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INTEGRALS AND FOURIER-BESSEL EXPANSIONS FOR PRODUCTS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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The aim of this paper is to evaluate an integral involving Bessel functions, generalized hypergeometric series and Fox's H-function and to use it for evaluating double integral involving Bessel functions, generalized hypergeometric series and the H-function. Further we use this integral to find a Fourier-Bessel expansion and a double Fourier-Bessel expansion for the products of generalized hypergeometric series and the H-function.

We shall deal with particular cases of our results and show how they reflect in generalizing many assertions some of which are new and others have been given earlier by S. D. Bajpai [5].

1. Introduction. The subject of expansion formulae and Fourier series of generalized hypergeometric functions is of great importance in the literature on special functions. Some expansion formulae and Fourier series of the generalized hypergeometric functions play an important role in the development of the theories of special functions and boundary value problems. It is interesting to note that there is a wide range of applications of the theory of expansion theorems and Fourier series to the fields of boundary value problems and applied mathematics. For example, some of the results in this paper can be used to obtain some solutions of the partial differential equation concerning the problem of free oscillations of water in a circular lake [16, pp. 45-47, 19, pp. 202-203].

The Fourier-Bessel expansions for generalized hypergeometric functions have been obtained by Bajpai [1—5], Goyal [11] and Taxak [20, 21]. The references given in this paper [14, 15, 19] provide good information on the subject. However, it is important to note that so far nobody has tried to find out the single and multiple Fourier-Bessel expansions for the products of the generalized hypergeometric functions. Therefore this paper appears to be one of the first attempts on the subject of single and multiple Fourier-Bessel expansions for the products of generalized hypergeometric functions.

The Fox's H-function is a generalization of Meijer's G-function [7, pp. 206-222] and therefore when specializing the parameters, the H-function may be reduced to almost all special functions appearing in pure and applied mathematics [15, pp. 144-159[. That is why the results obtained in this paper are of quite general nature. They are master or key formulae from which a large number of results for Meijer's G-function, MacRobert's E-function, hypergeometric functions, Bessel functions, Legendre functions, Whittaker functions, orthogonal polynomials, trigonometric functions and other related functions can be derived.

It is very important to note that most of the operations such as differentiation and integration could be performed more easily on the H-function than on the original functions, even though both are equivalent. Thus the H-function facilitates the analysis by permitting complicated expressions to be represented and handled more simply.

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The H-function introduced by F o x [9, p. 408] will be represented and defined as follows:

$$(1.1)\qquad H_{p,q}^{u,v}\left[z\,\middle|\,\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] = H_{p,q}^{u,v}\left[z\,\middle|\,\begin{matrix}(a_1, e_1), ---, (a_p, e_p)\\(b_1, f_1), ---, (b_q, f_q)\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_L X(s) z^s ds,$$

where L is a suitable Barnes contour and

$$X(s) = \frac{\prod_{j=1}^{u}\Gamma(b_j - f_j s)\prod_{j=1}^{v}\Gamma(1 - a_j + e_j s)}{\prod_{j=u+1}^{q}\Gamma(1 - b_j + f_j s)\prod_{j=v+1}^{p}\Gamma(a_j - e_j)}.$$

The asymptotic expansion and the analytic continuation of the H-function have been given by B r a a k s m a [6].

The following formulae are required in the proofs:

$$(1.2)\qquad \int_0^\infty x^{\sigma-1} J_m(ax) K_m(ax)\, {}_pF_Q\left[\begin{matrix}\alpha_p;\ cx^h\\ \beta_Q\end{matrix}\right] dx$$

$$= \frac{2^{\sigma-2}}{a^\sigma}\sum_{r=0}^{\infty}\frac{(\alpha_p)_r\, C^r\Gamma(\frac{\sigma+hr}{2})\Gamma(\frac{\sigma+hr}{4}+\frac{m}{2})}{(\beta_Q)_r\, r!\ \Gamma(1-\frac{\sigma+hr}{4}+\frac{m}{2})}\left(\frac{2}{a}\right)^{hr},$$

where α_p denotes $\alpha_1, ---, \alpha_p$; h is a positive integer; $P \leq Q$; none of β_Q is zero or a negative integer; $\operatorname{Re}\sigma > -2\operatorname{Re} m$, $|\arg a| < \pi/4$.

