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UNIFORM ASYMPTOTICS OF THE SPECTRAL FUNCTION FOR SOME GLOBALLY ELLIPTIC OPERATORS AND PERIODIC BICHARACTERISTICS

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We consider a class of globally elliptic essentially self-adjoint pseudodifferential operators in $L^2(\mathbb{R}^n)$. An asymptotics of the spectral function $e(\lambda, x, x)$ as $\lambda \to +\infty$, which is uniform with respect to the parameter $x \in \mathbb{R}^n$, is proved. Near the caustic points this asymptotics is expressed in terms of the Airy function. When the periodic bicharacteristics of the principal symbol are not too many a second term of the asymptotics is obtained.

1. Introduction and statement of the results. Let $A=a(x, D_x)$ be an elliptic essentially self-adjoint pseudodifferential operator in $L^2(\mathbb{R}^n)$ with a symbol

$$a(x, \xi) \sim \sum_{k=0}^{\infty} a_k(x, \xi),$$

where $a_k(x, \xi)$ is $C^\infty(\mathbb{R}^{2n}|0)$ positive homogeneous function:

$$a_k(\sqrt{\lambda}\, x, \sqrt{\lambda}\, \xi)=\lambda^{1-k} a_k(x, \xi),\ \lambda>0,\ k \geqq 0.$$

The spectrum of this operator is discrete and the function

$$e(\lambda, x, y)=\sum_{\lambda_j \leqq \lambda} \varphi_j(x) \overline{\varphi}_j(y)$$

is called spectral function of the operator A. Here $\lambda_j \to +\infty$ are the eigenvalues and φ_j — the corresponding orthonormalized eigenfunctions. The theory of such operators is developed in [3].

The aim of this paper is to find out the asymptotics of the function $e(\lambda, x, x)$ as $\lambda \to +\infty$, which is uniform with respect to the parameter $x \in \mathbb{R}^n$. We assume that the principal symbol $p(x, \xi)=a_0(x, \xi)$ satisfies the following conditions:

$$p(x, \xi)>0 \quad \text{if} \quad (x, \xi) \neq 0, \tag{1}$$

$$p(x, -\xi)=p(x, \xi) \quad \text{if} \quad x \neq 0, \tag{2}$$

$$\partial_\xi^2 p(x, \xi) \text{ is a positive definite matrix if } (x, \xi) \neq 0. \tag{3}$$

E x a m p l e 1. The function $p(x, \xi)=\xi^2+V(x)$, where V is a positive definite quadratic form, satisfies the conditions (1)-(3).

The assumptions (1)-(3) are sufficient to find the main term of the uniform asymptotics with an exact estimate of the rest. The second term of the asymptotics is obtained if in addition to (1)-(3) the following hypotheses are satisfied:

Let $n>1$ and $\Phi^t(y, \eta)=(x(t, y, \eta), \xi(t, y, \eta))$ be the hamiltonian flow of p, lying on the energy level $p(y, \eta)=1$.

(H_1) We say that the point y satisfies the hypothesis (H_1), if $1-p(y, 0) \geqq \delta>0$ and if the measure of the set $S(y)=\{\eta \in \mathbb{R}^n : p(y, \eta)=1,\ x(t, y, \eta)=y$ for some $t \neq 0\}$ is zero.

(H_2) We say that the point y satisfies the hypothesis (H_2) if $p(y, 0)=1$ and if the bicharacteristic $\Phi^t(y, 0)$ is not periodic.

From the homogeneity and ellipticity of the symbol $p(x, \xi)$ it follows that $z=z(x)=\partial_x p(x, 0)\neq 0$, if $x\neq 0$, in particular, the set of the points, satisfying the hypothesis (H_2), is open.

Example 2. Let $p(x, \xi)=\xi^2+\sum_{k=1}^{n}\alpha_k^2 x_k^2$ and α_i/α_j be not a rational number for some $i\neq j$. Then the points $x\in \mathbb{R}^n$, $p(x, 0)\leq 1-\delta$, $\delta>0$ satisfy the hypothesis (H_1) and the points $x\in \mathbb{R}^n$, $x_i\neq 0$, $x_j\neq 0$, $p(x, 0)=1$ satisfy the hypothesis (H_2).

Now we can formulate the main results of the paper. It is convenient to write the asymptotics of the function $E(\lambda, \sqrt{\lambda}x)=e(\lambda, \sqrt{\lambda}x, \sqrt{\lambda}x)$ as $\lambda\to+\infty$. All asymptotics and estimates are uniform with respect to the parameter x. Besides, if the hypotheses (H_1) or (H_2) are not satisfied, then in all estimates the quantity $0(1)$ should be replaced by $0(1)$ as $\lambda\to\infty$.

Theorem 1 (*the case* $p(x, 0)\leq 1-\delta$, $\delta>0$). *Let the points* x *satisfy the hypothesis* (H_1) *and* $n>2$. *Then*

$$E(\lambda, \sqrt{\lambda}x)=a_n(x)\lambda^{n/2}+\lambda^{n/2-1}(b_n(x)+o(1)),\ \lambda\to+\infty, \tag{4}$$

where

$$a_n(x)=(2\pi)^{-n}\operatorname{vol}\{\xi\in \mathbb{R}^n: p(x, \xi)\leq 1\}, \tag{5}$$

$$b_n(x)=-(2\pi)^{-n}\int \operatorname{Re}\, a_1(x, \xi)\frac{ds}{|\partial_\xi p|} \tag{6}$$

and ds *is the Riemann volume on the hypersurface* $\{\xi\in \mathbb{R}^n: p(x, \xi)=1\}$.

