Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

ON CURVES WITH A CONSTANT PERIMETER OF THE DESCRIBED POLYGON LINES

STANISŁAW GÓŹDŹ

A class of plane locally convex curves is considered. The conditions for constancy of the perimeter of some polygon lines described on curves belonging to the considered class are proved. For this purpose some properties of a periodic function are given.

Preliminaries. Let $r(t)$ be a periodic continuous and positive function with the period 2π. R. L. Tennison [4] has examined the curve represented by the well-known formula

$$\theta \to \left(\int_0^\theta r(t) \operatorname{const} dt,\ \int_0^\theta r(t) \sin t \, dt\right).$$

He has proved that the closed curve has a constant width whenever the even Fourier coefficients for $r(t)$ vanish. The case when the Fourier coefficients with indices $n\cdot l$ vanish for every integer l, where n is a fixed integer has been considered in [2]. In the present paper we generalize the results in [4] and [2]. We consider a condition of constancy for the following function

$$f(s)+f(\varphi((s))+\cdots+f(\underbrace{\varphi(\varphi(\ldots\varphi(s)\ldots)}_{m-1 \text{ times}})).$$

Here f is a continuous periodic function with a period L and φ is a function such that

1⁰ $\varphi(s+L)=\varphi(s)+L$

2⁰ $\underbrace{\varphi(\varphi(\ldots\varphi(s)\ldots}_{m\text{-times}}))=s+L$

3⁰ $k(\varphi(s))\,\varphi'(s)=k(s)$,

and $k(s)>0$ is a fixed continuous function defined for the whole real line R such that

(A) $k(s)>0$, for all $s\in R$,

(B) $k(s+L)=k(s)$, for all $s\in R$,

(C) $\int_0^L k(s)\,ds=2\pi j$, where $j\geq 1$ is an integer.

Using the above function f, we will define a family $\mathfrak{M}$ consisting of locally convex curves of a class C^1 (formula (18)). Next, conditions for constancy of the perimeter of some polygon lines described on curves belonging to $\mathfrak{M}$ are proved. The results are applied to star-shaped polygons which are described on simple convex and closed curves belonging to $\mathfrak{M}$.

Let us denote by $L^2\langle 0, L\rangle(k)$ a real Hilbert space with $k(s)$ weight. For $k(s)\equiv 1$ the space defined above is a usual Hilbert space $L^2\langle 0, 2\pi j\rangle$. Let us consider the following function

SERDICA Bulgaricae mathematicae publicationes, Vol. 16, 1990, p. 66-74

$$K(s)=\begin{cases}\int_0^s k(t)\,dt, & \text{for } s\geq 0\\ -\int_s^0 k(t)\,dt, & \text{for } s<0.\end{cases} \tag{1}$$

We shall need the following statement, formulated in [1]. "The set of all functions

$$\frac{1}{\sqrt{2\pi j}},\ \frac{1}{\sqrt{\pi j}}\cos\frac{n}{j}K(s),\ \frac{1}{\sqrt{\pi j}}\sin\frac{n}{j}K(s),\quad n=1,2,\ldots \tag{2}$$

is an orthonormal and complete system in $L^2\langle 0, L\rangle(k)$".

From [3] the equation

$$\varphi'(s)=\frac{k(s)}{k(\varphi(s))},\quad \text{for}\quad s\geq 0 \tag{3}$$

has a solution given by the formula $\varphi(s)=K^{-1}(K(s)+\alpha)$, where $\alpha\geq 0$ and K^{-1} is the inverse function for K. Now the function K is defined by (1) for the whole real line R. Therefore we can define the family of solutions of equation (3) by setfing

$$\varphi_\alpha(s)=K^{-1}(K(s)+\alpha j), \tag{4}$$

where $s\in R$ and the index $\alpha\in R$.