The integral (1.2) can easily be established by expressing the generalized hypergeometric series as [7, p. 181, (1)] and interchanging the order of integration and summation involved in the process, which is justified due to the absolute convergence of the integral and summation involved in the process and evaluating the integral with the help of [8, p. 333, (41)].

$$(1.3)\qquad \int_0^\infty x^{\sigma-1} J_m(ax) K_m(ax)\, {}_pF_Q\left[\begin{matrix}\alpha_p;\ cx^h\\ \beta_Q\end{matrix}\right] {}_uF_v\left[\begin{matrix}\nu_u;\ dx^w\\ \delta v\end{matrix}\right] dx$$

$$= \frac{2^{\sigma-2}}{a^\sigma}\sum_{r,t=0}^{\infty}\frac{(\alpha_p)_r\, c^r(\nu_u)_t\, d^t\Gamma(\frac{\sigma+hr+wt}{2})\Gamma(\frac{\sigma+hr+wt}{4}+\frac{m}{2})}{(\beta_Q)_r\, r!\ (\delta_v)_t\, t!\ \Gamma(1-\frac{\sigma+hr+wt}{4}+\frac{m}{2})}\left(\frac{2}{a}\right)^{hr+wt},$$

where in addition to the condition and the notations of (1.2), k is a positive integer; $U \leq V$, no one of the δ_v is zero or a negative integer.

To derive (1.3), we use the series representation for ${}_uF_v$, interchange the order of integration and summation and evaluate the resulting integral with the help of (1.2).

N o t e 1. By using the above procedure we can easily derive an integral analogous to (1.2) for the products of the n generalized hypergeometric series.

The orthogonality property of Bessel functions [13, p. 291, (6)] is

$$(1.4)\qquad \int_0^\infty x^{-1} J_{\nu+2n+1}(x)\, J_{\nu+2m+1}(x)\, dx$$

$$= \begin{cases}0, & \text{if } m \neq n;\\ (4n+2\nu+2)^{-1}, & \text{if } m = n,\ \operatorname{Re}\nu + m + 1 > -1.\end{cases}$$

Next for brevity we denote also by λ and μ some positive numbers and

$$\Phi(r) = \frac{(\alpha_p)_r\, c^r}{(\beta_Q)_r r\,!};\quad \psi(t) = \frac{(\nu_u)_t d^t}{(\delta_v)_t t\,!};$$

$$F_1(x) = {}_pF_Q\left[\begin{matrix}\alpha p\,;\; cx^h\\ \beta_Q\end{matrix}\right];\quad F_2(x) = {}_uF_v\left[\begin{matrix}\nu_u\,;\; dx^w\\ \delta_v\end{matrix}\right];$$

$$H(x) = H^{u,\,v}_{p,\,q}\left[zx^{4\lambda}\left|\begin{matrix}(a_p,\, e_p)\\ (b_q,\, f_q)\end{matrix}\right.\right];$$

$$H_1(m, r, t) = H^{u,\,v+2}_{p+3,\,q}\left[\left(\frac{2}{a}\right)^{4\lambda}_{z}\left|\begin{matrix}(1-\frac{\sigma+hr+wt}{2},\, 2\lambda),\ (1-\frac{(\sigma+hr+wt}{4}-\frac{m}{2},\, \lambda),\\ (a_p,\, e_p),\ (1-\frac{\sigma+hr+wt}{4}+\frac{m}{2},\, \lambda);\\ (b_q,\, f_q)\end{matrix}\right.\right];$$

