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ON THE ADMISSIBLE CONTROLS FOR SINGULARLY PERTURBED LINEAR SYSTEMS*

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A linear control system containing a real parameter ε in some derivatives is considered. The multivalued map "$\varepsilon \rightrightarrows$ set of admissible controls" is investigated, where admissible controls are those that drive the state to a target set at fixed final time. As an application an estimate for the marginal function of an optimal control problem is obtained.

1. Introduction. Consider a singularly perturbed control system with a "small" nonlinearity on the right-hand side

$$(1)\qquad \begin{aligned} \dot{x} &= A_1(t)x + A_2(t)y + B_1(t)u + \varepsilon f(x,u,t), \quad x(0)=x^0,\\ \varepsilon\dot{y} &= A_3(t)x + A_4(t)y + B_2(t)u + \varepsilon g(x,u,t), \quad y(0)=y^0, \end{aligned}$$

where $x \in R^n$, $y \in R^m$, $u \in R^k$, $t \in [0,T]$, $T>0$ and $\varepsilon>0$.

The set of controls is

$$U=\{u(\cdot) \text{ — measurable, } u(t)\in V \text{ for a. e. } t\in[0,T]\},\ V\subset R^k.$$

Let $C_\varepsilon \subset R^{m+n}$ be a "target" set, depending on $\varepsilon \geq 0$. We denote by $(x_\varepsilon(u)(\cdot), y_\varepsilon(u)(\cdot))$, $\varepsilon>0$, any solution of (1) corresponding to the control $u \in U$. For $\varepsilon>0$ let $F(\varepsilon)$ be the set of admissible controls, i. e. $F(\varepsilon)$ consists of all $u\in U$ that drive the corresponding solution of (1) at the moment T to C_ε, i. e.

$$F(\varepsilon)=\{u\in U \mid (x_\varepsilon(u)(T),\ y_\varepsilon(u)(T))\in C_\varepsilon\}.$$

In this paper the properties of the multivalued map $\varepsilon \rightrightarrows F(\varepsilon)$ at $\varepsilon=0$ are investigated.

Taking $\varepsilon=0$ in (1), one can formally define the set $F(0)$. However, in this case the map $F(\varepsilon)$ is not upper semicontinuous at $\varepsilon=0$. Therefore, in Section 2 we derive $F(0)$ in a special way and obtain that $F(\varepsilon)$ is α-upper Lipschitzian at $\varepsilon=0$ with $\alpha\in(0,1)$.

In Section 3 we suppose that $f\equiv g\equiv 0$ and the "target" set does not depend on ε (i. e. $C_\varepsilon = C$, $\varepsilon\geq 0$). An estimate of the Hausdorff distance between $F(\varepsilon)$ and $F(0)$ is derived. This estimate is as that for the attainable set of (1) found in [1].

In Section 4 we present an application to optimal control. It is proved that the optimal value of an optimal control problem is α-upper Lipschitzian at $\varepsilon=0$ for every $\alpha\in(0,1)$.

2. An upper estimate. In the sequel we use Dontchev's result [2] which we present in a form appropriate for our purpose.

Let $L^p(0,T)$, $1\leq p\leq\infty$, be the usual spaces of functions with p-integrable norm and let $|\cdot|$ be the Euclidean norm in R^l and $\|\cdot\|_p$ be the norm in $L^p(0,T)$. Denote by B the closed unit ball in R^{n+m} and by $\mathrm{proj}_x C$ the projection of $C\subset R^{n+m}$ on R^n. Let $(X,\|\cdot\|)$ be a linear normed space, $U\subset L^\infty(0,T)$ and let $P\colon U\rightrightarrows R^{n+m}$ and $Q\colon U\rightrightarrows R^{n+m}$ be multivalued maps. Given $C,D\subset X$ and $d\in X$ denote

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$$\mathrm{dist}\,(d,C)=\inf\{\|c-d\|\mid c\in C\},\ d(D,C)=\sup\{\mathrm{dist}\,(d,C)\mid d\in D\},$$
$$\mathrm{hausd}\,(D,C)=\mathrm{Hausdorff}\,(D,C)=\max\,(d(C,D),\ d(D,C)),$$
$$\rho(Q,P)=\sup\{d(Q(u),\ P(u))\mid u\in U\},\ P(U)=\bigcup P(u),\ u\in U,$$
$$\mathrm{dom}\ P=\{u\in U\mid P(u)\neq\emptyset\},\ \mathrm{diam}\ (U)=\sup\{\|u-v\|\mid u,v\in U\}.$$

If $X=R^{n+m}$, we use the same notations for dist $(\cdot,\cdot)$, $d(\cdot,\cdot)$, hausd $(\cdot,\cdot)$ and diam $(\cdot)$, but when $X=L^p(0,T)$ we add an index p, i. e. $\mathrm{dist}_p(\cdot,\cdot)$, $d_p(\cdot,\cdot)$, $\mathrm{hausd}_p(\cdot,\cdot)$ and $\mathrm{diam}_p(\cdot)$.

We are interested in the properties of the map

$$F(P,C)=\{u\in U\mid P(u)\cap C\neq\emptyset\}.$$

Theorem 1. *Let* $C\subset R^{n+m}$ *and* $P.\ U\rightrightarrows R^{n+m}$ *be such that*

(2) *U is bounded and convex, C and graph P are convex.*

(3) *There exists a real number* $a>0$ *such that*

$$(C+y)\cap P(U)\neq\emptyset \text{ for every } y\in aB.$$

Then for every $1\leq p\leq\infty$, *every set* $D\subset R^{n+m}$ *and every multivalued map* $Q\colon U\rightrightarrows R^{n+m}$ *with dom* $Q\subset$ *dom* P

(4) $$d_p(F(Q,D),\ F(P,C))\leq(\mathrm{diam}_p(U)/a)(\rho(Q,P)+d(D,C)).$$

There is an important remark to Theorem 1.

Remark 1. *A sufficient condition for* (3) *is*

(5) $$(P(U)\cap\mathrm{int}\,C)\cup(\mathrm{int}\,P(U)\cap C)\neq\emptyset.$$

We shall use also a proposition ([2]) closely related to Theorem 1.

Proposition. *Let U and P satisfy* (2) *and let the point* $c\in P(U)$. *Suppose that the constant* $a>0$ *and the point* $\bar d\in P(U)$ *are such that*

(6) $$\bar d+a(\bar d-c)/|\bar d-c|\in P(U).$$

Then for every $1\leq p\leq\infty$ *and every* $d=\theta c+(1-\theta)\bar d$, $\theta\in[0,1]$ *we have*

$$d_p(F(P,c),F(P,d))\leq(\mathrm{diam}_p(U)/a)\,|c-d|.$$

Throughout the paper we assume that

(A1). All eigenvalues of $A_4(t)$ have negative real parts for all $t\in[0,T]$ (which is denoted by $\mathrm{Re}\lambda\,(A_4(t))<0,\ t\in[0,T]$). The entries of $A_i(t)$ and $B_j(t)$ are continuous. Moreover, the entries of $A_4(t)$ and $B_2(t)$ are Lipschitzian on $[0,T]$, the entries of $A_2(t)\,A_4^{-1}(t)$ and $A_4^{-1}(t)\,A_3(t)$ are continuously differentiable on $[0,T]$.