Theorem 2 (*the case* $1-\delta\leq p(x, 0)\leq 1-\lambda^{-1/2+\varepsilon}$, $\varepsilon>0$, $\delta>0$). *Let the point* x_0 *satisfy the hypothesis* (H_2). *Then there exist a positive number* δ *and a neighbourhood* U *of* x_0 *such that the asymptotics* (4) *is valid, uniformly in* $x\in U$, $1-\delta\leq p(x, 0)\leq 1-\lambda^{-1/2+\varepsilon}$.

Theorem 3 (*the case* $1-\mathrm{const.}\,\lambda^{-1/2}\leq p(x, 0)\leq 1$). *Let the point* x_0 *satisfy the hypothesis* (H_2). *Then there exists a neighbourhood* U *of the point* x_0 *such that for every* $x\in U$ *with* $1-\mathrm{const}.\,\lambda^{-1/2}\leq p(x, 0)\leq 1$ *the following uniform asymptotics holds*

$$E(\lambda, \sqrt{\lambda}x)=a_n(\lambda, x)\lambda^{n/6}+b_n(\lambda, x)\lambda^{n/6-1/3}(b_n(x)+o(1)),\ \lambda\to+\infty, \tag{7}$$

where

$$a_n(\lambda, x)=(2\pi)^{-n}\operatorname{vol}\{\xi\in \mathbb{R}^n: p(x, \xi)\leq 1\}B^{-n/2}f_n(-B\lambda^{2/3}), \tag{8}$$

$$b_n(\lambda, x)=f_{n-2}(-B_0\lambda^{2/3}), \tag{9}$$

$$b_n(x)=-(2\pi)^{-n}V_n\langle E(x)z, z\rangle^{n/6-1/3}\operatorname{Re}\, a_1(x, 0)\operatorname{trace} E(x), \tag{10}$$

$E(x)$ is the matrix $\partial_\xi^2 p(x, 0)$ and V_n — the volume of the unit ball in $\mathbb{R}^n$. Further,

$$f_n(s)=\int_0^\infty Ai(\sigma+s)\sigma^{n/2}\,d\sigma,\quad Ai(\sigma)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-i(\sigma t+t^3/3)}\,dt \tag{11}$$

being the Airy function, and

$$B=B(x)=B_0(x)+0((1-p(x, 0))^2)\quad \text{as}\quad 1-p(x, 0)\to 0, \tag{12}$$

$$B_0=B_0(x)=2\langle E(x)z, z\rangle^{-1/3}(1-p(x, 0)). \tag{13}$$

Remark 1. The functions f_n, $n\geq 0$ are positive and satisfy the recurrence relations:

(14) $$f_n(s) = -sf_{n-2}(s) + f''_{n-2}(s),\ n \geqq 2;\ f_0(s) = \int_s^\infty Ai(\sigma)d\sigma;$$

(15) $$f_1(s) = \pi 2^{1/3}\{-4^{-1/3} s(Ai(4^{-1/3} s))^2 + (Ai'(4^{-1/3}s))^2\}.$$

Moreover, they decrease, if $n \geqq 1$ and

(16) $$f_n(s) = (-s)^{n/2} + R_n(s), \quad s \to -\infty,$$

where

(17) $$R_0(s) = 0(|s|^{-3/4}), \quad R_1(s) = 0(|s|^{-1}), \quad R_{2k}(s) = 0(|s|^{k-9/4}),$$
$$R_{2k+1}(s) = 0(|s|^{k-5/2}), \quad k \geqq 1.$$

Remark 2. The function $B(x)$ from (12) can be calculated exactly. Namely,

(18) $$B(x) = (\frac{3}{2}\ \psi(t(x),\ \xi(x),\ x))^{2/3},$$

where

(19) $$\psi(t,\ \xi,\ x) = t + \varphi(t,\ \xi,\ x) - \xi x,$$

(20) $$\partial_t\varphi + p(x,\ \partial_x\varphi) = 0, \quad \varphi(0,\ \xi,\ x) = \xi x$$

and the point $(t(x),\ \xi(x))$ is the critical point of the function ψ for which

(21) $$1 - p(x,\ 0) - \frac{t^2}{8}\langle E(x)z,\ z\rangle = 0(t^4), \quad \xi = \frac{z}{2}\ t + 0(t^3), \quad t \to 0.$$

Corollary. If $1 - \mathrm{const}\,\lambda^{-2/3} \leq p(x,\ 0) \leqq 1$, then the asymptotics (7) is fulfilled with a more simple coefficient a_n:

(22) $$a_n(\lambda,\ x) = (2\pi)^{-n} V_n \langle E(x)z,\ z\rangle^{n/6}\,(\det E(x))^{-1/2} f_n(-B_0\lambda^{2/3}).$$

Theorem 4 (*the case* $1 \leq p(x,\ 0) \leqq 1 + \mathrm{const.}\ \lambda^{-2/3}$). *Under the conditions of theorem 2 we have the asymptotics* (7), *where the coefficient* a_n *is given by* (22).

Theorem 5 (*the case* $p(x,\ 0) \geqq 1 + \lambda^{-2/3+\varepsilon},\ \varepsilon > 0$). *It this case*

(23) $$E(\lambda,\ \sqrt{\lambda}x) = 0(\lambda^{-\infty}), \quad \lambda \to +\infty.$$

3. Proof of theorem 1. Using appropriate tauberian arguments, we reduce the asymptotics of the function $E(\lambda,\ \sqrt{\lambda}x)$ to its averages:

$$e_\rho(\lambda,\ x) = \int \rho(\lambda-\mu)E(\mu,\ \sqrt{\lambda}x)d\mu, \quad e'_\rho(\lambda,\ x) = \int \rho(\lambda-\mu)dE(\mu,\ \sqrt{\lambda}x),$$

where ρ is a smooth, even, rapidly decreasing function on the real line, whose Fourier transform $\hat\rho(t) = \int^{-i\lambda t}\rho(\lambda)d\lambda$ has a compact support and $\hat\rho(0) = 1$.

