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A COEFFICIENT INEQUALITY FOR A SUB-CLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Let U be the class of functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k$$

regular in the unit disc $E = \{z: |z| < 1\}$ and satisfying the conditions $|w(z)| < 1$, $w(0) = 0$.

Let S^* denote the class of functions

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

which are regular, univalent and starlike in E.

Let the class $C(A, B)$ consists of regular functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

such that

$$\frac{zf'(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, \quad -1 \leqq B < A \leqq 1, \quad g \in S^*.$$

By definition of subordination, $f \in C(A, B)$ if and only if

$$(1) \qquad \frac{zf'(z)}{g(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \qquad w \in U.$$

The class $C(A, B)$ was studied by the second author in [3].

It is obvious that $C(1, -1) \equiv C$, the class of close-to-convex functions introduced by W. Kaplan in [1].

Let $C(\beta)$ denote the sub-class of C of regular functions $f(z)$ which satisfy the condition

$$\left| \frac{zf'(z)}{g(z)} - \beta \right| < \beta \left(\beta > \frac{1}{2}\right).$$

Note that $C(\beta) \equiv C\left(1, \frac{1}{\beta} - 1\right), \left(\beta > \frac{1}{2}\right)$.

$C(\alpha, -\alpha)$ is the sub-class of C of regular functions $f(z)$ satisfying

$$\left| \frac{\frac{zf'(z)}{g(z)} - 1}{\frac{zf'(z)}{g(z)} + 1} \right| < \alpha, \quad 0 < \alpha \leqq 1.$$

In this paper, we shall find the upper bounds of $|a_3 - \mu a_2^2|$ for the class $C(A, B)$, where μ is real. Results due to E. Keogh and E. Merkes [2] follow from our theorem.

We shall need the following

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Lemma. *If* $g\in S^*$, *then* $|b_3-\frac{3}{4}\mu b_2^2|\leq 1+\frac{1}{4}(|3-3\mu|-1)\ |b_2|^2$.
The result is sharp.
Proof. Since $g\in S^*$, therefore it follows that

$$\frac{zg'(z)}{g(z)}=\frac{1+w(z)}{1-w(z)},\qquad w\in U.$$

Expanding and equating the coefficients of z and z^2, we get

(2) $$b_2=2c_1$$

and

(3) $$b_3=b_2c_1+c_2+b_2^2/4.$$

From (2) and (3), we have

$$b_3-\frac{3}{4}\mu b_2^2=\frac{3}{4}(1-\mu)b_2^2+c_2,$$

(4) $$|b_3-\frac{3}{4}\mu b_2^2|\leq|c_2|+\frac{1}{4}|3-3\mu|\ |b_2|^2.$$

We also know (cf. [4]) that

(5) $$|c_2|\leq 1-|c_1|^2=1-\frac{|b_2|^2}{4}.$$

(4) and (5) together give

$$|b_3-\frac{3}{4}\mu b_2^2|\leq 1+\frac{1}{4}(|3-3\mu|-1)\ |b_2|^2.$$

The result is sharp for the function $g_0(z)$ defined by $g_0(z)=z/(1-z)^2$.