It is known from Flatto and Levinson [3] that $\mathrm{Re}\lambda\,(A_4(t))<0,\ t\in[0,T]$ implies existence of positive $\bar\varepsilon$, σ and σ_0 such that

(7) $$|Y_\varepsilon(t,s)|\leq\sigma_0\exp(-\sigma(t-s)/\varepsilon)$$

for all $0<\varepsilon\leq\bar\varepsilon$ and $0\leq s\leq t\leq T$, where $Y_\varepsilon(t,s)$ is the fundamental matrix solution of $\varepsilon\dot y=A_4(t)\,y$, principal at $t=s$.

Condition (A1) implies existence of $A_4^{-1}(t)$. Taking $\varepsilon=0$ in (1), we obtain the so-called reduced system

(8) $$\begin{gathered}\dot x=A_0(t)x+B_0(t)u(t),\ x(0)=x^0,\ u(\cdot)\in U,\\ A_0=A_1-A_2A_4^{-1}A_3,\ B_0=B_1-A_2A_4^{-1}B_2.\end{gathered}$$

Let $R(t)$, $t\in(0,T]$ be the attainable set of (8). Following A. Dontchev and V. Veljov [4], we define

$$K_0(t)=\{(x,y)\in R^{n+m}\mid x\in R(t),\ y\in-A_4^{-1}(t)\,A_3(t)x+\int_0^{+\infty}\exp(A_4(t)s)\,B_2(t)\,V\,ds\},$$

where the integral is Aumann's. As shown in [4], $K_0(t)$ is the Hausdorff limit for $\varepsilon\to0$ of the attainable set $K(\varepsilon,t)$ of (1) at $t\in(0,T]$ when $f\equiv g\equiv0$.

Suppose also that

(A2). V is compact and convex, C_0 is closed and convex and

$$(K_0(T)\cap \operatorname{int} C_0)\cup(\operatorname{int} K_0(T)\cap C_0)\neq\emptyset.$$

(A3). The functions $f(x,v,t)$ and $g(x,v,t)$ are measurable in $t\in[0,T]$ for every $x\in R^n$, $v\in V$ and continuous in $(x,v)\in R^n\times V$ for almost every $t\in[0,T]$. There exists $l(\cdot)\in L^\infty(0,T)$ such that

$$|f(x,v,t)|+|g(x,v,t)|\leq l(t)(1+|x|+|v|)$$

for every $x\in R^n$, $v\in V$ and almost every $t\in[0,T]$.

Let $x_0(u)(\cdot)$ be the solution of reduced system (8) with control $u\in U$ and

$$F(0)=\{u\in U\mid x_0(u)(T)\in \operatorname{proj}_x(K_0(T)\cap C_0)\}.$$

Then the following theorem holds:

Theorem 2. *For every* $\alpha\in(0,1)$ *and every* $1\leq p\leq\infty$ *there exist* $L_1>0$, $\varepsilon_1>0$, *such that for every* $\varepsilon\in(0,\varepsilon_1)$

$$d_p(F(\varepsilon),F(0))\leq L_1(\varphi(\varepsilon,\alpha)+d(C_\varepsilon,C_0)),$$

where $\varphi(\varepsilon,\alpha)=\varepsilon^\alpha+\exp(-\sigma\varepsilon^{\alpha-1})$ *and* σ *is as in* (7).

The following lemma is used in the proofs:

Lemma. *There exists a constant* $M>0$ *such that for any* $\varepsilon\in(0,\bar\varepsilon)$ ($\bar\varepsilon$ *is as in* (7)) *and* $u^1,u^2\in U$ *(possibly depending on* ε)

$$\max_{0\leq t\leq T}|x_\varepsilon(u^1)(t)-x_0(u^2)(t)|\leq M[\varepsilon+\max_{0\leq t\leq T}(|\int_0^t B_1(s)(u^1(s)-u^2(s))\,ds|$$

$$+|\int_0^t A_2(s)A_4^{-1}(s)B_2(s)(u^1(s)-u^2(s))\,ds|)]\cdot$$

Proof. The proof is similar to that of the analogous lemma in [1], but for completeness it is given here. Denote

$$\Delta x=x_\varepsilon^1-x_\varepsilon^2=x_\varepsilon(u^1)(\cdot)-x_0(u^2)(\cdot),\quad y_\varepsilon^2=-A_4^{-1}(A_3x_\varepsilon^2+B_2u^2),$$

$$\Delta y=y_\varepsilon^1-y_\varepsilon^2=y_\varepsilon(u^1)(\cdot)-y_\varepsilon^2(\cdot),\quad \Delta u=u^1-u^2.$$

Then

$$\Delta x(t)=\int_0^t A_1(s)\Delta x(s)\,ds+\int_0^t A_2(s)\Delta y(s)\,ds+\int_0^t B_1(s)\Delta u(s)\,ds+\varepsilon\int_0^t f(x_\varepsilon^1(s),u^1(s),s)\,ds,$$

$$\Delta y(t)=Y_\varepsilon(t,0)\,y^0+\frac{1}{\varepsilon}\int_0^t Y_\varepsilon(t,s)A_3(s)\Delta x(s)\,ds+\frac{1}{\varepsilon}\int_0^t Y_\varepsilon(t,s)B_2(s)\Delta u(s)\,ds$$

$$-\frac{1}{\varepsilon}\int_0^t Y_\varepsilon(t,s)A_4(s)y_\varepsilon^2(s)\,ds-y_\varepsilon^2(t)+\int_0^t Y_\varepsilon(t,s)\,g(x_\varepsilon^1(s),u^1(s),s)\,ds.$$

Furthermore,

$$\Delta x(t)=\int_0^t A_1(s)\Delta x(s)\,ds+\int_0^t B_1(s)\Delta u(s)\,ds+\varepsilon\int_0^t f(x_\varepsilon^1(s),u^1(s),s)\,ds \tag{9}$$

$$+\int_0^t A_2(s)Y_\varepsilon(s,0)\,y^0\,ds+\int_0^t A_2(s)[\frac{1}{\varepsilon}\int_0^s Y_\varepsilon(s,\tau)A_3(\tau)\Delta x(\tau)\,d\tau$$

$$+\frac{1}{\varepsilon}\int_0^s Y_\varepsilon(s,\tau)B_2(\tau)\Delta u(\tau)\,d\tau-\frac{1}{\varepsilon}\int_0^s Y_\varepsilon(s,\tau)A_4(\tau)y_\varepsilon^2(\tau)\,d\tau-y_\varepsilon^2(s)$$

$$+ \int_0^s Y_\varepsilon(s,\tau)\, g(x_\varepsilon^1(\tau), u^1(\tau), \tau)\, d\tau]\, ds.$$

