Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

THE NIELSEN NUMBER OF SET-VALUD MAPS. AN APPROXIMATION APPROACH

W. KRYSZEWSKI, D. MIKLASZEWSKI

In the paper the fixed point classes and the Nielsen number of a set-valued map from the recently introduced class J are defined. The class J contains, for example, u. s. c. maps of compact ANR-spaces taking contractible values. The Nielsen number of a J-map φ is homotopy invariant and equal to the classical Nielsen number of a sufficiently close single-valued approximation on the graph of φ. All results are formulated for compositions of J-maps satisfying an additional assumption (A. 2). The given examples show the necessity of (A. 2) and the dependence on a decomposition.

Introduction. In the recent article [7] (see also [8]) a new class J of set-valued maps of compact ANR-spaces was defined and studied from the view-point of the fixed point index theory. The class J is quite large and fairly general. For example, it contains upper semicontinuous maps taking values being R_δ-sets (the definition of an R_δ-set is recalled below).

Below, we try to apply the approximation techniques developed by [7] in order to define the fixed point classes and the Nielsen number $N(\varphi)$ of a map φ being a composition of J-maps. We prove that this number is homotopy invariant and constitutes a lower bound for the number of fixed points of φ.

We hope our approximation approach, being very simple and entirely elementary may be useful in a further development.

Let us remark that several different methods in the set-valued Nielsen theory were presented by H. Schirmer [10], [11], J. Jezierski [9], Z. Dzedzej [6]. As a general reference we use [2] (see also [4]).

We would like to express our gratitude to Professor Górniewicz for his kind encouragement during the preparation of this paper and to the referee for his suggestions.

1. Preliminaries. We denote by I the unit interval in R^1. By a space we shall understand a compact metric ANR-space and by a map an upper semicontinuous transformation of spaces whose values are non-empty compact connected sets. The composition of two maps is a map again [1].

If M is a class of maps and X, Y are spaces, then by $M(X, Y)$ we denote the totality of M-maps (i. e. maps from the class M) from X to Y.

By S we denote the class of single-valued maps.

Let X be a space with a metric d. For a subset $A\subset X$ and a number $\varepsilon>0$, $B^X(A, \varepsilon):=\{x\in X \mid \operatorname{dist}(x, A)<\varepsilon\}$ where $\operatorname{dist}(., A)$ stands for the distance from A. Unless it leads to ambiguity, we shall write $B(A, \varepsilon)$, too.

Let Y be a space. For any map $\varphi: X\to Y$, $x\in X$ and $\varepsilon>0$, let $\varphi(x, \varepsilon)=B^Y(\varphi(B^X(x, \varepsilon)), \varepsilon)$.

Let X, Y be spaces, $\varphi: X\to Y$ a map and let $\varepsilon>0$. A map $f\in S(X, Y)$ is an ε-approximation (on the graph) of φ — we write $f\in a(\varphi, \varepsilon)$ — if, for each $x\in X$, $f(x)\in\varphi(x, \varepsilon)$. (i. a. the graph of f lies in the ε-neighbourhood of the graph of φ).

One can easily verify the following lemmas.

(1.1) Lemma. *If Z is a space and $\gamma: Y\to Z$ is a map, then, for each $\varepsilon>0$, there is a $\delta>0$ such that a map $g\circ f\in S(X, Z)$ is an ε-approximation of $\gamma\circ\varphi$, provided $g\in a(\gamma, \delta)$ and $f\in a(\varphi, \delta)$.* □

Using symbols, we write down the assertion of (1.1) shortly:

$$a(\gamma, \delta) \circ a(\varphi, \delta) \subset a(\gamma \circ \varphi, \varepsilon).$$

(1.2). Lemma. *If* $\mathrm{Fi}\, x(\varphi) = \{x \in X \mid x \in \varphi(x)\}$ *is contained in an open set* $U \subset X$, *then there is a* $\eta > 0$ *such that* $\mathrm{Fi}\, x(f) \subset U$ *for* $f \in a(\varphi, \eta)$. □

We say that u is a path from x_0 to x_1 (resp. a loop at x) in $B \subset X$ if $u \in S(I, X)$, $u(i) = x_i$ for $i = 0, 1$ (resp. $u(0) = u(1) = x$) and $u(I) \subset B$. The constant loop at $x \in X$ will be denoted by $x^{\sim}$. If u is a path, then the reverse path u^{-} is given by $u^{-}(t) = u(1-t)$. If $u_1, \ldots, u_n$ are paths, then the product path u (given by $u(t) = u_i(nt - i + 1)$ for $n^{-1}(i-1) \leq t \leq n^{-1}i$) is denoted by $u_1 * \ldots * u_n$.

Let $B_0, B_1 \subset X$. We say that the paths $u_0, u_1 \in S(I, X)$ are homotopic with ends in B_0, B_1 and write

$$u_0 \approx u_1 \text{ end } (B_0, B_1)$$

if there is a map $u \in S(I \times I, X)$ such that $u(t, i) = u_i(t)$, $u(i, t) \in B_i$ for $i = 0, 1$ and $t \in I$.

Observe that $u_0 \approx u_1$ end (B_0, B_1) if and only if there are paths w_0, w_1 in B_0, B_1, respectively, such that $w_j(i) = u_i(j)$ for $i, j = 0,1$ and the loop $u_0 * w_1 * u_1^{-} * w_0^{-}$ is homotopic rel $\{0, 1\}$ to the constant loop $u_0(0)^{\sim}$.