To find the asymptotics of the function $e'_\rho(\lambda,\ x)$, we use the relation

(24) $$\int \rho(\lambda-\mu)de(\mu,\ x,\ y) = \frac{1}{2\pi}\int e^{i\lambda t}\,\hat\rho(t)\,\mathrm{U}(t,\ x,\ y)\,dt,$$

where $\mathrm{U}(t,\ x,\ y)$ is the kernel of the operator $\mathrm{U}(t) = e^{-itA}$, satisfying the Cauchy problem: $(\partial_t + iA\,\mathrm{U}(t) = 0,\ \mathrm{U}(0) = id.$

According to [3], we can construct a parametrix of this problem in the form

(25) $$Q(t)u(x) = (2\pi)^{-n}\int e^{i\varphi(t,\ \xi,\ x)} q(t,\ \xi,\ x)\hat u(\xi)\ d\xi, \quad u \in C_0^\infty(\mathbf{R}^n),$$

where t varies on a compact interval. Namely, if t is near zero, then the phase function

φ solves the problem (20) and $q(t, \xi, x) \sim \sum_{k=0}^{\infty} q_k(t, \xi, x)$, where $(\xi, x) \to q_k(t, \xi, x)$ is a positive homogeneous function of degree $-2k$. Moreover, $q_k(0, \xi, x)=0$ if $k \geq 1$,

(26) $$\partial_t q_0 + (\partial_\xi p)(x, \partial_x \varphi)\partial_x q_0 + b_0(x, \xi) q_0 = 0, \quad q_0(0, \xi, x) = 1$$

and

(27) $$b_0(x, \xi) = i a_1(x, \partial_x \varphi) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 p}{\partial \xi_i \partial \xi_j}(x, \partial_x \varphi) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \cdot$$

From (24), (25) it follows that:

(28) $$\int \rho(\lambda-\mu) de(\mu, x, x) \sim (2\pi)^{-n-1} \int e^{i(\lambda t + \varphi(t, \xi, x) - \xi x)} \hat{\rho}(t) q(t, \xi, x) dt d\xi,$$

where the equivalence "$A(\lambda, x) \sim B(\lambda, x)$" means that $A(\lambda, x) - B(\lambda, x) = 0(|\lambda| + x^2)^{-\infty}$. Hence

(29) $$e'_\rho(\lambda, x) \sim \lambda^{n/2} \int e^{i\lambda\psi} g(t, \sqrt{\lambda}\xi, \sqrt{\lambda} x) h(\xi) dt\, d\xi,$$

where ψ is given by (19) and $g(t, \xi, x) = (2\pi)^{-n-1} \hat{\rho}(t) q(t, \xi, x)$. Here $h \in C_0^\infty(\mathrm{R}^n)$ is a cutoff function, which is due to integration by parts, with the help of the estimate: $|\partial_t \psi| \geq C(1 + x^2 + \xi^2)$ for large ξ^2.

To evaluate the integral (29), we note that the critical points (t, ξ) of the phase function ψ satisfy the relations $\partial_\xi \varphi = x$, $p(x, \partial_x \varphi) = 1$, whence, using the property $\Phi^t(\partial_\xi \varphi, \xi) = (x, \partial_x \varphi)$, we get: $\Phi^t(x, \xi) = (x, \partial_x \varphi)$, $p(x, \partial_x \varphi) = 1$, $p(x, \xi) = 1$.

Since the rank of Hessian φ'' at the critical points is 2, the method of the stationary phase and the hypothesis (H_1) lead to

(30) $$e'_\rho(\lambda, x) = I(\lambda, x) + o(\lambda^{n/2-1}), \quad \lambda \to +\infty,$$

where $I(\lambda, x) = I_0(\lambda, x) + \lambda^{-1} I_1(\lambda, x)$, $I_k(\lambda, x) = \lambda^{n/2} \int e^{i\lambda\psi} g_k dt d\xi$, $g_k(t, \xi, x) = (2\pi)^{-n-1} \hat{\rho}(t)$ $q_k(t, \xi, x)\varkappa(t) h(\xi)$, $k = 0, 1$ and $\varkappa \in C_0^\infty(\mathrm{R})$ is a cutoff function with $\varkappa(0) = 1$ and small support.

Further, we can write

(31) $$\psi(t, \xi, x) = t(1 - r(t, \xi, x)), \; r(0, \xi, x) = p(x, \xi), \; \partial_t r(0, \xi, x) = -\frac{1}{2} \partial_x p \partial_\xi p.$$

Therefore

(32) $$I_k(\lambda, x) = \lambda^{n/2} \int e^{i\lambda t(1-\sigma)} \langle \delta_0(\sigma - r), g_k \rangle dt\, d\sigma,$$

where $\delta_0(\sigma - r)$ is the pullback of the Dirac measure δ_0 (see theorem 6.1.5 [4]). Hence the method of the stationary phase and (26), (27), (31) give

(33) $$I(\lambda, x) = b_0(x)\lambda^{n/2-1} + b_1(x)\lambda^{n/2-2} + o(\lambda^{n/2-2}), \quad \lambda \to +\infty,$$

where $b_0(x) = (2\pi)^{-n} \int_{p=1} \omega_x$, $b_1(x) = -(2\pi)^{-n} \int_{p=1} \operatorname{div}(a_1 \partial_\xi p |\partial_\xi p|^{-2})\omega_x$ and $\omega_x = \sum_{j=1}^{n} (-1)^{j-1}$ $\partial_{\xi_j} p |\partial_\xi p|^{-2} d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_n$ is the Leray-Gelfand form on the hypersurface $\{\xi \in \mathrm{R}^n : p(x, \xi) = 1\}$. In addition we have used the property (2).

