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ON THE TWO DIMENSIONAL WHITTAKER TRANSFORM

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This paper deals with a new theorem concerning the Whittaker transform of two variables. The result is derived by the application of two dimensional Erdélyi-Kober operators of Weyl type.

1. Introduction and Preliminaries. Following K. Miller [7, p. 82], let us denote by A the class of functions $f(x,y)$ which are differentiable any number of times and let they and all their partial derivatives be

$$0(|x|^{-\xi_1}, |y|^{-\xi_2}) \text{ for all } \xi_i\,(i=1,2) \text{ as } x\to\infty,\ y\to\infty.$$

With some modifications the Erdélyi-Kober operators of Weyl type in two dimensions of a function f are defined as follows:

$$K_x^{\eta,\alpha} K_y^{\delta,\beta} f(x,y)=\frac{(-1)^{m+n}x^\eta y^\delta}{\Gamma(m+\alpha)\Gamma(n+\beta)} D_{x,y}^{m+n}\int_x^\infty\int_y^\infty u^{-\eta-\alpha}v^{-\delta-\beta}(u-x)^{m+\alpha-1} \tag{1.1}$$

$$\times(v-y)^{n+\beta-1} f(u,v)\,du\,dv,$$

provided that $f(x,y)\in A$; α,β are real and $m,n=0,1,2,\ldots$; where $D_{x,y}^{m+n}$ stands for the operator $\partial^{n+m}/\partial x^m\,\partial y^n$.

For $\alpha>0$, $\beta>0$, $m=n=0$, (1.1) becomes a two-dimensional fractional integration operator:

$$K_x^{\eta,\alpha} K_y^{\delta,\beta} f(x,y)=\frac{x^\eta y^\delta}{\Gamma(\alpha)\Gamma(\beta)}\int_x^\infty\int_y^\infty u^{-\eta-\alpha}v^{-\delta-\beta}(u-x)^{\alpha-1}(v-y)^{\beta-1}f(u,v)\,du\,dv. \tag{1.2}$$

If we assume that $\alpha<0$, $\beta<0$ and m, n are positive integers such that $\alpha+m>0$ $\beta+n>0$, then (1.1) will yield the partial fractional derivatives of $f(x,y)$.

The Laplace transform $h(p,q)$ of a function f is defined as in [2].

$$h(p,q)=\mathscr{L}[f(x,y);\ p,q]=\int_0^\infty\int_0^\infty \mathrm{epx}(-px-qy)f(x,y)\,dx\,dy. \tag{1.3}$$

Analogously, the Laplace transform of $f(a\sqrt{x^2-b^2}, c\sqrt{y^2-d^2})$ is defined by the Laplace transform of $F(x,y)$, where

$$F(x,y)=\begin{cases} f(a\sqrt{x^2-b^2}, c\sqrt{y^2-d^2});\ x>b>0;\ y>d>0,\\ 0,\ \text{otherwise}.\end{cases} \tag{1.4}$$

We now define

$$h_1(p,q)=\mathscr{L}\{F(x,y);\ p,q\}=\int_b^\infty\int_d^\infty \exp(-px-qy)f(a\sqrt{x^2-b^2}, c\sqrt{y^2-d^2})\,dx\,dy, \tag{1.5}$$

where $R(p)>0$, $R(q)>0$.

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The Whittaker transform of two variables, $g(p,q)$ of a function F is defined by

$$(1.6)\qquad g(p,q)=W^{\lambda,\mu}_{\lambda_1,\mu_1}[F(x,y);\ \rho,\sigma,p,q]=\int_b^\infty\int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1}\exp\left(-\frac{1}{2}px-\frac{1}{2}qy\right)$$

$$\times W_{\lambda,\mu}(px)\,W_{\lambda_1,\mu_1}(qy)\,F(x,y)\,dx\,dy,$$

where $R(p)>0$, $R(q)>0$, g exists and belongs to A. Here $W_{\lambda,\mu}(z)$ is Whittaker's confluent hypergeometric function defined by [10, p. 340]

$$(1.7)\qquad \times W_{\lambda,\mu}(z)=\frac{e^{-\frac{1}{2}z}\,z^k}{\Gamma(\frac{1}{2}-\lambda+\mu)}\int_0^\infty t^{-\lambda-\frac{1}{2}+\mu}\left(1+\frac{t}{z}\right)^{\lambda+\mu-\frac{1}{2}}e^{-t}\,dt,$$

where $R(\frac{1}{2}-\lambda+\mu)>0$.