Using (7), (A3) and the boundedness of V, for an arbitrary $u \in U$, we have

$$|x_\varepsilon(u)(t)| \leq M_1(1 + \int_0^t |x_\varepsilon(u)(s)|\, ds + \int_0^t |y_\varepsilon(u)(s)|\, ds),$$

$$|y_\varepsilon(u)(t)| \leq M_2(1 + (1/\varepsilon) \int_0^t \exp(-\sigma(t-s)/\varepsilon)\, |x_\varepsilon(u)(s)|\, ds).$$

Using the second of these inequalities in the first one and applying Gronwall's lemma, we find that $\max_{0 \leq t \leq T} |x_\varepsilon(u)(t)|$ and $\max_{0 \leq t \leq T} |y_\varepsilon(u)(t)|$ are bounded uniformly in $\varepsilon \in (0, \bar\varepsilon)$ and $u \in U$. Then, by (7) and (A3), we get

$$\varepsilon \left| \int_0^t f(x_\varepsilon^1(s), u^1(s), s)\, ds \right| \leq M_3\, \varepsilon, \tag{10}$$

$$\left| \int_0^t \int_0^s A_2(s)\, Y_\varepsilon(s,\tau)\, g(x_\varepsilon^1(\tau), u^1(\tau), \tau)\, d\tau ds \right| \leq M_4 \int_0^t |A_2(s)| \int_0^s \exp(-\sigma(s-\tau)/\varepsilon)\, d\tau ds \leq M_5\, \varepsilon. \tag{11}$$

Also from (7) we derive

$$\left| \int_0^t A_2(s)\, Y_\varepsilon(s,0)\, y^0\, ds \right| \leq M_6\, \varepsilon \tag{12}$$

and

$$\left| (1/\varepsilon) \int_0^t \int_0^s A_2(s)\, Y_\varepsilon(s,\tau)\, A_3(\tau)\, \Delta x(\tau)\, d\tau ds \right|$$

$$\leq (1/\varepsilon)\, M_7 \int_0^t \left(\int_\tau^t |A_2(s)| \exp(-\sigma(s-\tau)/\varepsilon)\, ds\right) |A_3(\tau)|\, |\Delta x(\tau)|\, d\tau \leq M_8 \int_0^t |\Delta x(s)|\, ds. \tag{13}$$

In the sequel we use the following estimate. If $\zeta_\varepsilon(\cdot)$ and $\eta_\varepsilon(\cdot)$ are arbitrary (vector) functions with $\sup_{\varepsilon > 0} \|\zeta_\varepsilon\|_\infty < +\infty$, then

$$\left|(1/\varepsilon) \int_0^t \int_0^s A_2(s)\, Y_\varepsilon(s,\tau)\, \zeta_\varepsilon(\tau)\, d\tau ds + \eta_\varepsilon(t)\right| = \left| \int_0^t \left(\int_\tau^t A_2(s)\, A_4^{-1}(s) \frac{\partial}{\partial s} Y_\varepsilon(s,\tau)\, ds\right) \zeta_\varepsilon(\tau)\, d\tau + \eta_\varepsilon(t) \right|$$

$$\leq \left| \int_0^t A_2(\tau)\, A_4^{-1}(\tau)\, \zeta_\varepsilon(\tau)\, d\tau - \eta_\varepsilon(t) \right| + M_9 \int_0^t |Y_\varepsilon(t,\tau)\, \zeta_\varepsilon(\tau)|\, d\tau$$

$$+ M_{10} \int_0^t \left(\int_\tau^t |Y_\varepsilon(s,\tau)|\, ds\right) |\zeta_\varepsilon(\tau)|\, d\tau \leq M_{11}\varepsilon + \left| \int_0^t A_2(s)\, A_4^{-1}(s)\, \zeta_\varepsilon(s)\, ds - \eta_\varepsilon(t) \right|.$$

Using this inequality, we obtain

$$\left| (1/\varepsilon) \int_0^t \int_0^s A_2(s)\, Y_\varepsilon(s,\tau)\, B_2(\tau)\, \Delta u(\tau)\, d\tau ds \right| \tag{14}$$

$$\leq M_{12}\varepsilon + \max_{0 \leq t \leq T} \left| \int_0^t A_2(s)\, A_4^{-1}(s)\, B_2(s)\, \Delta u(s)\, ds \right|$$

and

$$\left| (1/\varepsilon) \int_0^t \int_0^s A_2(s)\, Y_\varepsilon(s,\tau)\, A_4(\tau)\, y_\varepsilon^2(\tau)\, d\tau ds + \int_0^t A_2(s)\, y_\varepsilon^2(s)\, ds \right| \leq M_{13}\, \varepsilon. \tag{15}$$

Applying (10)—(15) to (9) and using Gronwall's lemma, we complete the proof ■

Proof of Theorem 2. Let for every $u \in U$ $P_\varepsilon(u)=(x_\varepsilon(u)(T), y_\varepsilon(u)(T))$, $\varepsilon>0$ and

$$P_0(u)=x_0(u)(T)\times(-A_4^{-1}(T)A_3(T)x_0(u)(T)+\int_0^{+\infty}\exp(A_4(T)s)B_2(T)V\,ds).$$

We want to apply Theorem 1 to P_ε, $\varepsilon\geq 0$. Obviously dom $P_\varepsilon=$ dom $P_0=U$, $\varepsilon>0$. Moreover, the integral in $P_0(u)$ is a convex set and $x_0(\cdot)(T)$ is a linear map. Hence P_0 has a convex graph and condition (2) in Theorem 1 is fulfilled. By $P_0(U)=K_0(T)$ and Remark 1 condition (3) is also satisfied. Now we shall estimate $\rho(P_\varepsilon, P_0)$, $\varepsilon>0$.

Choose $\alpha\in(0,1)$ and $1\leq p\leq\infty$. Let $\varepsilon_1>0$ be such that $\varepsilon_1\leq\bar\varepsilon$ and $\varepsilon_1^\alpha<T$ (where $\bar\varepsilon$ is as in (7)). Take $u\in U$ and let $\bar v\in V$ be arbitrarily chosen.

Denote

$$v_\varepsilon(t)=\begin{cases} u(T-\varepsilon t), & t\in[0,T/\varepsilon), \\ \bar v, & t\in[T/\varepsilon,+\infty],\ \varepsilon\in(0,\varepsilon_1).\end{cases}$$

Let

$$x_0=x_0(u)(T),\ y_\varepsilon^0=-A_4^{-1}(T)A_3(T)x_0+\int_0^{+\infty}\exp(A_4(T)s)B_2(T)v_\varepsilon(s)\,ds.$$

Then $(x_0, y_\varepsilon^0)\in P_0(u)$. From the Lemma it follows that for every $\varepsilon\in(0,\varepsilon_1)$

$$\max_{0\leq t\leq T}|x_\varepsilon(u)(t)-x_0(u)(t)|\leq M\varepsilon. \tag{16}$$

Now let $\tilde y_\varepsilon$ solve $\varepsilon\dot y=A_4(t)y+B_2(t)u_\varepsilon(t)$, $y(0)=0$ and $\bar y_\varepsilon$ be the solution of $\varepsilon\dot y=A_4(T)y+B_2(T)u_\varepsilon(t)$, $y(0)=0$.

