N: $G_T = \Omega \cap \{t = T\}$, $x_N = (\frac{Nr}{4}, 0, \dots 0)$.

Let $E_N \subset Q^{x_N}_{r/4}$, $N = 1, 2 \dots$ be closed sets and for each N the following condition is fulfilled: $\mathrm{mes}\, E_N > a > 0$, where $c > 0$ is a constant.

Theorem 2. *There exists a constant* $c > 0$ *which depends on* λ, M, δ, T, a *and* r *such that if*

(H.7) $$|u(x, t)||_{E_N} < \exp(-\exp(cN)),$$

then $u(x, t) \equiv 0$ *in* G_T.

Proof of theorem 2. Let ε_0 and σ be the same constants as in Theorem 1. Let us choose constant c so great as the following estimates to be fulfilled: $\exp(-\exp(c)) < \varepsilon_0$, and

(H.8) $$c > 2 \ln \frac{1}{\sigma}.$$

So, because $Q^{x_{N-1}}_{r/4} \subset Q^{x_N}_{r/2}$, and by means of Theorem 1 and (H.8) we derive the estimate with respect to the balls $Q^{x_N}_{r_1} Q^{x_{N-1}}_{r} \dots Q^{x_2}_{r}$:

$$|u(x,t)||_{Q^{x_2}_{r/2}} < \exp(-\exp(cN) \cdot \sigma^N) = \exp(-\exp N(c - \ln\frac{1}{\sigma})) < \exp(-\exp(\frac{cN}{2})).$$

But integer N is arbitrary and it follows that $u(x, t) \equiv 0$ in $Q^{x_2}_{r_2}$.

As a consequence of the inequality (H.7) we derive that $u(x, t)$ is zero on characteristic G_T but it is not necessary identically equal to zero though it is bounded. Let us consider the following example:

Example 1. Let the function $u(x_2, t)$ be a solution of the following equation:

(H.9) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_2^2}$$

in a rectangle Π: $\Pi=\{0<t\leqq T; |x_2|<r\}$. Let $u(x_2, t)$ which is not identically equal to zero be equal to zero on the upper boundary of Π. A. P. Tikhonov has constructed in [2] the following solution of the equation (H. 9).

Let $f(t)\in C^{\infty}(0, T])$, $f(t)<0$ on $[0, T)$; $f^{(k)}(T)=0$, $k=0, 1, \ldots$ and

$$\max|f^{(k)}(t)|<k^{k(1+\varepsilon)},\quad k=1, 2, \ldots.$$

where $0<\varepsilon<1$.

We will define the function $u(x_2, t)$ as follows:

$$u(x_2, t)=\sum_{k=0}^{\infty} f^{(k)}(t)\frac{x_2^{2k}}{(2k)!}.$$

This function is a solution of the equation (H.9) in the rectangle Π and for $t=T$ $u(x_2, t)=0$. But it exists a constant $k>0$ such that $|u(x_2, t)|<k$. Let us determine the function $u(x_1, x_2, t)$ as follows: $u(x_1, x_2, t)\equiv u(x_2, t)$. Then $u(x_1, x_2, t)$ is a solution of the equation

$$\frac{\partial u}{\partial t}=\frac{\partial u^2}{\partial x_1^2}+\frac{\partial u^2}{\partial x_2^2}$$

in Ω, where $\Omega=\Pi\times\{0<x_1<\infty\}$. The function $u(x, x_2, t)$ is bounded in Ω and for $t=T$ $u(x_1, x_2, t)|_{t=T}=0$ but $u(x_1, x_2, t)\not\equiv 0$

For the strip $R^n x\{0<t\leqq T\}$ the following result can be proved:

Theorem 3. *Let Ω be the domain:*

$$\Omega=\{(x, t)\in R^{n+1}\,|\,x\in R^n;\ 0<t\leqq T\}.$$

Equation (H.1)-(H.2) *is defined in Ω under the same conditions and assumptions as in Theorem* 2 *and $u(x, t)$ is a bounded solution of it, continual for:* $|\mathrm{Jm}\, x_i|<\delta$, $=1,\ldots n$, $|\mathrm{Jm}\, t|<\delta$. *Denote by Ω_T the hyperplane $t=T$ and by $Q_\rho^{x_0}$ a ball (n-dimensional) in Ω_T with radius ρ and center in x_0.*

Let $r>0$, $x_N=(\frac{Nr}{4}, 0,\ldots 0)$ and $E_N\subset Q_{r/4}^{x_N}$ *be closed sets with*

$$\mathrm{mes}\, E_N>a>0,$$

where the constant $a>0$ is fixed and N is an arbitrary integer.

There exists a constant $c>0$ which depends on λ, δ, M, T, a and r such that if $|u(x,t)|_{E_N}<\exp(-\exp(NT))$, then $u(x, t)\equiv 0$ in Ω.

Proof of Theorem 3. The proof is a trivial consequence of Theorem 2, the analycity of $u(x, t)$ with respect to x and the uniqueness theorem for the inverse Cauchy problem for the parabolic equation. The solution $u(x, t)$ belongs to the class of the bounded functions [3].

Remark. The requirement $u(x, t)$ to be bounded in Ω may be changed to: $u(x, t)$ belongs to the Tihonov class of uniqueness: $u(x, t)<c_1\exp(c_2|x|^2)$, where $c_1>0$, $c_2>0$ are constants.

To prove that $u(x, t)|_{t=T}=0$ some obvious changes have to be done. In this case there is a theorem of uniqueness for the solution of the inverse Cauchy problem too [4].

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