2. Admissible maps. One sees easily that not all maps may be approximated on the graph, however, if X, Y are spaces and $\varphi\colon X \to Y$ is a J-map, i. e. satisfies the following condition

(J) for each $x \in X$, $\varepsilon > 0$, there is a $\delta = \delta(x, \varepsilon) > 0$, $\delta \leq \varepsilon$, such that, for any positive integer n and $y_0 \in B(\varphi(x), \delta)$, the inclusion $B(\varphi(x), \delta) \subset B(\varphi(x), \varepsilon)$ induces the trivia homomorphism

$$\pi_n(B(\varphi(x), \delta), y_0) \to \pi_n(B(\varphi(x), \varepsilon), y_0),$$

then φ has arbitrarily close approximations.

Observe that φ is a J-map if, for each $x \in X$, the set $\varphi(x)$ is proximally n-connected, in the sense of Dugundji [5], for any nonnegative integer n.

The class J was introduced in [7] where actually the following theorem was proved.

(2.1) Theorem. *Let* $\varphi \in J(X, Y)$.

(i) *For each* $\varepsilon > 0$, $a(\varphi, \varepsilon) \neq \varnothing$.

(ii) *For each* $\rho > 0$ *there is an* $\varepsilon > 0$ *such that any maps* $f_0, f_1 \in a(\varphi, \varepsilon)$ *are joined by a homotopy* $f \in S(X \times I, Y)$ (*i. e.* $f(\cdot, i) = f_i$, $i = 0, 1$) *such that* $f(\cdot, t) \in a(\varphi, \rho)$ *for any* $t \in I$. □

In the above situation, we say that f_0, f_1 are homotopic ρ-close to φ.

The maps $\varphi_0, \varphi_1 \in J(X, Y)$ are said to be J-homotopic (we write $\varphi_0 \approx_J \varphi_1$) if $\varphi_i = \varphi(\cdot, i)$ for $\varphi \in J(X \times I, Y)$, $i = 0, 1$.

Let X be a space. A map $\Psi\colon X \to X$, is called admissible (or Ψ is an A-map) if

(A.1) there exists a diagram of J-maps and spaces

$$D\colon X = X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \ldots \to X_n \xrightarrow{\varphi_n} X_{n+1} = X$$

(called a decomposition of Ψ) such that $\Psi = \varphi_n \circ \ldots \circ \varphi_2 \circ \varphi_1$,

(A.2) for $x \in \mathrm{Fi}\, x(\Psi)$ there is a neighbourhood W_x of $\Psi(x)$ which is trivial in the sense that, for each $y_0 \in W_x$, the inclusion $W_x \subset X$ induces the trivial homomorphism $\pi_1(W_x, y_0) \to \pi_1(X, y_0)$.

Remark. Observe that (A.2) is equivalent to the assertion that each loop in W_x is fixed ends homotopic to a constant loop. This condition was assumed in [9] where a different approach to the definition of a Nielsen number for a set-valued map was presented.

Let Ψ_0, $\Psi_1 \in A(X, X)$ and let $D_0: X = X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \ldots \to X_n \xrightarrow{\varphi_n} X_{n+1} = X$, $D_1: X = Y_1 \xrightarrow{\gamma_1} Y_2 \xrightarrow{\gamma_2} \ldots Y_m \xrightarrow{\gamma_m} Y_{m+1} = X$ be decompositions of Ψ_0, Ψ_1, respectively. We say that the pairs (Ψ_0, D_0), (Ψ_1, D_1) are *homotopic* if $n = m$, $Y_k = X_k$ and $\gamma_k \approx_J \varphi_k$ for each $k = 1, 2, \ldots, n$.

The class A seems to be quite large.

Example ([7]). Denote by $\mathbf{P}$ one of the following classes of subsets of X:

$\mathbf{C} = \{K \subset X \mid K$ is compact and contractible$\}$,

$\mathbf{R}_\delta = \{K \subset X \mid$ there are sets $K_i \in \mathbf{C}$, $K_{i+1} \subset K_i$ for $i = 1, 2, \ldots$, such that $K = \bigcap K_i\}$,

$\mathbf{F} = \{K \subset X \mid K$ is a fundamental absolute retract$\}$ (see [3]).

If $\varphi: X \to X$ and $\varphi(X) \in \mathbf{P}$ for each $x \in X$, then $\varphi \in J(X, X) \subset A(X, X)$. □

3. Properties of admissible maps. In order to make the further study and notation as clear as possible, we are going to deal only with admissible maps being compositions of two J-maps. It should be observed that passing to a more general situation causes no serious trouble.

Let X be a space, $\Psi \in A(X, X)$ and $D: X \xrightarrow{\varphi} Y \xrightarrow{\gamma} X$ be a decomposition of Ψ. Directly from (1.1) and (2.1) we get that for any $\xi > 0$, there exists $F \in a(\Psi, \xi)$; precisely $F = g \circ f$ where g and f are sufficiently close approximations of γ and φ, respectively. Next we have

(3.1) Lemma. (i) *There is a number $\alpha_0 = \alpha_0(\Psi) > 0$ such that, for any $y \in \mathrm{Fix}(\Psi)$, the neighbourhood $\Psi(y, \alpha_0)$ of $\Psi(y)$ is trivial (in the sense of* (A.2)).

(ii) *For $x \in X$ and an open set U, $\Psi(x) \subset U$, there is a $\eta > 0$ such that any two points $a, b \in \Psi(x, \eta)$ are joined by a path in U. In particular, there is a number $\beta_0 = \beta_0(\Psi) < 2^{-1}\alpha_0$ such that, for each $y \in \mathrm{Fix}(\Psi)$, points $a, b \in \Psi(y, \beta_0)$ are joined by a path in $\Psi(y, 2^{-1}\alpha_0)$.*

Proof. Let $K = \mathrm{Fix}(\Psi)$. For $x \in K$, there is a $\eta = \eta(x)$ such that $B(\Psi(x), \eta) \subset W_x$ (see (A.2)). By the upper semicontinuity of Ψ, there is a $\mu = \mu(x) < \eta$ such that $\Psi(B(x, \mu)) \subset B(\Psi(x), 2^{-1}\eta)$. Since K is compact, there are points $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup B(x_i, 2^{-1}\mu(x_i))$.