Denote $\Delta y_\varepsilon=\tilde y_\varepsilon-\bar y_\varepsilon$, $\Delta A_4(t)=A_4(t)-A_4(T)$, $\Delta B_2(t)=B_2(t)-B_2(T)$. Then some standard computations give us

$$|\Delta y_\varepsilon(T)|\leq(\sigma_0/\varepsilon)\int_0^{T-\varepsilon^\alpha}\exp(-\sigma(T-t)/\varepsilon)|\Delta A_4(t)\tilde y_\varepsilon(t)+\Delta B_2(t)u_\varepsilon(t)|\,dt \tag{17}$$

$$+(N_1/\varepsilon)\max_{T-\varepsilon^\alpha\leq t\leq T}(|\Delta A_4(t)|+|\Delta B_2(t)|)\int_{T-\varepsilon^\alpha}^{T}\exp(-\sigma(T-t)/\varepsilon)\,dt\leq N_2(\exp(-\sigma\varepsilon^{\alpha-1})+\varepsilon^\alpha),$$

where $N_1>0$, $N_2>0$ are constants. We have

$$|y_\varepsilon(u)(T)-y_\varepsilon^0|\leq|Y_\varepsilon(T,0)y^0|+|\int_0^T Y_\varepsilon(T,t)g(x_\varepsilon(u)(t),u(t),t)\,dt| \tag{18}$$

$$+|(1/\varepsilon)\int_0^T Y_\varepsilon(T,t)A_3(t)x_\varepsilon(t)\,dt+A_4^{-1}(T)A_3(T)x_0|+|\Delta y_\varepsilon(T)|$$

$$+|(1/\varepsilon)\int_0^T\exp(A_4(T)(T-t)/\varepsilon)B_2(T)u(t)\,dt-\int_0^{+\infty}\exp(A_4(T)s)B_2(T)v_\varepsilon(s)\,ds|.$$

From (7)

$$|Y_\varepsilon(T,0)y^0|\leq N_3\varepsilon,\ |\int_0^T Y_\varepsilon(T,t)g(x_\varepsilon(u)(t),u(t),t)\,dt|\leq N_4\varepsilon. \tag{19}$$

After an integration by parts we get

$$|\frac{1}{\varepsilon}\int_0^T Y_\varepsilon(T,t)A_3(t)x_\varepsilon(u)(t)\,dt+A_4^{-1}(T)A_3(T)x_0|\leq N_5(\varepsilon+|x_\varepsilon(u)(T)-x_0|). \tag{20}$$

Furthermore,

$$(21)\qquad |(1/\varepsilon)\int_0^T \exp(A_4(T)(T-t)/\varepsilon)\,B_2(T)\,u(t)\,dt-\int_0^{+\infty}\exp(A_4(T)\,s)\,B_2(T)\,v_\varepsilon(s)\,ds\,|$$

$$\leqq N_6\exp(-\sigma\varepsilon^{\alpha-1}).$$

Finally, from (16)—(21) we obtain

$$|\,y_\varepsilon(u)(T)-y_\varepsilon^0\,|\leqq N_7(\varepsilon^\alpha+\exp(-\sigma\varepsilon^{\alpha-1}))=N_7\varphi(\varepsilon,\alpha).$$

This, together with (16), gives $d(P_\varepsilon(u),P_0(u))\leqq N_8\varphi(\varepsilon,\alpha)$ for every $\varepsilon\in(0,\varepsilon_1)$ and every $u\in U$. The proof is completed ■

Remark 2. Theorem 2 holds for a more general system than (1), namely for the system with nonlinearity depending on y

$$\dot{x}=A_1(t)\,x+A_2(t)\,y+B_1(t)\,u+\varepsilon f(x,y,u,t),\; x(0)=x^0,$$

$$\varepsilon\dot{y}=A_3(t)\,x+A_4(t)\,y+B_2(t)\,u+\varepsilon g(x,y,u,t),\;\; y(0)=y^0.$$

Then we need more restrictive conditions than $(A1)$ and $(A3)$. More precisely, we suppose that in $(A1)$, instead of $\mathrm{Re}\lambda(A_4(t))<0,\ t\in[0,T]$, we have

$(\cdot)$ There exists a constant $\mu>0$ such that for every $y\in R^m$, $t\in[0,T]$

$$\langle y, A_4(t)\,y\rangle\leqq-\mu\,|\,y\,|^2.$$

Instead of (A3), suppose

$(A3')$. The functions $f(x,y,v,t)$ and $g(x,y,v,t)$ are measurable in $t\in[0,T]$ for every $x\in R^n$, $y\in R^m$, $v\in V$ and continuous in $(x,y,v)\in R^{n+m}\times V$ for almost every $t\in[0,T]$. There exists $m(\cdot)\in L^\infty(0,T)$ such that

$$|\,f(x,y,v,t)\,|+|\,g(x,y,v,t)\,|\leqq m(t)(1+|\,x\,|+|\,y\,|+|\,v\,|)$$

for every $x\in R^n$, $y\in R^m$, $v\in V$ and almost every $t\in[0,T]$. Then, by [5], it follows that $\max_{0\leq t\leq T}|\,x_\varepsilon(u)(t)\,|$ and $\max_{0\leq t\leq T}|\,y_\varepsilon(u)(t)\,|$ are bounded uniformly in $\varepsilon\in(0,\bar{\varepsilon})$ and $u\in U$. Then the proofs of the Lemma and Theorem 2 are one and the same.

3. An α-Lipschitz property. In this section we consider the system

$$(22)\qquad \dot{x}=A_1(t)\,x+A_2(t)\,y+B_1(t)\,u,\;\; x(0)=x^0,$$

$$\varepsilon\dot{y}=A_3(t)\,x+A_4(t)\,y+B_2(t)\,u,\;\; y(0)=y^0,\;\; t\in[0,T],\;\; \varepsilon>0,\;\; u\in U$$

with a constant target set C. We prove

Theorem 3. *For every $\alpha\in(0,1)$ and every $1\leqq p<\infty$ there exist constants $L_2>0$, $\varepsilon_2>0$ such that for every $\varepsilon\in(0,\varepsilon_2)$*

$$\mathrm{hausd}_p(F(\varepsilon),F(0))\leqq L_2\varepsilon^\alpha.$$

Proof. Choose $p\in[0,\infty)$ and $\alpha\in(0,1)$. By Theorem 2 it is sufficient to prove only that there exist $L_2>0$, $\varepsilon_2>0$ such that for every $\varepsilon\in(0,\varepsilon_2)$

$$(23)\qquad d_p(F(0),\;F(\varepsilon))\leqq L_2\varepsilon^\alpha.$$

Let $u_0\in F(0)$, i. e. there exists an integrable function $v_0(\cdot)$, $v_0(t)\in V$ for a. e. $t\in[0,+\infty)$ such that if

$$x_0=x_0(u_0)(T),\;\; y_0=-A_4^{-1}(T)\,A_3(T)\,x_0+\int_0^{+\infty}\exp(A_4(T)\,s)\,B_2(T)\,v_0(s)\,ds,$$

then $z_0=(x_0, y_0)\in K_0(T)\cap C$.