Theorem. *If* $f\in C(A,B)$, *then*

$$|a_3-\mu a_2^2|\leq\begin{cases}
\frac{1}{3}(3+(A-B)(2-B))-\mu[1+\frac{A-B}{2}]^2,\ \textit{for}\ \mu\leq\frac{-4B}{3(A-B+2)} & (6)\\
\quad \textit{and}\ B<0\ \textit{or}\ \mu<\frac{-4B}{3(A-B)}\ \textit{and}\ B>0;\\
\frac{1}{3}(1+A-B)+\frac{1}{3}(2-3\mu)[1+\frac{(2-3\mu)(A-B)}{3\mu(A-B)+4(1+B)}],\ \textit{for}\ \frac{-4B}{3(A-B+2)} & (7)\\
\quad \leq\mu\leq\frac{2}{3}\ (B<0,\ A+B<0)\ \textit{or}\ \frac{-4B}{3(A-B+2)}\leq\mu<\frac{-4B}{3(A-B)},\\
\quad (B<0,\ A+B>0);\\
\frac{1}{3}(1+A-B)+\frac{1}{3}(2-3\mu)[1+\frac{(2-3\mu)(A-B)}{4(1-B)-3\mu(A-B)}], & (8)\\
\quad \textit{for}\ \frac{-4B}{3(A-B)}\leq\mu\leq\frac{2}{3}(B>0);\\
\frac{1}{3}(1+A-B),\ \textit{for}\ \frac{2}{3}\leq\mu\leq\frac{4-(A+3B)}{3(A-B)}),\ (B<0,\ A>0 & (9)\\
\quad \textit{and}\ A+B<0);\\
\frac{1}{3}(1+A-B)+\frac{1}{3}(3\mu-4)+\frac{1}{3}\frac{(A-B)(3\mu-2)^2}{[4(1-B)-3\mu(A-B)]},\\
\quad \textit{for}\ \frac{4}{3}\leq\mu\leq\frac{4(2-B)}{3[2+A-B]}\ (A<0,\ B<0);\\
\mu[1+\frac{A-B}{2}]^2-\frac{1}{3}[3+(A-B)(2-B)],\ \textit{for}\ \mu\geq\frac{4(2-B)}{3[2+A-B]}. & (10)
\end{cases}$$

The estimates (6), (7), (8), (9) and (10) are sharp.

Proof. Expanding (1) and equating the coefficients of z^2 and z^3, we have

$$a_2 = \frac{1}{2}(b_2 + (A - B)c_1) \tag{11}$$

and

$$a_3 = \frac{b_3}{3} + \frac{(A - B)}{3}(b_2 c_1 + c_2 - Bc_1^2). \tag{12}$$

(11) and (12) yield.

$$a_3 - \mu a_2^2 = \frac{1}{3}\left(b_3 - \frac{3}{4}\mu b_2^2\right) + \frac{(A - B)}{6}(2 - 3\mu)b_2 c_1 + \frac{(A - B)}{3}\left[c_2 - \left(B + 3\mu\frac{(A - B)}{4}c_1^2\right)\right].$$

This implies

$$|a_3 - \mu a_2^2| \leqq \frac{1}{3}\left|b_3 - \frac{3}{4}\mu b_2^2\right| + \frac{(A - B)}{6}|2 - 3\mu|\,|b_2|\,|c_1|$$
$$+ \frac{(A - B)}{3}\left[1 - |c_1|^2 + \left|B + \frac{3\mu(A - B)}{4}\right|\,|c_1|^2\right].$$

Using the lemma, we get

$$|a_3 - \mu a_2^2| \leqq \frac{1}{3}(1 + A - B) + \frac{1}{12}(|3 - 3\mu| - 1)y^2 + \frac{(A - B)}{6}|2 - 3\mu|\,xy \tag{13}$$
$$+ \frac{(A - B)^2}{12}\left[\left|3\mu + \frac{4B}{(A - B)}\right| - \frac{4}{(A - B)}\right]x^2, \quad \text{where } x = |c_1| \leqq 1,\ y = |b_2| \leqq 2.$$

We consider the following cases:

Case I. $\mu \leq 2/3$.