Before presenting the theorem in the next section, we need the generalized Whittaker transform $h_1(p,q)$ of F defined by

$$(1.8)\qquad g_1(p,q)=G^{\lambda,\alpha,\eta,\mu}_{\lambda_1,\beta,\delta,\mu_1}[F(x,y);\ \rho,\sigma,p,q]=\int_b^\infty\int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1}G^{30}_{23}\left(px\,\middle|\,\begin{matrix}1-\lambda,\eta+\alpha-\rho+1\\ \eta+1-\rho,\frac{1}{2}+\mu,\frac{1}{2}-\mu\end{matrix}\right)$$

$$\times G^{30}_{23}\left(qy\,\middle|\,\begin{matrix}1-\lambda_1,\delta+\beta-\sigma+1\\ \delta+1-\sigma,\frac{1}{2}+\mu,\frac{1}{2}-\mu_1\end{matrix}\right)F(x,y)\,dx\,dy,$$

where $g_1(p,q)$ exists and belongs to $A, R(p)>0$, $R(q)>0$. Here the function $G^{30}_{23}(z)$ in (1.8) is Meijer's G-function.

In general, the G-function is defined by C. S. Meijer [6] by means of the Mellin-Barnes integral

$$(1.9)\qquad G^{m,n}_{p,q}(z)=G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)=\frac{1}{2\pi i}\int_L \chi(s)\,z^s\,ds,$$

where $i=(-1)^{1/2}$, $z\neq 0$,

$$(1.10)\qquad \chi(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)},$$

an empty product is interpreted as unity; $b_j,(j=1,\dots,q)$, $a_j,(j=1,\dots,p)$ are complex numbers such that none of the poles of $\Gamma(b_j-s)$, $j=1,\dots,m$ coincide with any of the poles of $\Gamma(1-a_j+s)$, $j=1,\dots,n$. The contour L separates these two sets of poles. General existence conditions are available from A. Mathai and R. Saxena [5] or from Y. Luke [4].

The object of this paper is to establish a theorem on the Whittaker transform of two variables which extends the results due to R. Saxena et al. [9], A. Arora et al. [1] and R. Raina and V. Kiryakova [8].

2. Theorem. *Let*

$$(2.1)\qquad g(p,q)=W^{\lambda,\mu}_{\lambda_1,\mu_1}[F(x,y);\rho,\sigma,p,q]=\int_b^\infty\int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1}\exp\left\{-\frac{1}{2}(px+qy)\right\}$$

$$\times W_{\lambda,\mu}(px)\,W_{\lambda_1,\mu_1}(qy)\,F(x,y)\,dx\,dy.$$

be the two-dimensional Whittaker transform, then for $\alpha>0$, $\beta>0$, *the following result holds*:

(2.2) $$K_p^{\eta,\alpha} K_q^{\delta,\beta}[g(p,q)]=G_{\lambda_1,\beta,\delta,\mu_1}^{\lambda,\alpha,\eta,\mu}[F(x,y);\rho,\sigma,p,q],$$

where R.H.S. of (2.2) *is defined by* (1.8).

Proof: Let $\alpha>0$, $\beta>0$. Then in view of (1.2) and (1.6), we find that

$$K_p^{\eta,\alpha} K_q^{\delta,\beta}[g(p,q)]=\frac{p^\eta q^\delta}{\Gamma(\alpha)\Gamma(\beta)}\int_p^\infty\int_q^\infty u^{-\eta-\alpha}v^{-\delta-\beta}(u-p)^{\alpha-1}(v-q)^{\beta-1}$$

$$\times g(u,v)\,du\,dv=\frac{p^\eta q^\delta}{\Gamma(\alpha)\Gamma(\beta)}\int_p^\infty\int_q^\infty u^{-\eta-\alpha}v^{-\delta-\beta}(u-p)^{\alpha-1}(v-q)^{\beta-1}$$

$$\times\Big[\int_b^\infty\int_d^\infty(ux)^{\rho-1}(vy)^{\sigma-1}\exp\Big(-\frac{1}{2}ux-\frac{1}{2}vy\Big)W_{\lambda,\mu}(ux)\,W_{\lambda_1,\mu_1}(vx)F(x,y)\,dx\,dy\Big]\,du\,dv.$$