Step 1. Let $\bar{\alpha}=\alpha p$ and $\bar{\varepsilon}_2>0$ be such that $\bar{\varepsilon}_2\le\bar{\varepsilon}$ and $\bar{\varepsilon}_2^{\bar{\alpha}}<T$ (where $\bar{\varepsilon}$ is as in (7)). Define the control

$$u_\varepsilon(t)=\begin{cases} u_0(t), & t\in[0, T-\varepsilon^{\bar{\alpha}}), \\ v_0((T-t)/\varepsilon), & t\in[T-\varepsilon^{\bar{\alpha}}, T], \end{cases} \tag{24}$$

for $\varepsilon\in(0,\bar{\varepsilon}_2)$. We shall prove that there exists a constant $G_1>0$ (independent of u_0 and v_0) such that for every $\varepsilon\in(0,\bar{\varepsilon}_2)$

$$|x_0-x_\varepsilon(u_\varepsilon)(T)|\le G_1\varepsilon^{\bar{\alpha}}, \tag{25 a}$$

$$|y_0-y_\varepsilon(u_\varepsilon)(T)|\le G_1\varepsilon^{\bar{\alpha}}. \tag{25 b}$$

The inequality (25 a) follows from the Lemma.

Let Δy_ε be defined as in the proof of Theorem 2 (see the rows between (16) and (17)). With arguments analogous to those in (17)—(21) and using (25 a), we get

$$|\Delta y_\varepsilon(T)|\le G_2(\varepsilon^{\bar{\alpha}}+\exp(-\sigma\varepsilon^{\bar{\alpha}-1}))=G_2\varphi(\varepsilon,\bar{\alpha}),$$

$$|y_\varepsilon(u_\varepsilon)(T)-y_0|\le G_3\varphi(\varepsilon,\bar{\alpha}).$$

Since $\varphi(\varepsilon,\bar{\alpha})\le G_4\varepsilon^{\bar{\alpha}}$ (and G_2-G_4 do not depend on u_0 and v_0), we obtain (25).

Step 2. Suppose that $K_0(T)\cap\operatorname{int} C\neq\emptyset$.

In the sequel, for every $D\subset R^{n+m}$ and $a>0$, we shall denote

$$]D[_a=\{z\in D\mid z+aB\subset D\}.$$

Choose $a>0$ so that $K_0(T)\cap\,]C[_a\neq\emptyset$. Now, let $\varepsilon_2>0$ be such that $\varepsilon_2\le\min(1,\bar{\varepsilon}_2)$ and $2G_1\varepsilon_2^{\bar{\alpha}}\le a$.

Case 1. Let $z_0=(x_0, y_0)\in\,]C[_a$. Since (25) is fulfilled it follows that $(x_\varepsilon(u_\varepsilon)(T), y_\varepsilon(u_\varepsilon)(T))\in C$ for all $\varepsilon\in(0,\varepsilon_2)$. This means that $u\in F(\varepsilon)$. Moreover,

$$\|u_\varepsilon-u_0\|_p\le G_5(\varepsilon^{\bar{\alpha}})^{1/p}=G_5\varepsilon^{\alpha}.$$

where G_5 do not depend on u_0 and v_0. Hence (23) it proved.

Case 2. Now, let $z_0=(x_0, y_0)\in C\setminus\,]C[_a$. We shall use a simple fact which for completeness is presented with a proof.

Sublemma. *Let H and K be closed and convex subsets of R^{n+m} such that $K\cap\operatorname{int} H\neq\emptyset$ and $K\cap H$ be bounded. Then there exists a constant $b_0>0$ such that for every $0<b<b_0$*

$$\operatorname{hausd}(K\cap H,\ K\cap\,]H[_b)\le(\operatorname{diam}(K\cap H)/b_0)\,b.$$

Proof. Take $b_0>0$ such that $K\cap\,]H[_{b_0}\neq\emptyset$. Let $0<b<b_0$ and let $z\in(K\cap H)\setminus\,]H[_b$ and $\bar{z}\in K\cap\,]H[_{b_0}$ be arbitrarily chosen. Let

$$z_b=(1-\lambda_b)z+\lambda_b\bar{z}\in\partial\,]H[_b,$$

where $\partial\,]H[_b$ is the boundary of $]H[_b$. Obviously the point z_b is unique and also $\operatorname{dist}(z_b,\partial H)=b$. We shall prove that

$$z_b+\lambda_b b_0 B\subset H. \tag{26}$$

Let $w_b\in z_b+\lambda_b b_0 B$ and let $\bar{w}_b=\bar{z}+(w_b-z_b)/\lambda_b$. Then $|\bar{w}_b-\bar{z}|\le b_0$, i. e. $\bar{w}_b\in\bar{z}+b_0B\subset H$. But

$$w_b=z_b+\lambda_b(\bar{w}_b-\bar{z})=(1-\lambda_b)z+\lambda_b\bar{w}_b\in H,$$

hence (26) is proved. Therefore $\lambda_b b_0 \leq b$, i. e. $\lambda_b \leq b/b_0$. This means that

$$|z-z_b|=\lambda_b|z-\bar{z}|\leq(\operatorname{diam}(K\cap H)/b_0)\,b$$

and the Sublemma is proved □

Take $\varepsilon\in(0,\varepsilon_2)$ and let $b=2G_1\varepsilon^{\bar{\alpha}}$. Denote by $z_0^\varepsilon=(x_0^\varepsilon, y_0^\varepsilon)$ the point from $K_0(T)\cap\,]C[_b$ for which

$$|z_0-z_0^\varepsilon|=\operatorname{dist}(z_0, K_0(T)\cap\,]C[_b).$$

Then, by the Sublemma, it follows that

$$(27)\qquad |z_0-z_0^\varepsilon|\leq(\operatorname{diam}(K_0(T)\cap C)/a)\,b=G_6\varepsilon^{\bar{\alpha}}.$$

Let $P_0\colon U\rightrightarrows R^{n+m}$ be the set valued map defined in the proof of Theorem 2. Let apply the Proposition for the maps

$$F(P_0, z_0)=\{u\in U \mid z_0\in x_0(u)(T)\times(-A_4^{-1}(T)\,A_3(T)\,x_0(u)(T)+I(T))\},$$

and

$$F(P_0, z_0^\varepsilon)=\{u\in U \mid z_0^\varepsilon\in x_0(u)(T)\times(-A_4^{-1}(T)\,A_3(T)\,x_0(u)(T)+I(T))\},$$

where $I(T)=\int_0^{+\infty}\exp(A_4(T)\,s)\,B_2(T)\,V ds$ and $\varepsilon\in(0,\varepsilon_2)$. Then we find $u_0^\varepsilon\in U$ such that $x_0(u_0^\varepsilon)(T)=x_0^\varepsilon\in\operatorname{proj}_x(K_0(T)\cap C)$, i. e $u_0^\varepsilon=F(0)$, and

$$\|u_0-u_0^\varepsilon\|_p\leq(\operatorname{diam}_p(U)/a)\,|z_0-z_0^\varepsilon|.$$

Using (27), we derive

$$\|u_0-u_0^\varepsilon\|_p\leq(\operatorname{diam}_p(U)/a)\,G_6\varepsilon^{\bar{\alpha}}\leq G_7\varepsilon^{\alpha},$$

where $G_7>0$ does not depend on u_0, v_0 and $\varepsilon\in(0,\varepsilon_2)$. In the above inequality we use the fact that $\varepsilon^{\bar{\alpha}}=\varepsilon^{\alpha p}\leq\varepsilon^{\alpha}$ which follows from $\varepsilon<\varepsilon_2\leq 1$.