Integrating the function $I(\lambda, x\lambda^{-1/2})$ over the interval $(0, \lambda)$ and using the homogeneity of p, we get the equality

$$\int_0^\lambda I(\mu, x\mu^{-1/2}) d\mu = (2\pi)^{-n} \int_{p \leq \lambda} d\xi - (2\pi)^{-n} \int_{p=\lambda} \operatorname{Re} a_1 \omega_x + o(\lambda^{n/2-1})$$

if $n>2$. Here we use the relation

(34) $$Im\, a_1+\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2 p}{\partial x_j\partial\xi_j}=0,$$

which follows from the self-adjointness of the operator A.

Thus, integrating (29) with respect to λ and repeating the arguments resulting in (30), we obtain

(35) $$e_\rho(\lambda,\ x)=a_n(x)\lambda^{n/2}+\lambda^{n/2-1}(b_n(x)+o(1)),\quad \lambda\to+\infty,$$

where the coefficients a_n, b_n are given by (5), (6), and

(36) $$e'_{\rho_T}(\lambda,\ x)=b_0(x)\lambda^{n/2-1}(1+o(1)),$$

where $\rho_T(\lambda)=\frac{1}{T}\rho(\frac{\lambda}{T})$, $0<T<1$. In particular, taking ρ positive, we derive from (36) the estimate

(37) $$|E(\lambda+\sigma T,\ \sqrt{\lambda}x)-E(\lambda,\ \sqrt{\lambda}x)|\leq \text{const}\,\lambda^{n/2-1}(T+o(1))\quad\text{if}\quad |\sigma|\leq 1.$$

On the other hand, the bounds of the eigenvalues and the eigenfunctions of the operator A [3], [5] imply the estimate:

$|E(\lambda+\mu,\ \sqrt{\lambda}x)|\leq\text{const}\,(1+\lambda+|\mu|)^{3n}$, $\lambda>0$, $\mu\in\mathbb{R}$. Therefore,

(38) $$e_{\rho_T}(\lambda,\ x)-E(\lambda,\ \sqrt{\lambda}x)=\int_0^{\lambda/2}[\Delta(\mu T)e+\Delta(-\mu T)e]\rho(\mu)d\mu+0(\lambda^{-\infty}),$$

where $\Delta(\mu T)e=E(\lambda+\mu T,\ \sqrt{\lambda}x)-E(\lambda,\ \sqrt{\lambda}x)$. From (37) we obtain the estimate $|\Delta(\pm\mu T)e|\leq\text{const}\,(1+|\mu|^{n/2-1})\lambda^{n/2-1}(T+o(1))$ if $0<\mu<\lambda/2$. Hence (38) gives

(39) $$|e_{\rho_T}(\lambda,\ x)-E(\lambda,\ \sqrt{\lambda}x)|\leq\text{const}\,\lambda^{n/2-1}(T+o(1)),\quad \lambda\to\infty.$$

Finally, the asymptotics (4) follows from (35) and (39).

4. Proof of theorem 2. We use the formula (29). The hypothesis (H_2) implies that for some $\delta>0$ and for some neighborhood U of x_0, if $x\in U$ and $1-\delta\leq p(x,\ 0)\leq 1$, the critical points $(t,\ \xi)$ of the phase function ψ are such that t is near zero. Hence instead of (30) now we can write: $e'_\rho(\lambda,\ x)=I(\lambda,\ x)+0(\lambda^{-\infty})$, $\lambda\to+\infty$. Further, we have the formula (32). When $p(x,\ 0)=1$ the integrand function $\langle\delta_0(\sigma-r),\ g_k\rangle$ has the singularities. So in the considered case, $1-p(x,\ 0)\geq\lambda^{-1/2+\varepsilon}$, $\varepsilon>0$, to obtain (33) we have to estimate the rest in the formula of the stationary phase. To this end we note, as in [5], that it is sufficient to integrate in (32) over the set $|t|+|1-\sigma|\leq c(d(x))^{1/2}$, where

(40) $$d(x)=1-p(x,\ 0).$$

Then in the integral $\langle\delta_0(\sigma-r),\ g_k\rangle$ the variable ξ satisfies the relations

(41) $$c_1d(x)\leq\xi^2\leq c_2d(x).$$

Since

(42) $$c_1|\xi|\leq|\partial_\xi p|\leq c_2|\xi|,$$

we obtain the estimate: $|\partial^j_{t,\sigma}\langle\delta_0(\sigma-r),\ g_k\rangle|\leq\text{const}\,(d(x))^{-j}$. Thus the method of the stationary phase ((7.7.12) [4]) shows that we have again the formula (33).

Next the proof of theorem 2 follows that of theorem 1.

5. Proof of theorem 3. Starting from (28) and integrating by parts with respect to each coordinate ξ_j, we get

$$e_\rho(\lambda, x) \sim \lambda^{n/2} \int e^{i\lambda\psi} g(t, \sqrt{\lambda}\,\xi, \sqrt{\lambda}\,x)\, h(\xi)\, dt\, d\xi,$$

where $g(t, \xi, x) = \frac{(2n)^{-n-1}}{n} \frac{\hat{\rho}(t)}{t} \{-\xi\partial_\xi\psi q(t, \xi, x) + i\xi\partial_\xi q(t, \xi, x)\}$. Using the hypothesis (H_2) as in the proof of theorem 2, we obtain

$$e_\rho(\lambda, x) = I(\lambda, x) + 0(\lambda^{-\infty}). \tag{43}$$

where

$$I(\lambda, x) = I_1(\lambda, x) + I_0(\lambda, x) + J(\lambda, x), \tag{44}$$

$$I_k(\lambda, x) = \lambda^{n/2+k} \int e^{i\lambda\psi} g_k(t, \xi, x)\, dt\, d\xi, \quad k = 0,\ 1, \tag{45}$$