$$H_2(xy) = H^{u,\,v}_{p,\,q}\left[zx^{4\lambda}y^{4\mu}\left|\begin{matrix}(a_p,\, e_p)\\ (b_q,\, f_q,\end{matrix}\right.\right];$$

$$H_3(m_1,\, m_2,\, r_1,\, t_1,\, r_2,\, t_2)$$

$$= H^{u,\,v+4}_{p+6,\,q}\left[\left(\frac{2}{a}\right)^{4(\lambda+\mu)} z\left|\begin{matrix}(1-\frac{\sigma_1+hr_1+wt_1}{2},\, 2\lambda),\ (1-\frac{\sigma_1+hr_1+wt_1}{4}-\frac{m_1}{2},\, \lambda),\\ (1-\frac{\sigma_2+hr_2+wt_2}{2},\, 2\mu),\ (1-\frac{\sigma_2+hr_2+wt_2}{2}-\frac{m_2}{2},\, \mu)\\ (a_p,\, e_p),\ (1-\frac{\sigma_1+hr_1+wt_1}{4}+\frac{m_1}{2},\, \lambda),\\ (1-\frac{\sigma_2+hr_2+wt_2}{2}+\frac{m_2}{2},\, \mu);\\ (b_q,\, f_q)\end{matrix}\right.\right]$$

$$A = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j;\quad B = \sum_{j=1}^{v} a_j - \sum_{j=v+1}^{p} a_j + \sum_{j=1}^{u} b_j - \sum_{j=u+1}^{q} b_j.$$

2(i) Integral. The integral to be evaluated is

(2.1)
$$\int_0^\infty x^{\sigma-1}J_m(ax)K_m(ax)\,F_1(x)F_2(x)H(x)dx$$

$$= \frac{2^{\sigma-2}}{a^\sigma}\sum_{r,t=0}^{\infty}\Phi(r)\psi(t)\left(\frac{2}{a}\right)^{hr+wt}H_1(m,\, r,\, t),$$

where $A \leq 0$, $B > 0$, $|\arg z| < \frac{1}{2}B\pi$, $|\arg a| < \pi/4$, $\mathrm{Re}\,(\sigma + \lambda b_j/f_j) > -2\,\mathrm{Rem}$, $j = 1, ---, u$, together with the conditions given in (1.2) and (1.3).

P r o o f. To establish (2.1), expressing the H-function in the integrand as a Mellin-Barnes type integral (1.1) and interchanging the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L X(s)z^s\int_0^\infty x^{\sigma+4\lambda s-1}J_m(ax)K_m(ax)F_1(x)F_2(x)dxds.$$

Evaluating the inner integral through (1.3), we get

$$\frac{2^{\sigma-2}}{a^\sigma}\sum_{r,\,t=0}^{\infty}\Phi(r)\psi(t)\left(\frac{2}{a}\right)^{hr+wt}$$

$$\times \frac{1}{2\pi i} \int_L \chi(s) \frac{\Gamma(\frac{\sigma+hr+wt}{2}+2\lambda s)\,\Gamma(\frac{\sigma+hr+wt}{4}+\frac{m}{2}+\lambda s)}{\Gamma(1-\frac{\sigma+hr+wt}{4}+\frac{m}{2}-\lambda s)} \left(\frac{2}{a}\right)^{4\lambda s} z^s ds.$$

Now, using (1.1), the value of the integral (2.1) is obtained.

Note 2. An integral analogous to (2.1), involving the product of n generalized hypergeometric series, the Bessel function and the H-function can be evaluated easily by means of the result mentioned in Note 1.

(ii) Particular cases. In (2.1), putting $d=0$, we get

$$(2.2)\qquad \int_0^\infty x^{\sigma-1} J_m(ax) K_m(ax) F_1(x) H(x) dx = \frac{2^{\sigma-2}}{a^\sigma} \sum_{r=0}^\infty \Phi(r) \left(\frac{2}{a}\right)^{hr} H_1(m, r, 0),$$

which holds for $d=0$ under the conditions of (2.1).