In this case

$$|a_3 - \mu a_2^2| \leqq \frac{1}{3}(1 + A - B) + \frac{1}{12}(2 - 3\mu)y^2 + \frac{(A - B)}{6}(2 - 3\mu)xy \tag{13'}$$
$$+ \frac{(A - B)^2}{12}\left[\left|3\mu + \frac{4B}{A - B}\right| - \frac{4}{(A - B)}\right]x^2.$$

If $\mu < \frac{-4B}{3(A - B)}$, then we have

$$|a_3 - \mu a_2^2| \leqq \frac{1}{3}(1 + A - B) + \frac{1}{3}(2 - 3\mu) + \frac{(A - B)}{6}(2 - 3\mu)x$$
$$- \frac{(A - B)}{12}[3\mu(A - B) + 4(1 + B)]x^2 = H_0(x), \text{ say,}$$

$$\frac{-4B}{3(A - B)} \lesseqgtr \frac{2}{3} \text{ according as } (A + B) \gtreqless 0.$$

$$H_0'(x) = \frac{(A - B)}{6}[3\mu(A - B) + 4(1 + B)]\left[\frac{2(2 - 3\mu)}{3\mu(A - B) + 4(1 + B)} - x\right]$$

and

$$H_0''(x) = -\frac{(A - B)}{6}[3\mu(A - B) + 4(1 + B)].$$

$H_0'(x)$ vanishes when $x = \frac{2(2 - 3\mu)}{3\mu(A - B) + 4(1 + B)} = x_0$, say, which gives the maximum value of $H_0(x)$ provided $x_0 < 1$, that is, when $\mu > \frac{-4B}{3(A-B+2)}$ and $H_0''(x) < 0$ which holds for $\mu > \frac{-4(1+B)}{3(A-B)}$. But $\frac{-4(1+B)}{3(A-B)} < \frac{-4B}{3(2+A-B)}$. Thus $x = x_0$ gives the maximum value of $H_0(x)$ provided $\mu > \frac{-4B}{3(2+A-B)}$. For $\mu < \frac{-4B}{3(A-B+2)}$, $x_0 > 1$ and $H_0'(x) > 0$.

$H_0(x)$ is monotonically increasing function of x and hence $\max H_0(x) = H_0(1)$

$$\frac{-4B}{3(2+A-B)} \lessgtr \frac{-4B}{3(A-B)} \quad \text{according as } B \lessgtr 0.$$

Thus we see that

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}[3 + (A-B)(2-B)] - \mu\left(1 + \frac{A-B}{2}\right)^2 \text{ for } B < 0 \text{ and } \mu < \frac{-4B}{3(A-B+2)}$$

$$\text{or for } B > 0 \text{ and } \mu < \frac{-4B}{3(A-B)}.$$

We have seen above that $x = x_0$ gives the maximum value of $H_0(x)$ provided $\mu > \frac{-4B}{3(2+A-B)}$.

But for $B > 0$, $\frac{-4B}{3(2+A-B)} > \frac{-4B}{3(A-B)}$.

Since we are interested in finding the maximum value of $H_0(x)$, when $\mu < \frac{-4B}{3(A-B)}$ and so this case has to be excluded. For $B < 0$, $A + B > 0$, $x = x_0$ gives the maximum value of $H_0(x)$ for

$$\frac{-4B}{3(2+A-B)} < \mu < \frac{-4B}{3(A-B)}.$$

For $B < 0$, $A + B < 0$, $x = x_0$ gives max $H_0(x)$ for

$$\frac{-4B}{3(2+A-B)} < \mu < \frac{2}{3}$$

Thus

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}(1 + A - B) + \frac{1}{3}(2 - 3\mu)\left[1 + \frac{(2-3\mu)(A-B)}{3\mu(A-B) + 4(1+B)}\right]$$

$$\text{for } \frac{-4B}{3(2+A-B)} < \mu \leq \frac{2}{3} \quad (B < 0,\ A + B < 0)$$

or

$$\text{for } \frac{-4B}{3(2+A-B)} < \mu < \frac{-4B}{3(A-B)} \quad (B < 0,\ A + B > 0).$$

In the case $\frac{-4B}{3(A-B)} < \mu < \frac{2}{3}$, $(A + B > 0)$, (13′) reduces to