$$J(\lambda, x) = \lambda^{n/2} \int e^{i\lambda\psi} r(t, \xi, x, \lambda)\, dx\, d\xi, \tag{46}$$

$$g_1(t, \xi, x) = -\frac{(2\pi)^{-n-1}}{n} \frac{\hat{\rho}(t)}{t} \xi\partial_\xi\psi q_0(t, \xi, x)\, \varkappa(t), \tag{47}$$

$$g_0(t, \xi, x) = \frac{(2\pi)^{-n-1}}{n} \frac{\hat{\rho}(t)}{t} [i\xi\partial_\xi q_0(t, \xi, x) - \xi\partial_\xi\psi \cdot q_1(t, \xi, x)]\varkappa(t),$$

$$r(t, \xi, x, \lambda) = 0(\lambda^{-1}).$$

To find the asymptotics of the integrals (45), (46), we note that

$$\begin{cases} \partial_t\psi = 1 - p + t\partial_x p\partial_\xi p - \frac{t^2}{2}(\langle\partial_\xi^2 p\partial_x p,\ \partial_x p\rangle + \langle\partial_x^2 p\partial_\xi p,\ \partial_\xi p\rangle + \langle\partial_{x\xi}^2 p\partial_x p,\ \partial_\xi p\rangle) + 0(t^3), \\ \partial_\xi\psi = t[-\partial_\xi p + \frac{t}{2}(\partial_\xi^2 p\partial_x p + \partial_{x\xi}^2 p\partial_\xi p)] + 0(t^3), \quad t \to 0. \end{cases} \tag{48}$$

From (19), (20), (2) it follows that the function $(t, \xi) \to \psi(t, \xi, x)$ is odd, hence the critical points of ψ are the points $\{(0, \xi) : p(x, \xi) = 1\}$ and $\{(\pm t(x), \pm\xi(x)) : p(x, \xi(x)) = 1\}$ where $(t(x), \xi(x))$ satisfies (21). Indeed, in view of (2), (3), (42), we have the estimates, (41), if $p(x, \xi) = 1$. Therefore the critical points exist only if $p(x, 0) \leq 1$ and $C_1 d(x) \leq |t(x)|^2 \leq C_2 d(x)$ if $d(x) \leq \delta$, δ is sufficiently small. Now the asymptotics (21) follows.

Let $p(x, 0) = 1$. Then $\psi(t, \xi, x) = t[\langle A(t, \xi, x)\xi, \xi\rangle + B(t, \xi, x)\xi t + C(t, \xi, x)t^2]$, where $A(0, 0, x) = -\frac{1}{2}E(x)$, $B(0, 0, x) = \frac{1}{2}E(x)z$, $C(0, 0, x) = -\frac{1}{6}\langle E(x)z, z\rangle$. There exists an odd and smooth change of variables $\xi \to \tilde{\xi}$ such that $\langle A(t, \xi, x)\xi, \xi\rangle = -\frac{1}{2}\sum_{j=1}^{n} \lambda_j(x)\tilde{\xi}_j^2$, where $\lambda_j(x)$ are the eigenvalues of the matrix $E(x)$. Hence there exists an odd change of coordinates $t = \tau p_1(\tau, \eta, x)$, $\xi = \xi_1(\tau, \eta, x)$ such that

$$\psi(t, \xi, x) = \tau\eta^2 + \tau^3/3 \text{ if } p(x, 0) = 1. \tag{49}$$

From the theory of the versal deformations [1], [6] it follows that the family $ct + t\xi^2 + t^3/3$ is a versal deformation of the function $t\xi^2 + t^3/3$ in the class $\mathcal{D}$ of all smooth functions $g(t, \xi)$, defined in a neighbourhood of the origin having the properties: $g(-t, -\xi) = -g(t, \xi)$, $g(0, \xi) = 0$. This class is invariant under the local diffeomorphisms $(\tau, \eta) = v(t, \xi)$ such that $v(-t, -\xi) = -v(t, \xi)$, $v(0, \xi) = (0, \eta)$.

Since $\psi \in \mathcal{D}$ and satisfies (49), we conclude that there exists an odd change of variables $(t, \xi) \to (\tau, \eta)$ with the properties:

$$t = \tau p(\tau, \eta, x), \quad \xi = \xi(\tau, \eta, x), \tag{50}$$

(51) $$\psi(t, \xi, x) = -B(x)\tau + \tau\eta^2 + \tau^3/3 \text{ if } |1-p(x, 0)| \leq \delta,$$

where δ is small enough. In addition, the coefficient $B(x)$ satisfies (18) and the asymptotics (12) if $p(x, 0) \leq 1$.

Using the principle of the stationary phase and the polar coordinates $\eta = \sigma\omega$, we can write

(52) $$I_k(\lambda, x) = \lambda^{n/2+k} \int_0^\infty \int e^{i\lambda(-B\tau+\tau\sigma^2+\tau^3/3)} \sigma^{n-1} g_k(\tau, \sigma)\, d\tau\, d\sigma,$$

where

(53) $$g_k(\tau, \sigma) = \int_{|\omega|=1} g_k(t, \xi, x)\, J(\tau, \sigma\omega)\, d\omega$$

and $J(\tau, \eta)$ is the Jacobian of the mapping (50). In particular, the function $\sigma \to g_k(\tau, \sigma)$ is even, hence by the Malgrange preparation theorem [1] we can write

(54) $$g_k(\tau, \sigma) = a_{0k} + a_{1k}\tau + a_{2k}\sigma^2 + (\tau^2 + \sigma^2 - B) f_{1k} + \tau\sigma^2 f_{2k}.$$