It is interesting to note that Singh and Varma [18] evaluated an integral involving the product of an associated Legendre function, a generalized hypergeometric series and the H-function [15, p. 40, (2. 9. 4)] by using the finite difference operator E [17, p. 33 with $w=1$]. It is also interesting to note that Gupta and Olkha [12] evaluated an integral involving the product of a generalized hypergeometric series and the H-function using an integral due to Goyal [10, p. 202].

Srivastava, Gupta and Goyal [19, pp. 61-63] presented some integrals based on the technique of Gupta and Olkha.

Thus having in mind the above discussion our integral appears to be newer and a more general one compared with [14, 15, 19] due to the new and simple technique of evaluating such integrals.

In (2.2), setting $c=0$, we obtain

$$(2.3)\qquad \int_0^\infty x^{\sigma-1} J_m(ax) K_m(ax) H(x) dx = \frac{2^{\sigma-2}}{a^\sigma} H_1(m, 0, 0),$$

which holds for $c=0$ under the conditions of (2.2).

In (2.3), assuming λ as a positive integer, putting $e_j=f_i=1$ ($j=1, ---, p$; $i=1, ---, q$) using [15, p. 10, (1. 7. 1)], viz.

$$(2.4)\qquad H_{p,q}^{u,v}\left[z \,\middle|\, \begin{matrix}(a_p, 1)\\(b_q, 1)\end{matrix}\right] = G_{p,q}^{u,v}\left[z \,\middle|\, \begin{matrix}a_p\\b_q\end{matrix}\right],$$

and simplifying by means of (1.1), [7, p. 4, (11)] and [7, p. 207, (1)], a result given by Bajpai [5, p. 37, (2.2)] is yielded.

3 (i) Double integral. The double integral to be evaluated is

$$(3.1)\quad \int_0^\infty \int_0^\infty x^{\sigma_1-1} y^{\sigma_2-1} J_{m_1}(ax) J_{m_2}(ay) K_{m_1}(ax) K_{m_2}(ay) F_1(x) F_2(x) F_1(y) F_2(y) H_2(xy) dx dy$$

$$= \frac{2^{\sigma_1+\sigma_2-4}}{a^{\sigma_1+\sigma_2}} \sum_{r_1, t_1=0}^\infty \sum_{r_2, t_2=0}^\infty \Phi(r_1)\psi(t_1)\Phi(r_2)\psi(t_2)$$

$$\times \left(\frac{2}{a}\right)^{h(r_1+r_2)+w(t_1+t_2)} H_3(m_1, m_2, r_1, t_1, r_2, t_2),$$

where $A \leq 0$, $B>0$, $|\arg z| < \frac{1}{2}B\pi$, $|\arg a| < \pi/4$, $\mathrm{Re}(\sigma_1+\lambda b_j/f_j) > -2\,\mathrm{Re}\,m_1$, $\mathrm{Re}(\sigma_2+\mu b_j/f_j) > -2\,\mathrm{Re}\,m_2$, $j=1, ---, u$, together with the conditions of (1.2).

Proof. To establish (3.1), evaluating the x-integral through (2.1) and interchanging the order of integration and summation, we get

$$\frac{2^{\sigma_1-2}}{a^{\sigma_1}} \sum_{r_1, t_1=0}^{\infty} \Phi(r_1)\psi(t_1)\left(\frac{2}{a}\right)^{hr_1+wt_1}$$

$$\times \int_0^\infty y^{\sigma_2-1} J_{m_2}(ay)K_{m_2}(ay)F_1(y)F_2(y)$$

$$\times H^{u,\, v+2}_{p+3,\, q}\left[\left(\frac{2}{a}\right)^{4\lambda} zy^{4\mu} \middle| \begin{matrix} (1-\frac{\sigma_1+hr_1+wt_1}{2},\ 2\lambda),\ (1-\frac{\sigma_1+hr_1+wt_1}{4}-\frac{m_1}{2},\ \lambda), \\ (a_p,\ e_p),\ (1-\frac{\sigma_1+hr_1+wt_1}{4}+\frac{m_1}{2},\ \lambda); \\ (b_q,\ f_q) \end{matrix}\right] dy.$$

Now, applying (2.1) to evaluate the y-integral, the value of the integral (3.1) is obtained.