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}(1 + A - B) + \frac{1}{3}(2 - 3\mu) + \frac{(A-B)}{3}(2 - 3\mu)x$$

$$- \frac{(A-B)^2}{12}\left[\frac{4(1-B)}{(A-B)} - 3\mu\right]x^2 = H_1(x), \text{ say.}$$

$$H_1'(x) = \frac{(A-B)}{6}[4(1-B) - 3\mu(A-B)]\left[\frac{2(2-3\mu)}{4(1-B) - 3\mu(A-B)} - x\right]$$

and

$$H_1''(x) = -\frac{(A-B)^2}{6}\left[\frac{4(1-B)}{(A-B)} - 3x\right] < 0 \quad \text{for} \quad \mu < \frac{2}{3}.$$

$$H_1'(x) \text{ vanishes at } x = \frac{2(2-3\mu)}{4(1-B) - 3\mu(A-B)} = x_1, \text{ say.}$$

$x = x_1$ gives max $H_1(x)$ provided $x_1 < 1$ which holds for $\mu > \frac{-4B}{3[2-(A-B)]}$.

Also
$$\frac{-4B}{3(A-B)} < \frac{-4B}{3[2-(A-B)]} \text{ for } B > 0.$$

Hence

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}(1 + A - B) + \frac{1}{3}(2 - 3\mu)\left[1 + \frac{(A-B)(2-3\mu)}{4(1-B) - 3\mu(A-B)}\right]$$

$$\text{for} \quad \frac{-4B}{3[2-(A-B)]} < \mu \leq \frac{2}{3} \text{ and } B > 0.$$

Case II. $\frac{2}{3} \leq \mu < 1$.

We have seen above that $\frac{2}{3} < \frac{-4B}{3(A-B)}$ for $B < 0$, $A + B < 0$. So we consider the case $\frac{2}{3} < \mu < \frac{-4B}{3(A-B)}$. (13) takes the form

$$(14) \qquad |a_3 - \mu a_2^2| \leq \frac{1}{3}(1 + A - B) - \frac{1}{12}(3\mu - 2)y^2 + \frac{(A-B)}{6}(3\mu - 2)xy$$
$$- \frac{(A-B)^2}{12}\left[3\mu + \frac{4(1+B)}{(A-B)}\right]x^2 = H_2(x, y), \text{ say.}$$

It is easy to see that in this case

$$\frac{\partial^2 H_2}{\partial x^2} < 0, \quad \frac{\partial^2 H_2}{\partial y^2} < 0$$

and

$$\left(\frac{\partial^2 H_2}{\partial x^2}\right)\left(\frac{\partial^2 H_2}{\partial y^2}\right) - \left(\frac{\partial^2 H_2}{\partial x \partial y}\right)^2 = \frac{(A-B)^2}{36}(3\mu - 2)\left[2 + \frac{4(1+B)}{(A-B)}\right] > 0.$$

Thus $H_2(x, y)$ has a maximum at $x = y = 0$ and from (14) we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}(1 + A - B), \quad B < 0, \quad A + B < 0.$$

It is easy to see that $\frac{2}{3} < \frac{-4B}{3(A-B)} < 1$ for $B < 0$, $A + B < 0$ and $3A + B > 0$. Now we consider the case $\frac{-4B}{3(A-B)} < \mu < 1$. (13) can be written as

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}(1 + A - B) - \frac{1}{12}(3\mu - 2)y^2 + (3\mu - 2)\frac{(A-B)}{6}xy$$

$$(15) \qquad - \frac{(A-B)^2}{12}\left[\frac{4(1-B)}{(A-B)} - 3\mu\right]x^2 = H(x, y).$$

An easy computation will show that $x = y = 0$ gives $\max H(x, y)$ and from (15) we get

$$|a_3 - \mu a_2^2| \leqq \frac{1}{3}(1 + A - B), \quad B < 0, \quad A + B < 0, \quad 3A + B > 0.$$

Case III. $1 \leqq \mu < 4/3$.