Solutions (4) have the following properties

$$\varphi_\alpha(\varphi_\beta(s))=\varphi_{\alpha+\beta}(s), \tag{5}$$

$$\varphi_\alpha(s+L)=\varphi_\alpha(s)+L. \tag{6}$$

We shall prove (6). By (4), we have

$$K(\varphi_\alpha(s+L))=K(s+L)+\alpha j=K(s)+2\pi j+\alpha j=K(\varphi_\alpha(s))+2\pi j$$

$$=K(\varphi_\alpha(s))+\int_{\varphi_\alpha(s)}^{\varphi_\alpha(s)+L}k(t)\,dt=\int_0^{\varphi_\alpha(s)}k(t)\,dt+\int_{\varphi_\alpha(s)}^{\varphi_\alpha(s)+L}k(t)\,dt$$

$$=\int_0^{\varphi_\alpha(s)+L}k(t)\,dt=K(\varphi_\alpha(s)+L).$$

Hence we obtain $\varphi_\alpha(s+L)=\varphi_\alpha(s)+L$.

Remark 1. Generally it is not possible to replace the function f belonging to $L^2\langle 0, L\rangle(k)$ by the function $\varphi_\alpha(s)$. We overcome this difficulty by defining some unitary group $\alpha\to T_\alpha$. For this purpose let us denote by $\mathcal{M}$ the set of all functions belonging to $L^2\langle 0, L\rangle(k)$ which are the restrictions of all continuous functions with a period L to the interval $\langle 0, L\rangle$. Obviously $\mathcal{M}$ is a dense linear subspace of $L^2\langle 0, L\rangle(k)$. At first let us define a semigroup

$$\widehat{T}_\alpha:\mathcal{M}\longrightarrow\mathcal{M} \tag{7}$$

$$(\widehat{T}_\alpha f)(s)=f(\varphi_\alpha(s)),\quad \text{for}\quad f\in\mathcal{M}. \tag{8}$$

Since by (6)

$$(\widehat{T}_\alpha f)(s+L)=f(\varphi_\alpha(s+L))=f(\varphi_\alpha(s)+L)=f(\varphi_\alpha(s))=(\widehat{T}_\alpha f)(s),$$

the restriction of function $\widehat{T}_\alpha f$ belongs to $\mathcal{M}$. Now we verify that $\widehat{T}_\alpha$ is an isometry for each $\alpha\in R$.

$$\|\widehat{T}_\alpha f\|^2=\int_0^L((\widehat{T}_\alpha f)(s))^2k(s)\,ds=\int_0^L f^2(\varphi_\alpha(s))\,k(s)\,ds$$

$$=\int_0^L f^2(\varphi_\alpha(s))\,k(\varphi_\alpha(s))\,\varphi_\alpha'(s)\,ds=\int_{\varphi_\alpha(0)}^{\varphi_\alpha(0)+L} f^2(t)\,k(t)\,dt$$

$$=\int_0^L f^2(t)\,k(t)\,dt=\|f\|^2.$$

Therefore we can extend each operation $\widehat{T}_\alpha$ to the whole space $L^2\langle 0, L\rangle(k)$. Let us denote this extension by T_α. Obviously by (5) the function $\alpha \longrightarrow T_\alpha$ is a semigroup, i. e.

(9) $$T_\alpha T_\beta = T_{\alpha+\beta}$$

(10) $$T_0 = I, \quad \text{where} \quad (If)(s)=f(s).$$

1. Fourier series. Let $f \in L^2\langle 0, L\rangle(k)$ and let the Fourier series for the function f be given by the formula

(11) $$f(s)=\frac{1}{2}A_0+\sum_{n=1}^{\infty} A_n \cos\frac{n}{j}K(s)+B_n \sin\frac{n}{j}K(s).$$

Lemma 1. *If $f \in L^2\langle 0, L\rangle(k)$, then the Fourier series for the function $T_\alpha f$ is given by*