Let $\alpha_0 = 2^{-1} \min\{\mu(x_i) \mid 1 \le i \le n\}$. Take any $y \in K$. There is j, $1 \le j \le n$, such that $B(y, \alpha_0) \subset B(x_j, \mu(x_j))$. Hence $\Psi(y, \alpha_0) \subset B(\Psi(x_j), 2^{-1}\eta + \alpha_0) \subset W_{x_j}$. Thus any loop in $\Psi(y, \alpha_0)$ is homotopic to a constant loop.

Assertion (ii) can be proved similarly, but, in place of (A.2), one should use the connectedness of values of Ψ and the uniform local contractibility property (ULC — see [4]) of ANR-spaces. □

In the sequel, for $y \in \mathrm{Fix}(\Psi)$, we put $B_y = \Psi(y, \alpha_0)$.

Now, we are going to define the *fixed point index* of A-maps. Let V be an open subset of X such that $\mathrm{Fix}(\Psi) \cap \partial(V) = \emptyset$.

In view of (1.2), there is a number $\eta_0 = \eta_0(\Psi, V) > 0$ such that no element of $a(\Psi, \eta_0)$ has fixed points on $\partial(V)$. By (1.1), there is a $\rho_0 = \rho_0(\Psi, V) > 0$ such that, $g \circ f \in a(\Psi, \eta_0)$ for $g \in a(\gamma, \rho_0)$ and $f \in a(\varphi, \rho_0)$. At last, by (2.1)(ii), there is an $\varepsilon_0 = \varepsilon_0(\Psi, V) > 0$ such that, for $f_0, f_1 \in a(\varphi, \varepsilon_0)$ and $g_0, g_1 \in a(\gamma, \varepsilon_0)$, there are maps $f \in S(X \times I, Y)$, $g \in S(Y \times I, X)$ such that $f(\cdot, i) = f_i$, $g(\cdot, i) = g_i$, $i = 0, 1$, and $f(\cdot, t) \in a(\varphi, \rho_0)$, $g(\cdot, t) \in a(\gamma, \rho_0)$.

The homotopy $F: X \times I \to X$ (given by the formula $F(x, t) = g(f(x, t), t)$) joins $F_0 = g_0 \circ f_0$ and $F_1 = g_1 \circ f_1$.

Moreover, for any $t \in I$, $F(\cdot, t) = g(\cdot, t) \circ f(\cdot, t) \in a(\Psi, \eta_0)$. Hence $x \ne F(x, t)$ for $x \in \partial(V)$, $t \in I$. By the homotopy invariance of the ordinary fixed point index of single-valued maps (see [4])

$$\mathrm{ind}(X, F_0, V) = \mathrm{ind}(X, F_1, V).$$

We may define the fixed point index of the pair (Ψ, D) on V by

(3.2) $$\operatorname{ind}(X, \Psi, D, V) = \operatorname{ind}(X, g\circ f, V)$$

where $g \in a(\gamma, \varepsilon)$, $f \in a(\varphi, \varepsilon)$ and $0<\varepsilon<\varepsilon_0(\Psi, V)$. The index is well defined since it does not depend on the choice of g and f.

Using the same reasoning, we are in a position to define a number $\bar{N}$ by

(3.3) $$\bar{N}(\Psi, D) = N(g\circ f)$$

where $g \in a(\gamma, \varepsilon_0(\Psi, X))$, $f \in a(\varphi, \varepsilon_0(\Psi, X))$, and $N(g\circ f)$ is the Nielsen number of $g\circ f$ (see [2] or [4]). The number $\bar{N}$ is well defined once again.

Remark. The above index in a natural way generalizes the index introduced in [7] (or [8]) for J-maps. One can easily prove that standard properties of an index are satisfied (maybe except that of commutativity). Observe that, to define the index (or the number $\bar{N}(\Psi, D)$), assumption (A.2) is superfluous.

4. The Nielsen number of an admissible map. Let X be a space, $\Psi \in A(X, X)$ and let D: $X \xrightarrow{\varphi} Y \xrightarrow{\gamma} X$ be a decomposition of Ψ. We say that points $x, y \in \operatorname{Fix}(\Psi)$ are D-equivalent, if there are:

(i) a path u from x to y,

(ii) a number $\delta>0$, trivial neighbourhoods W_x of $\Psi(x)$ and W_y of $\Psi(y)$ such that

$$u \approx g\circ f\circ u \text{ end } (W_x, W_y),$$

for any $f \in a(\varphi, \delta)$, $g \in a(\gamma, \delta)$.

If fixed points x, y are D-equivalent, then we write $x \sim^D y$.