We first consider the case $1 \leqq \mu < \frac{-4B}{3(A-B)}$ which holds for

$$B < 0, \ A > 0 \ \text{and} \ 3A + B < 0.$$

From (13), we have

$$|a_3 - \mu a_2^2| \leqq \frac{1}{3}(1 + A - B) - \frac{1}{12}(4 - 3\mu)y^2 + \frac{(A-B)}{6}(3\mu - 2)xy$$

$$- \frac{(A-B)^2}{12}\left[3\mu + \frac{4(1+B)}{(A-B)}\right]x^2 = H_3(x, y), \text{ say.}$$

It is easy to see that in this case $\frac{\partial^2 H_3}{\partial x^2} < 0$, $\frac{\partial^2 H_3}{\partial y^2} < 0$ and $F(\mu) = \left(\frac{\partial^2 H_3}{\partial x^2}\right)\left(\frac{\partial^2 H_3}{\partial y^2}\right)$ $- \left(\frac{\partial^2 H_3}{\partial x \partial y}\right)^2 = -2\,[9(A-B)\mu^2 + 6(1-(2A-3B))\mu - 2(4-(A-5B))]$.
If the roots μ_1 and μ_2 of $F(\mu) = 0$ are imaginary, then $F(\mu) > 0$ and the conditions for max $H_3(x, y)$ are not satisfied. If μ_1, μ_2 are real, then we are interested in the case, when μ_1 and μ_2 lie in $\left(1, \frac{-4B}{3(A-B)}\right)$. But a simple calculation will show that these conditions are not satisfied.

Hence $H_3(x, y)$ has no maxima for $1 \leqq \mu < \frac{-4B}{3(A-B)}$. Let us take the case $1 < \frac{-4B}{3(A-B)} < \mu \leqq \frac{4}{3}$ which holds for $B < 0$, $A > 0$, $3A + B < 0$. (13) reduces to

$$|a_3 - \mu a_2^2| \leqq \frac{1}{3}(1 + A - B) - \frac{1}{12}(4 - 3\mu)y^2 + \frac{(A-B)}{6}(3\mu - 2)xy$$

$$(16) \qquad - \frac{(A-B)^2}{12}\left[\frac{4(1-B)}{(A-B)} - 3\mu\right]x^2 = H_4(x, y), \text{ say.}$$

The extreme points of $H_4(x, y)$ are given by $x = y = 0$. An easy computation shows that $H_4(x, y)$ has a maximum at $x = y = 0$ provided

$$\frac{(A-B)^2}{36}(4 - 3\mu)\left[\frac{4(1-B)}{(A-B)} - 3\mu\right] - \frac{(A-B)^2}{36}(3\mu - 2) > 0$$

or when

$$\mu < \frac{4(A+3B)}{3(A-B)} = \mu_0, \text{ say.}$$

It is easy to verify that $\frac{-4B}{3(A-B)} < \mu_0 < \frac{4}{3}$ $(A > 0, B < 0)$. From (16), we obtain

$$|a_3 - \mu a_2^2| \leqq \frac{1}{3}(1 + A - B) \text{ where } 1 < \frac{-4B}{3(A-B)} < \mu_0 < \frac{4}{3}$$

which holds for $A > 0$, $B < 0$ and $3A + B < 0$.

Case IV. $\mu \geqq \frac{4}{3}$.

Consider the case $\frac{4}{3} \leqq \mu < \frac{-4B}{3(A-B)}$ $(B < 0, A < 0)$. From (13), we get

$$|a_3 - \mu a_2^2| \leqq \frac{1}{3}(1 + A - B) + \frac{1}{3}(3\mu - 4) + \frac{(A-B)}{3}(3\mu - 2)x$$

$$- \frac{(A-B)^2}{12}[3\mu + \frac{4(1+B)}{(A-B)}]x^2 = H_5(x), \text{ say.}$$

$$H_5'(x) = \frac{(A-B)}{6}[2(3\mu-2) - (3\mu(A-B) + 4(1+B)x)]$$

and

$$H_5''(x) = -\frac{(A-B)}{6}[3\mu(A-B) + 4(1+B)] < 0.$$

The only extreme point of $H_5(x)$ is $x = \frac{2(3\mu-2)}{3\mu(A-B)+4(1+B)} = x_2$, say. Now $x_2 < 1$ provided $\mu < \frac{4(2+B)}{3[2-(A-B)]}$ $(A < 0,\ B < 0)$.