Note 3. The multiple analogous to (3.1) can be obtained easily by applying the above technique $(n-1)$ times.

(ii) Particular cases. Putting $d=0$ in (3.1), we get

(3.2) $$\int_0^\infty \int_0^\infty x^{\sigma_1-1} y^{\sigma_2-1} J_{m_1}(ax)K_{m_1}(ax)J_{m_2}(ay)K_{m_2}(ay)F_1(x)F_1(y)H_2(xy)dxdy$$

$$= \frac{2^{\sigma_1+\sigma_2-4}}{a^{\sigma_1+\sigma_2}} \sum_{r_1, r_2=0}^{\infty} \Phi(r_1)\Phi(r_2)\left(\frac{2}{a}\right)^{hr_1+wt_1} H_3(m_1, m_2, r_1, 0, r_2, 0),$$

which holds for $d=0$ under the conditions of (3.1).

In (3.2), setting $c=0$, we obtain

(3.3) $$\int_0^\infty \int_0^\infty x^{\sigma_1-1} y^{\sigma_2-1} J_{m_1}(ax)K_{m_1}(ax)J_{m_2}(ay)K_{m_2}(ay)H_2(xy)dxdy$$

$$= \frac{2^{\sigma_1+\sigma_2-4}}{a^{\sigma_1+\sigma_2}} H_3(m_1, m_2, 0, 0, 0, 0),$$

which holds for $c=0$ under the conditions of (3.2).

Note 4. The integrals in this sections may be used to find out the double and multiple Fourier-Bessel series for the products of generalized hypergeometric series and the H-function.

4(i) Fourier-Bessel series. The Fourier-Bessel series to be found is as follows:

(4.1) $$x^{\sigma} K^{(ax)}_{\nu+2m+1} F_1(x)F_2(x)H(x)$$

$$= \frac{2^{\sigma-1}}{a^{\sigma}} \sum_{n=0}^{\infty} NJ_N(ax) \sum_{r,t=0}^{\infty} \Phi(r)\psi(t)\left(\frac{2}{a}\right)^{(hr+wt)} H_1(N, r, t),$$

where $N=\nu+2n+1$, $\mathrm{Re}\, N>0$ and the other conditions of validity are the same as in (2.1).

Proof. To prove (4.1), let

(4.2) $$f(x) = x^{\sigma}F_1(x)F_2(x)H(x) = \sum_{n=0}^{\infty} C_n J^{(ax)}_{\nu+2n+1}.$$

Since $f(x)$ is continuous and of bounded variation in the open interval $(0, \infty)$, then the equation (4.1) holds.

Multiplying both sides of (4.2) by $x^{-1}J^{(ax)}_{\nu+2m+1}$ and integrating with respect to x from 0 to ∞, we get

$$\int_0^\infty x^{\sigma-1} J^{(ax)}_{\nu+2m+1} K^{(ax)}_{\nu+2m+1} F_1(x) F_2(x) H(x) dx = \sum_{n=0}^\infty C_n \int_0^\infty x^{-1} J^{(ax)}_{\nu+2m+1} J^{(ax)}_{\nu+2n+1} dx.$$

Now using (2.1) and (1.4), we get

$$C_m = \frac{2^{\sigma-1}}{a^\sigma} M \sum_{r,\, t=0}^\infty \Phi(r)\psi(t)\left(\frac{2}{a}\right)^{(hr+wt)} H_1(M, r, t), \tag{4.3}$$

where $M = \nu + 2m + 1$.

The Fourier-Bessel series (4.1) follows from (4.2) and (4.3).