Integrating by parts the integral in (52) and using (54), we get

(55) $$I_k(\lambda, x) = \lambda^{n/2+k} \int_0^\infty \int e^{i\lambda(-B\tau+\tau\sigma^2+\tau^3/3)} \sigma^{n-1} (a_{0k} + a_{1k}\tau + a_{2k}\sigma^2)\, d\tau\, d\sigma + R_k.$$

The coefficients a_{jk} satisfy the formulas:

(56) $$\begin{cases} a_{0k} = \frac{1}{2} [g_k(\sqrt{B}, 0) + g_k(-\sqrt{B}, 0)], \quad a_{1k} = \frac{1}{2\sqrt{B}} [g_k(\sqrt{B}, 0) - g_k(-\sqrt{B}, 0)], \\ a_{2k} = \frac{1}{B} [g_k(0, \sqrt{B}) - a_{0k}]. \end{cases}$$

Since $(\pm\sqrt{B}, 0)$ are images of the critical points $(\pm t(x), \pm\xi(x))$ it follows from (53), (47) that $g_1(\pm\sqrt{B}, 0) = 0$, hence $a_{01} = a_{11} = 0$. The points $(0, \sqrt{B}\omega)$ are images of the critical points $(0, \xi)$, $p(x, \xi) = 1$, therefore (56), (53), (47), (48) give

$$a_{21} = \frac{(2\pi)^{-n-1}}{n} \frac{1}{B} \int_{|\omega|=1} \xi(0, \sqrt{B}\omega)\, \partial_\xi p(x, \xi(0, \sqrt{B}\omega))\, J(0, \sqrt{B}\omega)\, d\omega.$$

To compute this integral we use the relations, following from (50), (51), (48):

(57) $$J(0, \eta) = \det\left(\frac{\partial\xi}{\partial\eta}\right) \frac{\partial t}{\partial\tau},$$

(58) $$-\frac{\partial\xi}{\partial\eta}\, \partial_\xi p\, \frac{\partial t}{\partial\tau}(0, \eta) = 2\eta \text{ if } \eta^2 = B.$$

Since $\omega \to \xi(0, \sqrt{B}\omega)$ is a parametrization of the surface $p(x, \xi) = 1$, it follows that $a_{21} = \frac{(2\pi)^{-n-1}}{n} 2B^{-n/2} \int_{p=1} \xi\partial_\xi p |\partial_\xi p|^{-1} ds$, whence

(59) $$a_{21} = (2\pi)^{-n-1}\, 2B^{-n/2} \operatorname{vol}\{\xi \in \mathbb{R}^n : p(x, \xi) \leq 1\}.$$

In particular,

(60) $$a_{21}(x) = (2\pi)^{-n-1}\, 2V_n \langle E(x)z, z\rangle^{n/6} (\det E(x))^{-1/2} + 0(B(x)) \text{ if } B(x) \to 0.$$

Hence (55), (59) and (11) imply

(61) $$I_1(\lambda, x) = a_n(\lambda, x)\lambda^{n/6} + R_1(\lambda, x),$$

where the coefficient a_n is given by (8). Since

$$R_1(\lambda, x) = i\lambda^{n/2}\int_0^\infty\int e^{i\lambda(-B\tau+\tau\sigma^2+\tau^3/3)}\sigma^{n-1}h_1(\tau, \sigma)d\tau\, d\sigma, \text{ where}$$

(62) $$h_1(\tau, \sigma) = \partial_\tau f_{11}(\tau, \sigma) + \frac{n}{2} f_{21}(\tau, \sigma) + \frac{\sigma}{2}\partial_\sigma f_{21}(\tau, \sigma),$$

we obtain analogously to (61)

(63) $$R_1(\lambda, x) = i\pi C_0 f_{n-2}(-B\lambda^{2/3})\lambda^{n/6-1/3} + (\widetilde{b}_n(\lambda, x) + 0(1))0(\lambda^{n/6-2/3}),$$

(64) $$\widetilde{b}_n(\lambda, x) = |f'_{n-2}(-B\lambda^{2/3})| + \lambda^{-2/3} f_n(-B\lambda^{2/3})$$

and $C_0 = \frac{1}{2}[h_1(\sqrt{B}, 0) + h_1(-\sqrt{B}, 0)]$. In particular,

(65) $$C_0 = h_1(0, 0) + 0(B) \text{ as } 1 - p(x, 0) \to 0.$$

From (62) and (54) it follows that

$$C_0 = \frac{1}{6}\partial_\tau^3 g_1(0, 0) - \frac{n}{4}\partial_\tau^2 g_1(0, 0) + \frac{n}{4}\frac{\partial^3 g_1}{\partial\tau\partial\sigma^2}(0, 0) + 0(B).$$

Since $g_1(\tau, 0) = a(\tau)b(\tau)$, where $a(\tau) = \frac{1}{t}\xi\partial_\xi\psi$, and $a(0) = 0$, $a'(0) = 0$, $a''(0) = 0(B)$, $a'''(0) = 0$ we obtain

(66) $$\partial_\tau^2 g_1(0, 0) = 0(B),\ \partial_\tau^3 g_1(0, 0) = 0(B).$$

Therefore $C_0 = \frac{n}{4}\frac{\partial^3 g_1}{\partial\tau\partial\sigma^2}(0, 0) + 0(B)$. Further, $\frac{\partial^3 g_1}{\partial\tau\partial\sigma^2}(0, 0) = \int_{|\omega|=1}\langle\frac{\partial^3 q_1(0, 0, x)}{\partial\tau\partial\eta^2}\omega, \omega\rangle d\omega\, J(0, 0)$ and (53), (47), (50), (51), (26), (27) give $\frac{\partial^3 g_1(0, 0, x)}{\partial\tau\,\partial\eta^2} = i\frac{(2\pi)^{-n-1}}{n}4a_1(x, 0)E(x)$. Hence