But for $A < 0,\ B < 0$, $\frac{4(2+B)}{3[2-(A-B)]} < \frac{4}{3}$ and hence we reject the value $x = x_2$.

Now we take the case $\mu > \frac{-4B}{3(A-B)} > \frac{4}{3}$ $(A < 0,\ B < 0)$. (13) reduces to

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}(1+A-B) + \frac{1}{3}(3\mu-4) + \frac{(A-B)}{6}(3\mu-2)x$$

$$- \frac{(A-B)^2}{12}[\frac{4(1-B)}{(A-B)} - 3\mu]x^2 = H_6(x), \text{ say.}$$

$$H_6'(x) = \frac{(A-B)}{6}[4(1-B) - 3\mu(A-B)][\frac{2(3\mu-2)}{4(1-B)-3\mu(A-B)} - x],$$

$$H_6''(x) = \frac{-(A-B)}{6}[4(1-B) - 3\mu(A-B)] < 0 \text{ for } \mu < \frac{4(1-B)}{3(A-B)}.$$

$$H_6'(x) = 0 \text{ gives } x = \frac{2(3\mu-2)}{4(1-B)-3\mu(A-B)} = x_3, \text{ say.}$$

$$x_3 \lesseqgtr 1 \text{ according as } \mu \lesseqgtr \frac{4(2-B)}{3[2+(A-B)]}.$$

Also

$$\frac{4(2-B)}{3[2+(A-B)]} < \frac{4(1-B)}{3(A-B)}\ (A < 0,\ B < 0) \text{ and } \frac{4(2-B)}{3(2+A-B)} < \frac{-4B}{3(A-B)}\ (A < 0,\ B < 0).$$

Hence for $\frac{4}{3} \leq \mu \leq \frac{2(2-B)}{3[2+(A-B)]}$, $H_6(x)$ has a maximum at $x = x_3$ and hence

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}(1+A-B) + \frac{1}{3}(3\mu-4) + \frac{1}{3}\frac{(A-B)(3\mu-2)^2}{[4(1-B)-3\mu(A-B)]}.$$

For $\mu > \frac{4(2-B)}{3|2+A-B|}$, $x = x_3 > 1$ and $H_0'(x) > 0$ and hence $\max H_6(x) = H_6(1)$.

Thus

$$|a_3 - \mu a_2^2| \leq \mu[1 + \frac{(A-B)}{2}]^2 - \frac{1}{3}[3 + (A-B)(2-B)],$$

$$\frac{4(2-B)}{3(2+A-B)} < \mu < \frac{4(1-B)}{3(A-B)}.$$

For $\mu \geq \frac{4(1-B)}{3(A-B)}$, we at once get

$$|a_3 - \mu a_2^2| \leq \mu(1 + \frac{(A-B)}{2})^2 - \frac{1}{3}[3 + (A-B)(2-B)].$$

This completes the proof of the theorem. The bounds (6) and (7) coincide at $\mu = -4B/3(A-B+2)$. The bounds (7), (8) and (9) coincide at $\mu = 2/3$. The results (6) and (10) are sharp for the function $f(z)$ defined by

$$f'(z) = \frac{1}{(1-z)^2} \frac{1+Az}{1+Bz}.$$

The results (7), (8) and (9) are sharp for the function $f(z)$ defined, respectively, by

$$f'(z) = \frac{1}{(1-z)^2} \left[\frac{1+Az\left(\frac{c_1+z}{1+c_1 z}\right)}{1+Bz\left(\frac{c_1+z}{1+c_1 z}\right)} \right], \quad \text{where } c_1 = \frac{2(2-3\mu)}{3\mu(A-B)+4(1+B)},$$