On changing the order of integrations which is permissible and evaluating the inner integrals through the integral [3, p. 212, eq. 76]

$$\int_p^\infty x^{-\rho_1}(x-p)^{\sigma_1-1}e^{-\frac{1}{2}ax}W_{\lambda,\mu}(ax)\,dx=\Gamma(\sigma_1)p^{\sigma_1-\rho_1}G_{23}^{30}\Big(ax\,\Big|\,{\rho_1,\,1-\lambda \atop \rho_1-\sigma_1,\,\frac{1}{2}+\mu,\,\frac{1}{2}-\mu}\Big),$$

where $R(\sigma_1)>0$, we obtain the following result:

3. Corollary 1. *Let*

(3.1) $$g_2(p,q)=W_{\lambda_1,\mu_1}^{\lambda,\mu}\Big[F(x,y);\ \mu+\frac{1}{2},\ \mu_1+\frac{1}{2},\ p,q\Big]$$

$$=\int_b^\infty\int_d^\infty(px)^{\mu-\frac{1}{2}}(qy)^{\mu_1-\frac{1}{2}}\exp\Big(-\frac{1}{2}px-\frac{1}{2}qy\Big)\,W_{\lambda,\mu}(px)\,W_{\lambda_1,\mu_1}(qy)\,F(x,y)\,dx\,dy$$

exists and belongs to A, then for $\alpha>0$, $\beta>0$, *the following interesting result holds*:

(3.2) $$K_p^{\alpha,-\alpha}K_q^{\beta,-\beta}[g_2(p,q)]=K_p^{\alpha,-\alpha}K_q^{\beta,-\beta}\Big\{W_{\lambda_1,\mu_1}^{\lambda,\mu}\Big[F(x,y);\mu+\frac{1}{2},\ \mu_1+\frac{1}{2},\ p,q\Big]\Big\}$$

$$=W_{\lambda_1-\frac{\beta}{2},\,\mu_1+\frac{\beta}{2}}^{\lambda-\frac{\alpha}{2},\,\mu+\frac{\alpha}{2}}\Big[F(x,y);\ \mu-\frac{\alpha}{2}+\frac{1}{2},\ \mu_1-\frac{\beta}{2}+\frac{1}{2},\ p,q\Big].$$

Corollary 2. *Let*

(3.3) $$g_3(p,q)=W_{\lambda_1\mu_1}^{\lambda,\mu}[F(x,y);\ \eta+\lambda,\ \delta+\lambda_1,\ p,q]$$

$$=\int_b^\infty\int_d^\infty(px)^{\eta+\lambda-1}(qy)^{\delta+\lambda_1-1}\exp\Big(-\frac{1}{2}px-\frac{1}{2}qy\Big)W_{\lambda,\mu}(px)\,W_{\lambda_1,\mu_1}(qy)\,F(x,y)\,dx\,dy$$

exists and belongs to A, then for $\alpha>0$, $\beta>0$, *the following interesting result holds*:

(3.4) $$K_p^{\eta,\alpha}K_q^{\delta,\beta}[g_3(p,q)]=K_p^{\eta,\alpha}K_q^{\delta,\beta}\{W_{\lambda_1,\mu_1}^{\lambda,\mu}[F(x,y);\ \eta+\lambda,\ \delta+\lambda_1,\ p,q]\}$$

$$=W_{\lambda_1-\beta,\,\mu_1}^{\lambda-\alpha,\,\mu}[F(x,y);\ \eta+\lambda,\ \delta+\lambda_1,\ p,q].$$

Next, if we take $\rho=\sigma=1$ and use the identity

$$W_{m+\frac{1}{2},\,\pm m}(x)=x^{m+\frac{1}{2}}e^{-\frac{1}{2}x},$$

the two-dimensional Whittaker transform reduces to a two-dimensional Laplace transform and consequently, we have a result recently given by Saxena et al. [9]. Further if we take $\eta=-\alpha$, $\delta=-\beta$, we obtain the result due to Arora et al. [1] which itself is a generalization of the result given by R. Raina and V. Kiryakova [8] to which it reduces for $a=c=1$, $b=d=0$.

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