(67) $$C_0 = i(2\pi)^{-n-1}a_1(x, 0)V_n \operatorname{trace} E(x) + 0(B).$$

Later on we shall prove that

(68) $$\widetilde{b}_n(\lambda, x) \geq \text{const} > 0 \text{ and } \widetilde{b}_n(\lambda, x) \leq \text{const}\, b_n(\lambda, x).$$

Thus (61), (63), (64), (67), (68), (34) imply

(69) $$I_1(\lambda, x) = a_n(\lambda, x)\lambda^{n/6} + b_n(\lambda, x)\lambda^{n/6-1/3}(b_n(x) + 0(\lambda^{-1/6})),$$

where the coefficients a_n, b_n are given by (8), (9). Here we use the property $f_{n-2}(-B\lambda^{2/3}) = f_{n-2}(-B_0\lambda^{2/3})(1 + 0(\lambda^{-1/6}))$, which follows from the asymptotics (13), (16), (17) and the relations (70), (77), (80) obtained further on.

To estimate the functions $I_0(\lambda, x)$ and $J(\lambda, x)$ from (44), we proceed as before and use the relations:

(70) $$C_1 \max(1, |s|) \leq f_n(s)/f_{n-2}(s) \leq C_2 \max(1, |s|), \quad s \leq 0.$$

Since $a_{02} = 0(B)$ we have $I_0(\lambda, x) + J(\lambda, x) = b_n(\lambda, x)0(\lambda^{n/6-2/3})$ and (43), (44), (69) yield

(71) $$e_{\rho_T}(\lambda, x) = a_n(\lambda, x)\lambda^{n/6} + b_n(\lambda, x)\lambda^{n/6-1/3}(b_n(x) + 0(\lambda^{-1/6})).$$

Analogously,

(72) $$|e'_{\rho_T}(\lambda, x)| \leq \text{const}\, b_n(\lambda, x)\lambda^{n/6-1/3}(1 + 0(\lambda^{-1/6})).$$

Hence

(73) $$|E(\lambda+\sigma T,\ \sqrt{\lambda}x)-E(\lambda,\ \sqrt{\lambda}x)|\leq \text{const}\ b_n(\lambda,\ x)\lambda^{n/6-1/3}(T+0(\lambda^{-1/6}))$$

if $|\sigma|\leq 1$. Later on we shall prove that

(74) $$b_n(\lambda\pm\mu,\ x)\leq \text{const}\ b_n(\lambda,\ x)\ \text{if}\ 0<\mu<\lambda/2.$$

Therefore, using the formulas (38), (68) and (73), (74), we obtain instead of (39) the following estimate

(75) $$|e_{\rho_T}(\lambda,\ x)-E(\lambda,\ \sqrt{\lambda}x)|\leq \text{const}\ b_n(\lambda,\ x)\lambda^{n/6-1/3}(T+0(\lambda^{-1/6})).$$

Evidently, the asymptotics (7) follows from (75) and (71). Finally we have to verify the properties (68), (70), (74). First we shall prove the asymptotics (16), (17). From the asymptottics of the Airy function it follows that

$$f_0(s)=1-\frac{1}{\sqrt{\pi}}(-s)^{-3/4}\cos\left(\frac{2}{3}(-s)^{3/2}+\frac{\pi}{4}\right)+0(|s|^{-9/4})\ \text{and}$$

$$f_0''(s)=\frac{1}{\sqrt{\pi}}(-s)^{1/4}\cos\left(\frac{2}{3}(-s)^{3/2}+\frac{\pi}{4}\right)+0(|s|^{-5/4})\ \text{as}\ s\to-\infty.$$

Hence (14) implies that $f_2(s)=-s+0(|s|^{-5/4})$. Similarly

(76) $$f_{2k}(s)=(-s)^k+0(|s|^{k-9/4}),\quad s\to-\infty\ \text{if}\ k=2,\ 3.$$

Now (76) follows inductively for every $k\geq 1$ in view of (14). Analogously,

$$f_1(s)=(-s)^{1/2}+\frac{1}{2}(-s)^{-1}\cos\frac{4}{3}(-s)^{3/2}+0(|s|^{-5/2})\ \text{and}$$

$$f_1''(s)=-\frac{1}{2}\cos\frac{4}{3}(-s)^{3/2}+0(|s|^{-3/2})\ \text{as}\ s\to-\infty.\ \text{Therefore}$$

$$f_{2k+1}(s)=(-s)^{k+1/2}+0(|s|^{k-5/2}),\ s\to-\infty,\ k\geq 1.$$

Further, we need the bound

(77) $$f_0(s)>0$$

Clearly it suffices to prove

(78) $$J_k=\int_{s_{2k+2}}^{s_{2k}} Ai(\sigma)d\sigma>0,\quad k\geq 0,\quad s_0=\infty,$$

where $0>s_1>s_2>\ldots$ are all the zeros of the Airy function $Ai(s)$, so $s_{n+1}-s_{n+2}<s_n-s_{n+1}$ and $Ai(s)>0$ on the intervals $(s_{2k+1},\ s_{2k})$, $k\geq 0$. Since

$$J_k=\int_{s_{2k+1}}^{s_{2k}} Ai(\sigma)d\sigma+\int_{s_{2k+1}}^{r_k} Ai(2s_{2k+1}-\sigma)d\sigma,$$

where $r_k=2s_{2k+1}-s_{2k+2}$, then

(79) $$J_k\geq\int_{s_{2k+1}}^{r_k}(Ai(\sigma)-f(\sigma))\,d\sigma,$$

where $f(\sigma)=-Ai(2s_{2k+1}-\sigma)$, if $\sigma\in(s_{2k+1},\ r_k)$. To compare the functions $Ai(\sigma)$ and $f(\sigma)$ on the interval $(s_{2k+1},\ r_k)$, we notice that there

$$f''(\sigma)+(\sigma-2s_{2k+1})f(\sigma)=0,\quad Ai''(\sigma)+(-\sigma)Ai(\sigma)=0,$$
$$-\sigma<\sigma-2s_{2k+1},\quad \sigma-2s_{2k+1}>0,\quad f(\sigma)>0,\quad Ai(\sigma)>0,$$
$$f(s_{2k+1})=Ai(s_{2k+1})=0,\quad f'(s_{2k+1})=Ai'(s_{2k+1}).$$

Hence $f(\sigma)<Ai(\sigma)$ on the same interval. Then (78) follows from (79).