First, we shall prove the following

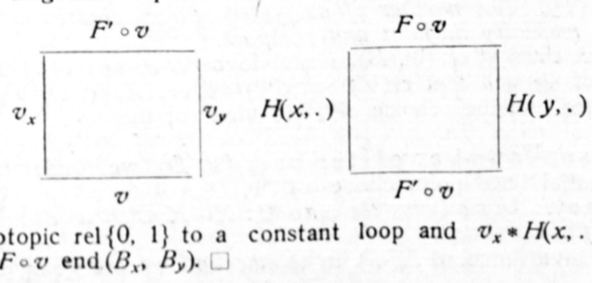
(4.1) Lemma. *There exists $\delta_0 = \delta_0(\Psi)$ such that, for arbitrarily chosen points $x, y \in \operatorname{Fix}(\Psi)$, $x \sim^D y$ if and only if there is a path v from x to y such that $v \approx g\circ f\circ v$ end (B_x, B_y) for any $f \in a(\varphi, \delta_0)$, $g \in a(\gamma, \delta_0)$.*

Proof. The "only if" part is to be proved. By (1.1), (2.1)(ii), choose $\delta_0 = \delta_0(\Psi)$ such that for any $f, f' \in a(\varphi, \delta_0)$, $g, g' \in a(\gamma, \delta_0)$ the compositions $g\circ f$ and $g'\circ f'$ are homotopic α_0-close to Ψ.

Let $x, y \in \operatorname{Fix}(\Psi)$, $x \sim^D y$. There is a path u from x to y, $\delta>0$, and trivial neighbourhoods W_x of $\Psi(x)$ and W_y of $\Psi(y)$ such that $u \approx g'\circ f'\circ u$ end (W_x, W_y) for $g' \in a(\gamma, \delta)$, $f' \in a(\varphi, \delta)$.

By (3.1)(ii), choose $\eta>0$ such that any two points from $\Psi(x, \eta)$ (resp. from $\Psi(y, \eta)$) may be joined by a path in $B_x \cap W_x$ (resp. in $B_y \cap W_y$)

Let $v := u$ and let $g \in a(\gamma, \delta_0)$, $f \in a(\varphi, \delta_0)$. Take $\delta_1>0$, $\delta_1<\min(\delta_0, \delta)$, such that $a(\gamma, \delta_1)\circ a(\varphi, \delta_1) \subset a(\Psi, \eta)$. Let $g' \in a(\gamma, \delta_1)$, $f' \in a(\varphi, \delta_1)$, $F' := g'\circ f'$. Obviously, $x, F'(x) \in \Psi(x, \eta)$ and $y, F'(y) \in \Psi(y, \eta)$. Hence there are paths v_x from x to $F'(x)$ in $B_x \cap W_x$ and v_y from y to $F'(y)$ in $B_y \cap W_y$. Moreover, by the choice of δ_0, there is a homotopy $H \in S(X\times I, X)$, $H(\cdot, 0) = F'$, $H(\cdot, 1) = F = g\circ f$ such that, for any $t \in I$, $H(\cdot, t) \in a(\Psi, \alpha_0)$. In the diagrams of paths



any loop is homotopic rel$\{0, 1\}$ to a constant loop and $v_x * H(x, \cdot) \subset B_x$, $v_y * H(y, \cdot) \subset B_y$. Hence $v \approx F\circ v$ end (B_x, B_y). □

The relation "^{D}" is non empty. Moreover, we prove

(4.2) Theorem. *The relation "^{D}" is an equivalence.*

Proof. Let $x \in \mathrm{Fi}\,x(\Psi)$. By (1.1), there is a $\delta>0$ such that $a(\gamma, \delta) \circ a(\varphi, \delta) \subset a(\Psi, \beta_0)$, $(\beta_0=\beta_0(\Psi))$. Let $f \in a(\varphi, \delta)$, $g \in a(\gamma, \delta)$ and $F=g \circ f$. Obviously, x, $F(x) \in \Psi(x, \beta_0)$. By (3.1)(ii), there is a path v from x to $F(x)$ in $\Psi(x, 2^{-1}\alpha_0) \subset B_x$. The homotopy $h \in S(X \times I, X)$, $h(s, t)=v(t)$, shows that $x^{\sim} \approx F \circ x^{\sim}$ end(B_x, B_x) and $x \sim^D x$.

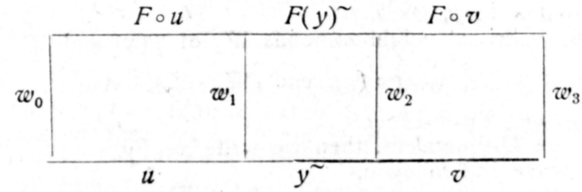
If $x \sim^D y$, then, reversing the path, we easily see that $y \sim^D x$.

Now, assume that $x \sim^D y$, $y \sim^D z$, $x, y, z \in \mathrm{Fi}\,x(\Psi)$. There are paths u from x to y, v from y to z and a number $\varepsilon>0$ such that

$$u \approx F \circ u \ \mathrm{end}\,(B_x, B_y), \quad v \approx F \circ v \ \mathrm{end}\,(B_y, B_z)$$

for any $F=g \circ f$ where $g \in a(\gamma, \varepsilon)$, $f \in a(\varphi, \varepsilon)$. Let $h_1, h_2 \in S(I \times I, X)$ join u and $F \circ u$, v and $F \circ v$, respectively. Let $w_i=h_1(i, .)$, $i=0, 1$, $w_j=h_2(j-2, .)$, $j=2,3$. The loop $y^{\sim} * w_2 * F(y)^{\sim} * w_1^{-}$ lies in B_y, so by (3.1)(i), it is homotopic to a constant one.

The diagram



shows that $x \sim^D z$. □

The equivalence classes of the relation "$\sim^D$" are called **fixed point classes of** (Ψ, D) and denoted by $\boldsymbol{\Psi}_D$ (below we shall omit the subscript D unless it leads to ambiguity).

(4.3) Theorem. *Any fixed point class* $\boldsymbol{\Psi}$ *of* (Ψ, D) *is an open subset of the set* $\mathrm{Fi}\,x(\Psi)$.