An integrable function $v_0^\varepsilon(\cdot), v_0^\varepsilon(t)\in V$ for a. e. $t\in[0,+\infty)$, is associated to y_0^ε Define

$$u_\varepsilon(t)=\begin{cases} u_0^\varepsilon(t), & t\in[0, T-\varepsilon^{\bar{\alpha}}),\\ v_0^\varepsilon((T-t)/\varepsilon), & t\in[T-\varepsilon^{\bar{\alpha}}, T).\end{cases}$$

Then, by Step 1, we find

$$|x_0^\varepsilon-x_\varepsilon(u_\varepsilon)(T)|\leq G_1\varepsilon^{\bar{\alpha}},\quad |y_0^\varepsilon-y_\varepsilon(u_\varepsilon)(T)|\leq G_1\varepsilon^{\bar{\alpha}},$$

so that

$$|z_0^\varepsilon-(x_\varepsilon(u_\varepsilon)(T),\ y_\varepsilon(u_\varepsilon)(T))|\leq 2G_1\varepsilon^{\bar{\alpha}}.$$

Therefore $(x_\varepsilon(u_\varepsilon)(T), y_\varepsilon(u_\varepsilon)(T))\in C$, i. e. $v_\varepsilon\in F(\varepsilon)$. On the other hand,

$$\|u_0-u_\varepsilon\|_p\leq\|u_0-u_0^\varepsilon\|_p+\|u_0^\varepsilon-u_\varepsilon\|_p\leq G_7\varepsilon^{\alpha}+G_5(\varepsilon^{\bar{\alpha}})^{1/p}\leq G_8\varepsilon^{\alpha}$$

and (23) is proved in this case.

Step 3. Now, suppose that $\operatorname{int}K_0(T)\cap C\neq\emptyset$ and choose $a>0$ so that $]K_0(T)[_{3a}\cap C\neq\emptyset$. Take $\varepsilon_2>0$ such that $\varepsilon_2\leq\min(1,\varepsilon_2)$ and $2G_1\varepsilon_2^{\bar{\alpha}}\leq a$ (where G_1 is as in (25)).

Case 1. Let $z_0=(x_0, y_0)\in\,]K_0(T)[_{3a}$. Then for every $\varepsilon\in(0,\varepsilon_2)$ there exists a cube $N_\varepsilon\subset\operatorname{int}K_0(T)$, which is centered at $z_0=(x_0,y_0)$ and have vertices $z_\varepsilon^i=(x_\varepsilon^i, y_\varepsilon^i)$ such that $|z_0-z_\varepsilon^i|=4G_1\varepsilon^{\bar{\alpha}}$, $i=\overline{1,r}$, $r=2^{n+m}$. It is a standard observation that

(28) if $|z^i - z^i_\varepsilon| \leq 2G_1\varepsilon^{\bar{\alpha}}$, $i=\overline{1,r}$ then $z_0 \in \mathrm{co}\{z^i\}^r_{i=1}$.

Since $z^i_\varepsilon = (x^i_\varepsilon, y^i_\varepsilon) \in K_0(T)$, there exist controls $u^i_\varepsilon \in U$ and integrable functions $v^i_\varepsilon, v^i_\varepsilon(t) \in V$ for a. e. $t \in [0, +\infty)$, $i=\overline{1,r}$ such that

$$x^i_\varepsilon = x_0(u^i_\varepsilon)(T),$$

$$y^i_\varepsilon = -A_4^{-1}(T)\, A_3(T)\, x^i_\varepsilon + \int_0^{+\infty} \exp(A_4(T)\, s)\, B_2(T)\, v^i_\varepsilon(s)\, ds.$$

Let us fix $i \in \{1, 2, \ldots, r\}$ and apply again the Proposition (as in Step 1) for set valued maps

$$F(P_0, z_0) = \{u \in U \mid z_0 \in x_0(u)(T) \times (-A_4^{-1}(T)\, A_3(T)\, x_0(u)(T) + I(T))\},$$

and

$$F(P_0, z^i_\varepsilon) = \{u \in U \mid z^i_\varepsilon \in x_0(u)(T) \times (-A_4^{-1}(T)\, A_3(T)\, x_0(u)(T) + I(T))\},$$

where $\varepsilon \in (0, \varepsilon_2)$. Then we find a number $G_9 > 0$ (independent of u_0 and v_0) and controls $\bar{u}^i_\varepsilon \in U, i=\overline{1,r}$, $\varepsilon \in (0, \varepsilon_2)$ such that

$$x_0(\bar{u}^i_\varepsilon)(T) = x^i_\varepsilon, \; i=\overline{1,r},$$

and

$$\| \bar{u}^i_\varepsilon - u_0 \|_p \leq G_9 \, | z^i_\varepsilon - z_0 | \leq G_{10}\varepsilon^{\bar{\alpha}}, \; i=\overline{1,r}.$$

Now, let

$$\tilde{u}^i_\varepsilon(t) = \begin{cases} \bar{u}^i_\varepsilon(t), & t \in [0, T-\varepsilon^{\bar{\alpha}}), \\ v^i_\varepsilon((T-t)/\varepsilon), & t \in [T-\varepsilon^{\bar{\alpha}}, T], \; i=\overline{1,r}. \end{cases}$$

By Step 1 ((24) and (25)) it follows that

$$| x_\varepsilon(\tilde{u}^i_\varepsilon)(T) - x^i_\varepsilon | \leq G_1\varepsilon^{\bar{\alpha}}, \; | y_\varepsilon(\tilde{u}^i_\varepsilon)(T) - y^i_\varepsilon | \leq G_1\varepsilon^{\bar{\alpha}}, \; i=\overline{1,r}.$$

Hence, by (28), there is $\beta^i_\varepsilon \geq 0$, $\sum_{i=1}^r \beta^i_\varepsilon = 1$ such that

$$(x_0, y_0) = \sum_{i=1}^r \beta^i_\varepsilon (x_\varepsilon(\tilde{u}^i_\varepsilon)(T), y_\varepsilon(\tilde{u}^i_\varepsilon)(T)).$$

Define

$$u_\varepsilon(t) = \sum_{i=1}^r \beta^i_\varepsilon \tilde{u}^i_\varepsilon(t),$$

then $(x_0, y_0) = (x_\varepsilon(u_\varepsilon)(T), y_\varepsilon(u_\varepsilon)(T))$, i. e. $u_\varepsilon \in F(\varepsilon)$. Moreover,

$$\| u_0 - u_\varepsilon \|_p \leq \| u_0 - \sum_{i=1}^r \beta^i_\varepsilon \bar{u}^i_\varepsilon \|_p + \| \sum_{i=1}^r \beta^i_\varepsilon \bar{u}^i_\varepsilon - \sum_{i=1}^r \beta^i_\varepsilon \tilde{u}^i_\varepsilon \|_p \leq G_{10}\varepsilon^{\alpha} + G_5(\varepsilon^{\bar{\alpha}})^{1/p},$$

so that $\| u_0 - u_\varepsilon \|_p \leq G_{11}\varepsilon^{\alpha}$, where G_{11} does not depend on u_0 and v_0. Therefore (23) is proved in this case.