(ii) Particular cases. In (4.1), putting $d = 0$, we obtain

$$x^\sigma K^{(ax)}_{\nu+2m+1} F_1(x) H(x) = \frac{2^{\sigma-1}}{a^\sigma} \sum_{n=0}^\infty N J_N(ax) \sum_{r=0}^\infty \Phi(r)\left(\frac{2}{a}\right)^{hr} H_1(N, r, 0), \tag{4.4}$$

which holds for $d = 0$ under the conditions of (4.1).

In (4.4), setting $c = 0$, we get

$$x^\sigma K^{(ax)}_{\nu+2m+1} H(x) = \frac{2^{\sigma-1}}{a^\sigma} \sum_{n=0}^\infty N H_1(N, 0, 0) J_N(ax), \tag{4.5}$$

which holds for $c = 0$ under the conditions of (4.4). If in (4.1), we assume that λ is a positive integer, put $e_j = f_i = 1 (j = 1, ---p;\ i = 1, ---, q)$, $a = 1$, use (2.4) and simplify by means of (1.1), [7, p. 4, (11)] and [7, p. 207, (1)], then 4.1 is reduced to a result given by B a j p a i [5, p. 37, (3.2)].

5 (i) Double Fourier-Bessel series. The double Fourier-Bessel series to be found is

$$\begin{aligned} & x^{\sigma_1} K^{(ax)}_{\nu_1+2m_1+1} y^{\sigma_2} K^{(ay)}_{\nu_2+2m_2+1} F_1(x) F_2(x) F_1(y) F_2(y) H_2(xy) \\ & = \frac{2^{\sigma_1+\sigma_2-2}}{a^{\sigma_1+\sigma_2}} \sum_{n_1,\, n_2=0}^\infty N_1 N_2 J_{N_1}(ax) J_{N_2}(ay) \\ & \times \sum_{r_1,\, t_1=0}^\infty \sum_{r_2,\, t_2=0}^\infty \Phi(r_1)\psi(t_1)\Phi(r_2)\psi(t_2)\left(\frac{2}{a}\right)^{(hr_1+wt_1+hr_2+wt_2)} H_3(N_1, N_2, r_1, t_1, r_2, t_2), \end{aligned} \tag{5.1}$$

where $N_1 = \nu_1 + 2n_1 + 1$, $N_2 = \nu_2 + 2n_2 + 1$ and it holds for the conditions in (3.1)

P r o o f. To obtain (5.1), let

$$\begin{aligned} f(x, y) &= x^{\sigma_1} K^{(ax)}_{\nu_1+2m_1+1} y^{\sigma_2} K^{(ay)}_{\nu_2+2m_2+1} F_1(x) F_2(x) F_1(y) F_2(y) H_2(xy) \\ &= \sum_{n_1,\, n_2=0}^\infty A_{n_1, n_2} J^{(ax)}_{\nu_1+2n_1+1} J^{(ay)}_{\nu_2+2n_2+1}. \end{aligned} \tag{5.2}$$

Since $f(x, y)$ is continuous and of bounded variation in the open interval $(0, \infty)$, then equation (5.2) holds.

Series (5.2) is an example of what is called a double Fourier-Bessel series. Next we show a method of finding A_{n_1, n_2} from (5.2) instead of discussing the theory. For a fixed x, we note that $\sum_{n_1=0}^\infty A_{n_1, n_2} J^{(ax)}_{\nu_1+2n_1+1}$ depends only on n_2, which should be the Fourier-Bessel series coefficient in y of $f(x, y)$ over $0 < y < \infty$.