Proof. Let $x \in \boldsymbol{\Psi}$. By the ULC property of the space X, there is a number $\eta>0$ such that any point $y \in B(x, \eta) \cap \mathrm{Fi}\,x(\Psi)$ is joined with x by a path u lying in $B(x, \beta_0)$. Observe that $u(I) \subset \Psi(x, \beta_0)$. Choose an $\varepsilon>0$ such that $a(\gamma, \varepsilon) \circ a(\varphi, \varepsilon) \subset a(\Psi, \beta_0)$ and let $F=g \circ f$ where $g \in a(\gamma, \varepsilon)$, $f \in a(\varphi, \varepsilon)$. For each $t \in I$, $F \circ u(t) \in \Psi(u(t), \beta_0) \subset \Psi(x, \alpha_0)$. Obviously, x, $F(x) \in \Psi(x, \beta_0)$, y, $F(y) \in \Psi(y, \beta_0)$.

By (3.1)(ii), there are paths w_x from x to $F(x)$ and w_y from y to $F(y)$ lying in $\Psi(x, 2^{-1}\alpha_0)$ and $\Psi(y, 2^{-1}\alpha_0)$. But $\Psi(x, 2^{-1}\alpha_0) \cup \Psi(y, 2^{-1}\alpha_0) \subset \Psi(x, \alpha_0)$. The loop $u * w_y * (F \circ u)^{-} * w_x^{-}$ lies in $\Psi(x, \alpha_0)$, so $x \sim^D y$ since $u \approx F \circ u$ end(B_x, B_y). Thus $B(x, \eta) \cap \mathrm{Fi}\,x(\Psi) \subset \boldsymbol{\Psi}$. □

Recall that as Ψ is upper semi-continuous $\mathrm{Fi}\,x(\Psi)$ is compact; hence we have

(4.4) Corollary. *The number of all fixed point classes of* (Ψ, D) *is finite. These classes are mutually disjoint and compact.* □

The fixed point class $\boldsymbol{\Psi}$ of (Ψ, D) is said to be D-**essential** if, for each open neighbourhood V of $\boldsymbol{\Psi}$ such that $\mathrm{cl}(V) \cap \mathrm{Fi}\,x(\Psi)=\boldsymbol{\Psi}$, $\mathrm{ind}(X, \Psi, D, V) \neq 0$. This definition does not depend on the choice of V in view of the excision property of the index.

By the **Nielsen number of the pair** (Ψ, D) we understand the number $N(\Psi, D)$ of D-essential fixed point classes of (Ψ, D).

(4.5) Corollary. *Any admissible map* $\Psi: X \to X$ *with a decomposition* D *has at least* $N(\Psi, D)$ *fixed points.* □

5. Homotopy invariance of $N(\cdot,\cdot)$. In this section we are going to prove that the

Nielsen number $N(\Psi, D)$ is homotopy invariant and $N(\Psi, D)=\bar{N}(\Psi, D)$. As above, let X be a space and let $\Psi \in A(X, X)$ have a decomposition $D: X \xrightarrow{\varphi} Y \xrightarrow{\gamma} X$.

Let $\mathbf{\Psi}_1, \ldots, \mathbf{\Psi}_n$ be all fixed point classes of (Ψ, D). There exists a number $\xi_0=\xi_0(\Psi)$, $0<\xi_0<\beta_0(\Psi)$, such that $B(\Psi_i, \xi_0) \cap B(\mathbf{\Psi}_j, \xi_0)=\emptyset$ for $i \neq j$, $1 \leq i, j \leq n$, and such that for any $x \in X$ whatever $a, b \in B(x, \xi_0)$, may be joined by a path in $B(x, 2^{-1}\alpha_0)$.

(5.1). Lemma. *There is a number* $0<\varepsilon_1=\varepsilon_1(\Psi)<\delta_0(\Psi)$ *such that, for* $f \in a(\varphi, \varepsilon_1)$, $g \in a(\gamma, \varepsilon_1)$ *and an arbitrary fixed point class* $\mathbf{F}$ *of* $F=g \circ f$ (see [2] or [4]), *there is* $i \in \{1, 2, \ldots, n\}$ *such that* $\mathbf{F}=B(\mathbf{\Psi}_i, \xi_0) \cap \operatorname{Fi} x(F)$.

Proof. By (1.2), there is $\mu<\xi_0$ such that if $G \in a(\Psi, \mu)$, then $\operatorname{Fi} x(G) \subset B(\operatorname{Fi} x(\Psi), \xi_0)$. Choose an $\varepsilon_1=\varepsilon_1(\Psi)<\delta_0(\Psi)$ such that $a(\gamma, \varepsilon_1) \circ a(\varphi, \varepsilon_1) \subset a(\Psi, \mu)$. Let $F=g \circ f$ where $g \in a(\gamma, \varepsilon_1)$, $f \in a(\varphi, \varepsilon_1)$. Denote by $\mathbf{F}$ an arbitrary fixed point class of F and take $x' \in \mathbf{F}$, $y' \in \operatorname{Fi} x(F)$. There are points $x, y \in \operatorname{Fi} x(\Psi)$ such that $x' \in B(x, \xi_0)$, $y' \in B(y, \xi_0)$. Hence $x, x', F(x) \in \Psi(x, \beta_0)$, $y, y', F(y) \in \Psi(y, \beta_0)$. There are paths u_x from x to x' in $B(x, 2^{-1}\alpha_0)$ and u_y from y to y' in $B(y, 2^{-1}\alpha_0)$. By (3.1)(ii), we have paths w_x from x to $F(x)$ in $\Psi(x, 2^{-1}\alpha_0)$ and w_y from y to $F(y)$ in $\Psi(y, 2^{-1}\alpha_0)$. Observe that the paths $F \circ u_x$, $F \circ u_y$ lie in $\Psi(x, \alpha_0)$ and $\Psi(y, \alpha_0)$, respectively.