Now we shall prove that

(80) $f_n(s)$ is a positive and decreasing function if $n\geq 1$.

This is obvious if $s\geq 0$. Since $f_1'(s)=-\pi 2^{-1/3}(Ai(4^{-1/3}s))^2\leq 0$ and $f_2'(s)=-f_0(s)<0$, we have (80) for $n=1, 2$. Now (80) follows inductively, having in mind the equality

(81) $$f_n'(s)=-\frac{n}{2}f_{n-2}(s),\quad n\geq 2.$$

Evidently, the properties (68), (70), (74) follow from the asymptotics (16), (17) and (77), (80), (81), (64).

Theorem 3 is proved.

6. Proof of theorem 4. Following the proof of the theorem 3, we first note that in the case considered, $p(x, 0)\geq 1$, there are not critical points of the phase function ψ if $p(x, 0)>1$. Therefore we can not compute the coefficient $B(x)$ in the formula (51) as before. But we can prove that the asymptotics (12) is again valid. Namely, (50), (51), (48) show that

(82) $$(1-p(x, 0))\frac{\partial t}{\partial \tau}(0, 0, x)=-B(x)\quad \text{if}\quad |1-p(x, 0)|\leq\delta,$$

whence the asymptotics (12) leads to $\frac{\partial t}{\partial\tau}(0, 0, x_0)=-2\langle E(x_0)z(x_0), z(x_0)\rangle^{-1/3}$ if $p(x_0, 0)=1$. Let $d(x)=p(x, 0)-1$. Since $d(x)=(x-x_0)z(x_0)+0(d^2)$ we have the bound $|x-x_0|\leq Cd(x)$ and the Taylor formula gives the asymptotics $B(x)=(x-x_0)\partial_x B(x_0)+0(d^2)$. On the other hand, from (82) it follows that $\partial_x B(x_0)=z(x_0)\frac{\partial t}{\partial\tau}(0, 0, x_0)$ and $B(x)=-2\langle E(x_0)z(x_0), z(x_0)\rangle^{-1/3}d(x)+0(d^2)$, so the Taylor formula yields the asymptotics (12).

Further, we have to know the smooth coefficients $a_{jk}(x)$ from (54). The Taylor formula and (60) give

$$a_{21}(x)=(2\pi)^{-n-1}\,2V_n\langle E(x)z(x), z(x)\rangle^{n/6}(\det E(x))^{-1/2}+0(d(x)).$$

Since the asymptotics (65) is valid, we see that the formula (67) is also true. To estimate the coefficients a_{0k}, a_{1k}, we note that (54) implies the relations: $a_{0k}=Bf_{1k}(0, 0)$, $a_{11}=B\partial_\tau f_{11}(0, 0)$, $\partial_\tau^2 g_k(0, 0)=2f_{1k}(0, 0)+0(B)$, $\partial_\tau^3 g_1(0, 0)=6\partial_\tau f_{11}(0, 0)+0(B)$. Hence (66) gives: $a_{01}=0(B^2)$, $a_{11}=0(B^2)$, $a_{02}=0(B)$. Thus we have again the relations (71), (72), where the coefficients $a_n(\lambda, x)$, $b_n(\lambda, x)$, $b_n(x)$ are given by (22), (9), (10).

7. Proof of theorem 5. First let us consider the case $1+\lambda^{-2/3+\varepsilon}\leq p(x, 0)\leq 1+\delta$, where δ is taken from (51). As in the proof of theorem 3 we obtain the estimate

(83) $$e_\rho(\lambda, x)=B_n(\lambda, x)0(\lambda^{n/6+2/3}),$$

where $B_n(\lambda, x)=f_{n-2}(-B\lambda^{2/3})+|f'_{n-2}(-B\lambda^{2/3})|+f_n(-B\lambda^{2/3})$. Since $-B(x)\lambda^{2/3}\geq c.\lambda^\varepsilon$, $c>0$, it follows from the asymptotics of the Airy function and its derivative that $B_n(\lambda, x)=0(\lambda^{-\infty})$, uniformly in x. Therefore (83) gives

(84) $$e_\rho(\lambda, x)=0(\lambda^{-\infty}).$$

Analogously

(85) $$e'_\rho(\lambda, x) = O(\lambda^{-\infty}).$$

Now let $p(x, 0) \geqq 1+\delta$. Then there are not critical points of the phase function ψ and

(86) $$|\partial_t\psi| + |\partial_\xi\psi| \geqq c > 0.$$

Indeed, if x varies on the compact $1+\delta \leqq p(x, 0) \leqq b$, then (86) is obvious. If $p(x, 0) \geqq b$ and the constant b is large enough, we obtain from (1) and (19), (20) the estimate $|\partial_t\psi| \geqq c(x^2 + (\partial_x\varphi)^2 - 1) \geqq c_1 > 0$. Hence (86) is verified. Finally, integrating by parts as usual in the integral (29), we get (84), (85).

Theorem 5 follows from (84), (85) and the usual Tauberian arguments.

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