Multiplying both sides of (5.2) by $y^{-1} J^{(ay)}_{\nu_2+2m_2+1}$, integrating with respect to y from 0 to ∞ and using (2.1) and (1.4), we get

$$x^{\sigma_1} K^{(ax)}_{\nu_1+2m_1+1} F_1(x) F_2(x) \frac{2^{\sigma_2-1}}{a^{\sigma_2}} M_2 \sum_{r_2,\, t_2=0}^\infty \Phi(r_2)\psi(t_2)\left(\frac{2}{a}\right)^{(hr_2+wt_2)} \tag{5.3}$$

$$\times H_{p+3,\,q}^{u,\,v+2}\left[\left(\frac{2}{a}\right)^{4\mu} x^{4\lambda} z \,\middle|\, \begin{matrix} (1-\frac{\sigma_2+hr_2+wt_2}{2},\ 2\mu),\ (1-\frac{\sigma_2+hr_2+wt_2}{2}-\frac{M_2}{2},\ \mu), \\ (a_p,\ e_p),\ (1-\frac{\sigma_2+hr_2+wt_2}{4}+\frac{M_2}{2},\ \mu); \\ (b_q,\ f_q) \end{matrix}\right]$$

$$= \sum_{n_1=0}^{\infty} A_{n_1,\,m_2} J_{\nu_1+2n_1+1}^{(ax)},$$

where $M_2 = \nu_{2+2m_2+1}$.

Multiplying both sides of (5.3) by $x^{-1}J_{\nu_1+2m_1+1}^{(ax)}$, integrating with respect to x from 0 to ∞ and using (2.1) and (1.4), we obtain

(5.4)
$$A_{m_1,\,m_2} = \frac{2^{\sigma_1+\sigma_2+2}}{a^{\sigma_1+\sigma_2}} M_1 M_2 \sum_{r_1,\,t_1=0}^{\infty} \sum_{r_2,\,t_2=0}^{\infty} \Phi(r_1)\psi(t_1)\Phi(r_2)\psi(t_2)$$
$$\times \left(\frac{2}{a}\right)^{(hr_1+\omega t_1+hr_2+\omega t_2)} H_3(M_1,\ M_2,\ r_1,\ t_1,\ r_2,\ t_2),$$

where $M_1 = \nu_1 + 2m_1 + 1$.

From (5.2) and (5.4), the double Fourier-Bessel series (5.1) is obtained.

(ii) Particular cases. In (5.1), putting $d = 0$, we get

(5.5)
$$x^{\sigma_1} K_{\nu_1+2m_1+1}^{(ax)}\, y^{\sigma_2} K_{\nu_2+2m_2+1}^{(ay)} F_1(x) F_1(y) H_2(xy)$$
$$= \frac{2^{\sigma_1+\sigma_2-2}}{a^{\sigma_1+\sigma_2}} \sum_{n_1,\,n_2=0}^{\infty} N_1 N_2 J_{N_1}(ax)\, J_{N_2}(ay)$$
$$\times \sum_{r_1,\,r_2=0}^{\infty} \Phi(r_1)\Phi(r_2) \left(\frac{2}{a}\right)^{h(r_1+r_2)} \times H_3(N_1,\ N_2,\ r_1,\ 0,\ r_2,\ 0),$$

which holds for $d = 0$ under the conditions in (5.1).

In (5.5), taking $c = 0$, we obtain

(5.6)
$$x^{\sigma_1} K_{\nu_1+2m_1+1}^{(ax)}\, y^{\sigma_2} K_{\nu_2+2m_2+1}^{(ay)} H_2(xy)$$
$$= \frac{2^{\sigma_1+\sigma_2-2}}{a^{\sigma_1+\sigma_2}} \sum_{n_1,\,n_2=0}^{\infty} N_1 N_2 H_3(N_1,\ N_2,\ 0,\ 0,\ 0,\ 0) J_{N_1}(ax)\, J_{N_2}(ay),$$

which holds for $c = 0$ under the conditions in (5.5).

Note 5. Multiple Fourier-Bessel series analogous to (5.1) can be obtained by applying the above technique repeatedly.

Note 6. The results analogous to our main results (2.1), (3.1), (4.1) and (5.1) involving the H-function of several complex variables [19, pp. 251-255] can be derived easily by following the technique given in this paper.

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