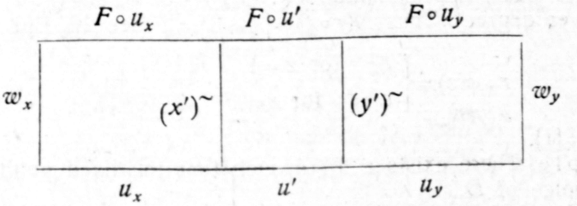
Let $i \in \{1, 2, \ldots, n\}$ be such that $x \in \Psi_i$. We shall end the proof by showing that:

(i) if $y' \in \mathbf{F}$, then $y \in \mathbf{\Psi}_i$; and

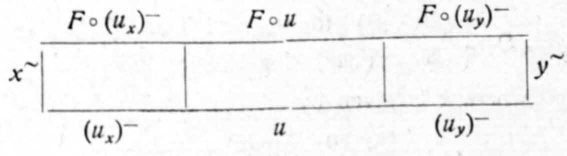
(ii) if $y \in \mathbf{\Psi}_i$, then $y' \in \mathbf{F}$.

Ad (i). There is a path u' from x' to y' such that $u' \approx F \circ u'$ rel$\{0, 1\}$.

The diagram



shows that $u_x * u' * u_y \approx F \circ (u_x * u' * u_y)$ end(B_x, B_y), hence $x \sim^D y$, $y \in \mathbf{\Psi}_i$. Ad (ii). Since $x \sim^D y$, therefore, by lemma (4.1), there is a path u from x to y such that $u \approx F \circ u$ end(B_x, B_y). The diagram



shows that $w \approx F \circ w$ rel$\{0, 1\}$, $(w=(u_x)^- * u * (u_y)^-)$, i. e. $x \sim y$, $y \in \mathbf{F}$. □

(5.2) Theorem. $N(\Psi, D)=\bar{N}(\Psi, D)$.

Proof. Let $\varepsilon<\varepsilon_0(\Psi, B(\operatorname{Fi} x(\Psi), \xi_0))$ and $\varepsilon<\varepsilon_1(\Psi)$. By (3.4), $\bar{N}(\Psi, D)=N(F)$ where $F=g \circ f$, $g \in a(\gamma, \varepsilon)$, $f \in a(\varphi, \varepsilon)$. Let $\mathbf{\Psi}$ be a D-essential class of (Ψ, D). Then, by (3.2), $\operatorname{ind}(X, \Psi, D, B(\mathbf{\Psi}, \xi_0))=\operatorname{ind}(X, F, B(\mathbf{\Psi}, \xi_0)) \neq 0$. Hence there is $x \in B(\mathbf{\Psi}, \xi_0) \cap \operatorname{Fi} x(F)$. Let $\mathbf{F}$ be a fixed point class of F that contains x. By (5.1), we find a class $\mathbf{\Psi}'$ of (Ψ, D) such that $\mathbf{F}=B(\mathbf{\Psi}', \xi_0) \cap \operatorname{Fi} x(F)$. Therefore $\mathbf{\Psi}=\mathbf{\Psi}'$ and $\mathbf{F}$ is essential. Thus $\mathbf{\Psi} \to B(\mathbf{\Psi}, \xi_0) \cap \operatorname{Fi} x(F)$ defines a bijection between the sets of all essential fixed point classes of (Ψ, D) and F. (For its surjectivity see (5.1)). □

To prove the homotopy invariance of the Nielsen number $N(\Psi, D)$, we shall need the following lemma which is proved in [8].

(5.3) Lemma. *Let* $\varphi \in J(X\times I, Y)$. *For any* $\varepsilon>0$, $t\in I$, *there is* $\lambda>0$ *such that for* $h\in a(\varphi, \lambda)$, $h(\cdot, t)\in a(\varphi(\cdot, t), \varepsilon)$. □

(5.4) Theorem. *The Nielsen number* $N(\Psi, D)$ *is homotopy invariant.*

Proof. Let $\Psi_0, \Psi_1 \in A(X, X)$ and let $D_0: X\xrightarrow{\varphi_0} Y'\xrightarrow{\gamma_0} X$, $D_1: X\xrightarrow{\varphi_1} Y''\xrightarrow{\gamma_1} X$ be decompositions of Ψ_0, Ψ_1, respectively. Assume that $(\Psi_0, D_0)\approx(\Psi_1, D_1)$, i. e. $\varphi_0\approx J\varphi_1$, $\gamma_0\approx J\gamma_1$ and $Y'=Y''$. We have $\varphi\in J(X\times I, Y')$ and $\gamma\in J(Y'\times I, X)$ such that $\varphi(\cdot, i)=\varphi_i$ and $\gamma(\cdot, i)=\gamma_i$ for $i=0,1$.

By (5.2), (3.3), there is $\varepsilon>0$ such that $N(\Psi_i, D_i)=N(g_i\circ f_i)$ for arbitrary $g_i\in a(\gamma_i, \varepsilon)$, $f_i\in a(\varphi_i, \varepsilon)$, $i=0, 1$. By (5.3) take $\lambda>0$ such that, for $h\in a(\varphi, \lambda)$ and $k\in a(\gamma, \lambda)$, $h(\cdot, i)\in a(\varphi_i, \varepsilon)$ and $k(\cdot, i)\in a(\gamma_i, \varepsilon)$, $i=0, 1$. Such maps h, k exist in view of (2.1)(i). Obviously, $N(\Psi_0, D_0)=N(k(\cdot, 0)\circ h(\cdot, 0))=N(k(\cdot, 1))\circ h(\cdot, 1))=N(\Psi_1, D_1)$. □

6. Examples. In this section we shall show several examples.

(6.1) Example. First, we shall prove that condition (A.2) in the definition of an admissible map is really not superfluous.