Case 2. Let $z_0 \in K_0(T) \setminus]K_0(T)[_{3a}$. Take $\varepsilon \in (0, \varepsilon_2)$ and let $b = 6G_1\varepsilon^{\bar{\alpha}}$. Denote by $z^\varepsilon_0 = (x^\varepsilon_0, y^\varepsilon_0)$ the point from $]K_0(T)[_b \cap C$ for which

$$| z_0 - z^\varepsilon_0 | = \mathrm{dist}(z_0, \;]K_0(T)[_b \cap C).$$

Then applying arguments analogous to those in Step 2. Case 2 (with the Sublemma), we find a number $G_{12}>0$, independent of u_0, v_0 and $\varepsilon\in(0,\varepsilon_2)$ and a control $u_0^\varepsilon\in F(0)$ such that

$$\|u_0-u_0^\varepsilon\|_p\leq G_{12}\varepsilon^\alpha. \tag{29}$$

Since

$$\{z\in R^{n+m}\mid |z-z_0^\varepsilon|\leq 6G_1\varepsilon^{\alpha}\}\subset K_0(T),$$

we have Case 1 for $z_0^\varepsilon=(x_0^\varepsilon, y_0^\varepsilon)$ and can find a number $G_{13}>0$ (independent of u_0, v_0 and $\varepsilon\in(0,\varepsilon_2)$) and $u_\varepsilon\in F(\varepsilon)$ such that

$$\|u_\varepsilon-u_0^\varepsilon\|_p\leq G_{13}\varepsilon^\alpha. \tag{30}$$

Hence, (23) is proved by (29) and (30). The proof of the theorem is completed ■

Remark 3. Using the Proposition and arguments like those in the proof of the Sublemma, we are able to prove that if

$$H=\{u\in U\mid x_0(u)(T)\in \mathrm{proj}_x\ ((K_0(T)\cap \mathrm{int}\, C)\cup(\mathrm{int}\, K_0(T)\cap C))\},$$

then for every $1\leq p\leq\infty$ $F(0)$ is the closure of H in $L^p(0,T)$.

Remark 4. Condition (A2) implies $F(0)\neq\emptyset$. From the proof of Theorem 3 it follows that $F(\varepsilon)\neq\emptyset$ for $\varepsilon\in(0,\varepsilon_2)$.

In the general case the number α in Theorem 3 should be less than 1 which is shown by the following

Example 1.

$$\dot{x}=u,\ x(0)=0,\ T=1,\ u(t)\in[-1,1],$$

$$\varepsilon\dot{y}=-y+u,\ y(0)=0,\ C=\{(x,y)\in R^2\mid x+y=0\}.$$

We have $K_0(1)=[-1,1]x[-1,1]$. Since the system for $\varepsilon>0$ is normal, by the bang-bang principle every point at the boundary of $K(\varepsilon, 1)$ can be reached by means of a bang-bang control having one switching point $\tau\in[0,1]$. Take

$$u_\tau(t)=\begin{cases}\beta, & t\in[0,\tau),\\ -\beta, & t\in[\tau,1],\ \beta=1 \text{ or } -1.\end{cases}$$

Then, setting $r(\varepsilon,\tau)=\exp((\tau-1)/\varepsilon)$ we get the boundary of $K(\varepsilon,1)$

$$x_\varepsilon(1)=\beta(1+2\varepsilon ln(r(\varepsilon,\tau))),$$

$$y_\varepsilon(1)=\beta(2r(\varepsilon,\tau)-\exp(-1/\varepsilon)-1),\ \tau\in[0,1].$$

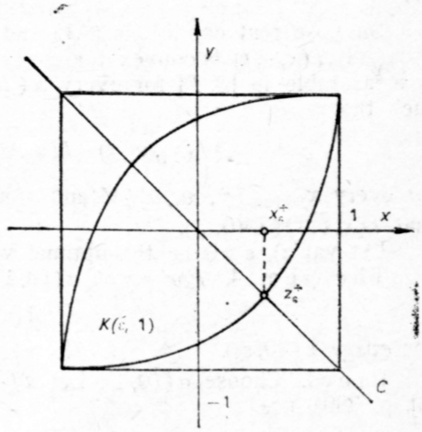

Fig. 1

Depending on β there are two points from both the boundary of $K(\varepsilon,1)$ and C. For $\beta=1$, the point $z_\varepsilon^+=(x_\varepsilon^+, y_\varepsilon^+)$ which belongs to C and the boundary of $K(\varepsilon,1)$, satisfies $z_\varepsilon^+\to(1,-1)$ as $\varepsilon\to 0$ and hence $r_\varepsilon^+=r(\varepsilon,\tau_\varepsilon)\to 0$ as $\varepsilon\to 0$. If $u_0(t)\equiv 1$, then $x_0(u_0)(1)=\int_0^1 1dt=1$ and since $(1,-1)\in C$, then $u_0\in F(0)$. Let $\varepsilon>0$ and $u\in F(\varepsilon)$ be arbitrarily chosen. We have for $1\leq p<\infty$ that

$$\|u-u_0\|_p\geq\int_0^1|u(t)-u_0(t)|\,dt=1-\int_0^1 u(t)\,dt=1-x_\varepsilon(u)(1)\geq 1-x_\varepsilon^+=-2\varepsilon ln(r_\varepsilon^+).$$

Consequently

$$\frac{1}{\varepsilon}\ \text{hausd}_\rho(F(\varepsilon), F(0)) \geqq -2(\ln r_\varepsilon^+) \to +\infty \ \text{ as } \ \varepsilon\to 0.$$

4. An application to optimal control. Consider the following problem depending on the singular parameter ε

(31) $$I_\varepsilon(u)=\int_0^T L(x(t), u(t), t)\,dt \to \inf_u$$

subject to

$$\dot{x}=A_1(t)\,x+A_2(t)\,y+B_1(t)\,u,\ x(0)=x^0,$$
$$\varepsilon\dot{y}=A_3(t)\,x+A_4(t)\,y+B_2(t)\,u,\ y(0)=y^0,$$
$$(x_\varepsilon(T),\ y_\varepsilon(T))\in C,$$

(32) $$U=\{u(\cdot)\ \text{— measurable},\ u(t)\in V \text{ for a. e. } t\in[0, T]\}.$$

For $\varepsilon=0$ we get the reduced system

(33) $$\dot{x}=A_0(t)\,x+B_0(t)\,u(t),\ x(0)=x^0$$

and the corresponding limit problem is (31)—(33) and

(34) $$x_0(T)\in \text{proj}_x(K_0(T)\cap C).$$

Suppose that conditions (A1) and (A2) are fulfilled and moreover

(A4) $L(x,\cdot,t)$ is convex for every $x\in R^n$ and every $t\in[0,T]$. The function $L(x,v,\cdot)$ is measurable in $[0,T]$ for every $x\in R^n$, $v\in V$ and there exists a function $l(\cdot)\in L^2(0,T)$ such that

$$|L(x_1, v_1, t)-L(x_2, v_2, t)|\leq l(t)(|x_1-x_2|+|v_1-v_2|)$$

for every $x_1, x_2\in R^n$, $v_1, v_2\in V$ and almost every $t\in[0,T]$. There exist $\bar{x}\in R^n$, $\bar{v}\in V$ such that $L(\bar{x},\bar{v},\cdot)\in L^1(0,T)$.