$$f'(z) = \frac{1}{(1-z)^2} \left[\frac{1+Az\left(\frac{c_1+z}{1+c_1 z}\right)}{1+Bz\left(\frac{c_1+z}{1+c_1 z}\right)} \right], \quad \text{where } c_1 = \frac{2(2-3\mu)}{4(1-B)-3\mu(A-B)},$$

and

$$f'(z) = \frac{1}{(1-z^2)} \frac{1+Az^2}{1+Bz^2}.$$

Corollary 1. *If* $f \in C = C(1, -1)$, *then*

$$|a_3 - \mu a_2^2| \le \begin{cases} 3-4\mu, & \mu \le 1/3, \\ \frac{1}{3} + \frac{4}{9\mu}, & \frac{1}{3} \le \mu \le \frac{2}{3}, \\ 1, & \frac{2}{3} \le \mu \le 1. \\ 4\mu - 3, & \mu \ge 1, \end{cases}$$

which are the results due to E. Keogh and E. Merkes [2].

Corollary 2. *If* $f \in C(\alpha, -\alpha)$, $0 < \alpha \le 1$, *then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{1}{3}[3+2\alpha(2+\alpha)] - \mu(1+\alpha)^2, & \mu \le \frac{2\alpha}{3(1+\alpha)}; \\ \frac{1}{3}(1+2\alpha) + \frac{1}{3}(2-3\mu)\left[1+\frac{(2-3\mu)\alpha}{3\mu\alpha+2(1-\alpha)}\right], & \frac{2\alpha}{3(1+\alpha)} \le \mu \le \frac{2}{3}; \\ \frac{1}{3}(1+2\alpha), & \frac{2}{3} \le \mu \le \frac{2+\alpha}{3\alpha} \\ \mu(1+\alpha)^2 - \frac{1}{3}[3+2\alpha(2+\alpha)], & \mu > \frac{2(2+\alpha)}{3(1+\alpha)}. \end{cases}$$

Corollary 3. *If* $f \in C(\beta) = C(1, \frac{1}{\beta} - 1)$, *then*

$$
|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{3}[3 + (2 - \frac{1}{\beta})(3 - \frac{1}{\beta})] - \mu(2 - \frac{1}{2\beta})^2, \quad \mu \leq \frac{4(\beta - 1)}{3(4\beta - 1)} \text{ and } \beta > 1 ; \\ \frac{1}{3}(3 - \frac{1}{\beta}) + \frac{1}{3}(2 - 3\mu)[1 + \frac{(2 - 3\mu)(2\beta - 1)}{3\mu(2\beta - 1) + 4}] \quad \text{for} \\ \frac{4(\beta - 1)}{3(4\beta - 1)} \leq \mu \leq \frac{2}{3} \text{ and } \beta > 1, \\ \text{or } \frac{4(\beta - 1)}{3(4\beta - 1)} \leq \mu < \frac{4(\beta - 1)}{3(2\beta - 1)} \text{ and } \frac{1}{2} < \beta < 1 ; \\ \frac{1}{3}(3 - \frac{1}{\beta}) + 2\frac{(2 - 3\mu)(1 - \mu)}{(4 - 3\mu)}, \quad \frac{4(\beta - 1)}{3(2\beta - 1)} \leq \mu \leq \frac{2}{3} \\ \qquad \text{and } \frac{1}{2} < \beta < 1 ; \\ \frac{1}{3}(3 - \frac{1}{\beta}), \text{ for } \frac{2}{3} \leq \mu \leq 1 \text{ and } \beta > 1 ; \\ \mu(2 - \frac{1}{2\beta})^2 - \frac{1}{3}[3 + (2 - \frac{1}{\beta})(3 - \frac{1}{\beta})], \text{ for } \mu > \frac{4(3\beta - 1)}{3(4\beta - 1)} . \end{cases}
$$

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