Let $\mathbf{C}$ be the complex plane, $X=S^1\subset\mathbf{C}$, where S^1 is the unit circle, and $Y=\{(e^{6\pi it}, e^{2\pi it})\in\mathbf{C}\times\mathbf{C}\mid t\in I\}$.Define a J-map $\varphi: X\to Y$ by

$$\varphi(z)=\begin{cases}\{(e^{6\pi it}, e^{2\pi it})\mid t\in[0,2/3]\} & \text{for } z=1\\ (e^{6\pi i(t^2+2)/3}, e^{2\pi i(t^2+2)/3}) & \text{for } z=e^{2\pi it},\ t\in(0, 1)\end{cases}$$

and $r\in S(Y, X)$ by $r(z, s)=z$. Observe that the pair (Y, r) is a 3-fold covering of X. For a sufficiently close approximation $f\in S(X, Y)$ of φ, $\deg(r\circ\varphi)=3$ (by deg(.) we denote the Brouwer degree). Thus $N(r\circ f)=|1-3|=2$ (see [2]). But

$$r\circ\varphi(z)=\begin{cases}S^1 & \text{for } z=1\\ e^{2\pi it^2} & \text{for } z=e^{2\pi it},\ t\in(0, 1).\end{cases}$$

Hence $\mathrm{Fix}(r\circ\varphi)=\{1\}$.

(6.2) Example. There exists a $\Psi\in A$ such that the fixed point classes of (Ψ, D) depend on the choice of D.

Let $X=S^1\times I$ and define maps $\varphi\in J(X, X)$, f, $g\in S(X, X)$ by the formulae

$$\varphi(z, t)=\begin{cases}(z, t) & \text{for } 0\leq t<1/3 \text{ or } 2/3<t\leq 1\\ \{ze^{2\pi is}\mid 0\leq s\leq 1/2\}\times\{t\} & \text{for } 1/3\leq t\leq 2/3,\end{cases}$$

$$f(z, t)=\begin{cases}(k(z), t^2) & \text{for } 0\leq t<1/3 \text{ or } 2/3<t\leq 1\\ (k(z)e^{6\pi it}, t^2) & \text{for } 1/3\leq t\leq 2/3,\end{cases}$$

and $g(z, t)=(k(z), t^2)$ where k is given by

$$k(z)=\begin{cases}z^4, & 0\leq \mathrm{Arg}(z)\leq 3\pi/2\\ z^{-8}, & 3\pi/2\leq \mathrm{Arg}(z)\leq 2\pi.\end{cases}$$

It is easy to see that the diagrams $D: X\xrightarrow{\varphi} X\xrightarrow{f} X$ and $D': X\xrightarrow{\varphi} X\xrightarrow{g} X$ are decompositions of the same map $\Psi\in A(X, X)$. Let $x=(1,0)$, $y=(1, 1)$. Then $x, y\in\mathrm{Fix}(\Psi)$ and $x\sim^{D'}y$. We shall show that x is not D-equivalent to y.

First, observe that $(z, t)\in\varphi(z, t)$ for any $(z, t)\in X$. Thus, for each $\varepsilon>0$, $F=f\circ\mathrm{id}_X\in a(\Psi, \varepsilon)$ and $x, y\in\mathrm{Fix}(F)$. Suppose to the contrary that $x\sim^{D}y$. Then, by (5.1), x, y belong to the same fixed point class of F, so there is a path u from x to y such that $u\approx F\circ u$ rel$\{0, 1\}$. If $r: X\to S^1$ is given by $r(z, t)=z$, then the maps $r\circ u$ and $r\circ F\circ u$ are homotopic to each other. But this is a contradiction since, as it is easy to compute $\deg(r\circ F\circ u)=\deg(r\circ u)+1$ (the Brouwer degree is homotopy invariant). □

Let X be a space. We shall say that a map $\Psi \in A(X, X)$ is **strongly admissible** if condition (A.2) holds for each $x \in X$ (not only for $x \in \mathrm{Fix}(\Psi)$).

(6.3) Proposition. *If Ψ: $X \to X$ is strongly admissible, then the fixed point classes of (Ψ, D) do not depend on D.*