Let $\text{val}(\varepsilon)$, $\varepsilon\geqq 0$ be the optimal value of the above problems.

Theorem 4. *For every* $\alpha\in(0,1)$ *there exist* $L_3>0$, $\varepsilon_3>0$ *such that*

$$|\text{val}(\varepsilon)-\text{val}(0)|\leq L_3\varepsilon^\alpha$$

for every $\varepsilon\in(0,\varepsilon_3)$.

Proof. Choose $\alpha\in(0,1)$. Let $\hat{u}_\varepsilon(\cdot)$, $\varepsilon\geqq 0$ be the optimal control which exists (see [6], p. 389), i. e.

$$\text{val}(\varepsilon)=I_\varepsilon(\hat{u}_\varepsilon),\ \varepsilon\geqq 0.$$

By Theorem 3 there are $L_2>0$, $\varepsilon_2>0$ such that for every $\varepsilon\in(0,\varepsilon_2)$ there exists $u_\varepsilon\in F(\varepsilon)$ such that

$$\|u_\varepsilon-\hat{u}_0\|_2\leq L_2\varepsilon^\alpha.$$

Then by (A4) and the Lemma it follows that

$$\text{val}(\varepsilon)-\text{val}(0)=I_\varepsilon(\hat{u}_\varepsilon)-I_0(\hat{u}_0)\leq I_\varepsilon(u_\varepsilon)-I_0(\hat{u}_0)$$
$$\leq M\,(\max_{0\leq t\leq T}|x_\varepsilon(u_\varepsilon)(t)-x_0(\hat{u}_0)(t)|+\|u_\varepsilon-\hat{u}_0\|_2)\leq L_3\varepsilon^\alpha,$$

when $\varepsilon\in(0,\varepsilon_2)$.

With analogous argument for $\widehat{u}_\varepsilon$ we find

$$\mathrm{val}(0)-\mathrm{val}(\varepsilon)\leq L_3\varepsilon^{\alpha},\quad \varepsilon\in(0,\varepsilon_2),$$

which completes the proof ∎

Remark 5. When we define the limit problem (for $\varepsilon=0$) the most important question is how to choose the terminal condition. If we take (34), then the optimal control problem considered is well-posed, i. e. $\mathrm{val}(\varepsilon)\to\mathrm{val}(0)$ as $\varepsilon\to 0$. But replacing $K_0(T)$ by the set

$$\bar{K}(T)=\{(x,y)\in R^{n+m}\mid x\in R(T),\ y\in -A_4^{-1}(T)(A_3(T)\,x+B_2(T)\,V)\}$$

(which results from the formal substituting $\varepsilon=0$), we may obtain ill-posedness Consider, for example, the following problem

$$\int_0^1 |u(t)|^2\,dt\to\inf.$$

$$\dot{x}=u,\ x(0)=0,\ T=1,$$

$$\varepsilon\dot{y}_1=-y_1+u,\ y_1(0)=0,\ u(t)\in[-1,1],$$

$$\varepsilon\dot{y}_2=-2y_2+u,\ y_2(0)=0,\ C=\{(x,y_1,y_2)\in R^3\mid y_1-2y_2=-1/8\}.$$

With analoguous arguments like in Example 1 (see also [7], p. 79) we find that

$$K_0(1)=\{(x,y_1,y_2)\in R^3\mid |x|\leq 1,\ |y_1|\leq 1,\ y_2\in[\tfrac{1}{4}(y_1+1)^2-\tfrac{1}{2},\ -\tfrac{1}{4}(y_1-1)^2+\tfrac{1}{2}]\}$$

and

$$\bar{K}(1)=\{(x,y_1,y_2)\in R^3\mid |x|\leq 1,\ y_1=2y_2\}.$$

The limit problem

$$\int_0^1 |u(t)|^2\,dt\to\inf.$$

$$\dot{x}=u,\ x(0)=0,\ T=1,\ u(t)\in[-1,1],$$

with the following terminal condition

$$x_0(1)\in\mathrm{proj}_x(K_0(1)\cap C)=[-1,1]$$

has a solution $\widehat{u}_0=0$, $\mathrm{val}(0)=0$. Since $\bar{K}(1)\cap C=\varnothing$, see fig. 2, the value of the limit problem with a terminal condition

$$x_0(1)\in\mathrm{proj}_x(\bar{K}(1)\cap C)$$

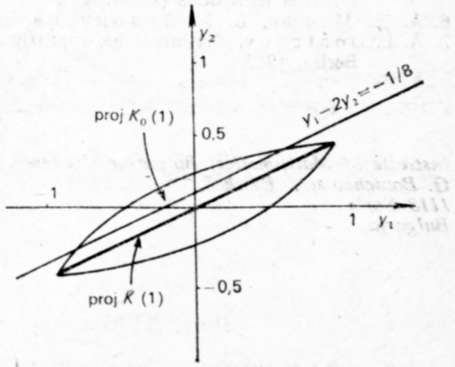

Fig 2

is $\overline{\mathrm{val}}(0)=+\infty$.

We finish with an example showing that in the general case $\alpha<1$.

Example 2. Minimize

$$\int_0^1 |u(t)-1|\,dt$$

subject to the same system and target set as in Example 1, i. e. for $\varepsilon>0$

$$\dot{x}=u,\ x(0)=0,\ T=1,\ u(t)\in[-1,1],$$

$$\varepsilon\dot{y}=-y+u,\ y(0)=0,\ C=\{(x,y)\in R^2 \mid x+y=1\}.$$

The limit problem for $\varepsilon=0$ is

$$\int_0^1 |u(t)-1|\,dt \to \inf,$$

$$\dot{x}=u,\ x(0)=0,\ u(t)\in[-1,1],$$

$$x_0(1)\in \mathrm{proj}_x([-1,1]\times[-1,1]\cap C)=[-1,1].$$

It is clear that the optimal control for the limit problem is $\hat{u}_0\equiv 1$ and $\mathrm{val}(0)=0$. Using the same arguments as in Example 1, we find

$$\frac{1}{\varepsilon}(\mathrm{val}(\varepsilon)-\mathrm{val}(0))=\frac{1}{\varepsilon}\int_0^1 |\hat{u}_\varepsilon(t)-1|\,dt \to +\infty \text{ as } \varepsilon\to 0.$$

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