Proof. Using the strong admissibility of Ψ, we prove (comp. (3.1)) that there are numbers $\alpha_1 = \alpha_1(\Psi) < \alpha_0(\Psi)$, $\beta_1 = \beta_1(\Psi) < 2^{-1}\alpha_1$ such that: for $x \in X$, $\Psi(x, \alpha_1)$ is trivial and any two points from $\Psi(x, \beta_1)$ may be joined by a path in $\Psi(x, 2^{-1}\alpha_1)$. Let D_i: $X \xrightarrow{\varphi_i} Y_i \xrightarrow{\gamma_i} X$, $i=0, 1$, be two distinct decompositions of Ψ. Take $x, y \in \mathrm{Fix}(\Psi)$ and assume that there is a path u from x to y. By (1.1), there is $\varepsilon > 0$ such that, for $f_i \in a(\varphi_i, \varepsilon)$, $g_i \in a(\gamma_i, \varepsilon)$, $F_i = g_i \circ f_i \in a(\Psi, \beta_1)$, $i=0, 1$. Hence, for any $t \in I$, $F_i(u(t)) \in \Psi(u(t), \beta_1)$. By compactness, there are points $t_0 = 0 < t_1 < \ldots < t_n = 1$ such that, for each $j = 1, \ldots, n$, $F_i \circ u([t_{j-1}, t_j]) \subset \Psi(u(\tau_j), \beta_1)$ for some $\tau_j \in I$. Additionally, we assume that $u(\tau_j) \in B(u(\tau_{j-1}), 2^{-1}\alpha_1)$ for $j = 1, \ldots, n+1$, where $\tau_0 = 0$ and $\tau_{n+1} = 1$. Points $F_0 \circ u(t_j)$ and $F_1 \circ u(t_j)$ may be joined by a path w_j in $\Psi(u(\tau_j), 2^{-1}\alpha_1)$. Points $F_0 \circ u(t_{j-1})$ and $F_1 \circ u(t_{j-1})$ may be joined by a path v_j in $\Psi(u(\tau_j), 2^{-1}\alpha_1)$. Hence the loops $w_{j-1} * (v_j)^-$ and $(F_0 \circ u | [t_{j-1}, t_j]) * w_j * (F_1 \circ u | [t_{j-1}, t_j])^- * (v_j)^-$ are homotopic rel$\{0, 1\}$ to constant loops. Thus we see that

$$F_0 \circ u \approx F_1 \circ u \ \mathrm{end}\,(B_x, B_y),$$

which shows that $x \sim^{D_0} y$ if and only if $x \sim^{D_1} y$. □

(6.4) Example. There exists a strongly admissible map Ψ such that $N(\Psi, D)$ depends on the choice of D.

Indeed, define maps $\varphi' \in J(S^1, S^1)$, f', $g' \in S(S^1, S^1)$ by the formulae:

$$\varphi'(z) = \{z e^{2\pi i t} \mid 0 \leq t \leq 1/2\},$$

$$f'(z) = z^2$$

$$g'(z) = \begin{cases} z^4 & \text{for } 0 \leq \mathrm{Arg}\,(z) \leq 3\pi/2 \\ z^{-16} & \text{for } 3\pi/2 \leq \mathrm{Arg}\,(z) \leq 2\pi \end{cases}$$

and let $f, g \in S(S^2, S^2)$ be the suspensions of f', g', respectively.

Next, let φ be the suspension of φ' (i. e. φ maps the segment $[e_-, z, e_+]$ linearly onto the join of $\varphi'(z)$ with e_-, e_+, where e_- is the south pole and e_+ the north pole of S^2). Then $\varphi \in J(S^2, S^2)$, and D: $S^2 \xrightarrow{\varphi} S^2 \xrightarrow{f} S^2$, D': $S^2 \xrightarrow{\varphi} S^2 \xrightarrow{g} S^2$ are decompositions of the same strongly admissible map Ψ: $S^2 \to S^2$, $\Psi(x) = S^2$ for each $x \in S^2$. Obviously, $S^2 = \mathrm{Fix}(\Psi)$ is the single fixed point class of (Ψ, D) and (Ψ, D'). By (3.2), $\mathrm{ind}\,(S^2, \Psi, D, S^2) = \mathrm{ind}\,(S^2, f \circ id, S^2) = 1 + \deg(f') = 3$ and $\mathrm{ind}\,(S^2, \Psi, D', S^2) = 1 + \deg(g') = 0$. Thus $N(\Psi, D) = 1$ and $N(\Psi, D') = 0$. □

(6.5) Proposition. *If Ψ is a J-map, then $N(\Psi, D)$ does not depend on the choice of D.*

The proof follows easily from (1.1), (2.1)(ii), (3.3) and (5.2). □

Finally let us note that the notions of the Nielsen theory given above, related to a single-valued map will be equivalent to the classical ones. This is a simple consequence of (5.1) and (5.2).

REFERENCES

1. C. Berge. Espaces Topologiques. Fonctions Multivoques. Paris, 1959.
2. Boju Jiang. Lectures on Nielsen Fixed Point Theory. (Contemporary Math., vol. 14. AMS). Providence, 1983.
3. K. Borsuk. Theory of shape. Warszawa, 1975.

4. R. F. Brown. The Lefschetz Fixed Point Theorem. London, 1971.
5. J. Dugundji. Modified Vietoris theorems for homotopy. *Fund. Math.*, **114**, 1970, 223-235.
6. Z. Dzedzej. Fixed point index theory for a class of nonacyclic multivalued maps. *Dissertationes Math.*, **253**, 1985.
7. L. Górniewicz, A. Granas, W. Kryszewski. Sur la méthode de l'homotopie dans le théorie des points fixes pour les applications multivoques. (Partie 1, 2) *C. R. Acad. Sci. Paris*, **307**, 1988, 489-492; **308**, 1989, 449-452.
8. L. Górniewicz, A. Granas, W. Kryszewski. Approximation methods in the fixed point theory of multi-valued mappings of compact ANR's. University of N. Copernicus — preprint, 1987.
9. J. Jezierski. The Nielsen relation for multivalued maps. *Serdica Bulg. Math. Publ.*, **13**, 1987, 174-181.
10. H. Schirmer. A Nielsen number for fixed points and near points of small multifunctions. *Fund. Math.*, 88, 1975, 145-156.
11. H. Schirmer. An index and a Nielsen number for *n*-valued multifunctions. *Fund. Math.*, **124**, 1984, 207-219.

Institute of Mathematics
University of Łódż,
Banacha 22, 90-238, Łódż POLAND
Institute of Mathematics
Nicholas Copernicus University
Chopina 12/18,87-100 Toruń, POLAND

Received 24. 03. 89