(12) $$(T_\alpha f)(s)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}([A_n\cos(n\alpha)+B_n\sin(n\alpha)]\cos\frac{n}{j}K(s)$$

$$+[-A_n\sin(n\alpha)+B_n\cos(n\alpha)]\sin\frac{n}{j}K(s)).$$

Proof. We shall compute the Fourier coefficients a_n and b_n for the function $T_\alpha f$, where f belongs to $\mathscr{M}$. Then $(T_\alpha f)(s)=f(\varphi_\alpha(s))$ and

$$a_n=\frac{1}{\sqrt{\pi j}}\int_0^L f(\varphi_\alpha(s))\,k(s)\cos\frac{n}{j}K(s)\,ds$$

$$=\frac{1}{\sqrt{\pi j}}\int_0^L f(\varphi_\alpha(s))\,k(s)\cos\frac{n}{j}(K(\varphi_\alpha(s))-\alpha j)\,ds$$

$$=\frac{1}{\sqrt{\pi j}}\int_0^L f(\varphi_\alpha(s))\,k(\varphi_\alpha(s))\,\varphi_\alpha'(s)\cos\frac{n}{j}(K(\varphi_\alpha(s))-\alpha j)\,ds$$

$$=\frac{1}{\sqrt{\pi j}}\int_{\varphi_\alpha(0)}^{\varphi_\alpha(0)+L} f(t)\,k(t)\cos\frac{n}{j}(K(t)-\alpha j)\,dt$$

$$=[\frac{1}{\sqrt{\pi j}}\int_0^L f(t)\,k(t)\cos\frac{n}{j}K(t)\,dt]\cos(n\alpha)+[\frac{1}{\sqrt{\pi j}}\int_0^L f(t)\,k(t)\sin\frac{n}{j}K(t)\,dt]\sin(n\alpha)$$

$$=A_n\cos n\alpha+B_n\sin n\alpha.$$

Similarly, we obtain

$$b_n=\frac{1}{\sqrt{\pi j}}\int_0^L f(\varphi_\alpha(s))\,k(s)\sin\frac{n}{j}K(s)\,ds=-A_n\sin n\alpha+B_n\cos n\alpha.$$

Since the set $\mathscr{M}$ is dense in $L^2\langle 0, L\rangle(k)$, it follows that the formula (12) holds for the whole space $L^2\langle 0, L\rangle(k)$. Q. E. D.

Let us denote by m an integer greater than 1 and let $\beta=\frac{2\pi}{m}$. Let us consider the function

(13) $$F_f(s)=([I+T_\beta+T_{2\beta}+\cdots+T_{(m-1)\beta}]f)(s).$$

Theorem 1. *For each function f belonging to $L^2(0, L)(k)$ the Fourier series for the function F_f is expressed by the formula*

(14) $$F_f(s)=\frac{m}{2}A_0+m\sum_{\substack{n=1\\ m|n}}^{\infty}\left(A_n\cos\frac{n}{j}K(s)+B_n\sin\frac{n}{j}K(s)\right),$$

where $m|n$ denotes that n is divided by m.

Proof. From (12) we have

$$(T_{l\beta}f)(s)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}([A_n\cos(nl\beta)+B_n\sin(nl\beta)]\cos\frac{n}{j}K(s)$$

$$+[-A_n\sin(nl\beta)+B_n\cos nl\beta]\sin\frac{n}{j}K(s)),$$

for $l=1, 2, \ldots, m-1$. Hence we can compute the Fourier coefficients for the function

$$F_f(s)=\frac{1}{2}c_0+\sum_{n=1}^{\infty}(c_n\cos\frac{n}{j}K(s)+d_n\sin\frac{n}{j}K(s))$$

as follows

$$c_n=\sum_{l=1}^{m-1}A_n\cos(nl\beta)+B_n\sin(nl\beta)$$

and

$$d_n=\sum_{l=0}^{m-1}[-A_n\sin(nl\beta)+B_n\cos(nl\beta)].$$

If $m|n$, then it is easy to see that

$$c_n=\sum_{l=0}^{m-1}[A_n\cos 2\pi lr+B_n\sin 2\pi rl]=mA_n$$

and similarly $d_n=mB_n$, where $n=m\cdot r$ and $nl\beta=2\pi rl$, $l=0, 1, \ldots, m-1$. If $m|n$ (n is not divided by m), then $c_n=d_n=0$. Obviously $c_0=mA_0$. Hence equation (14) holds.

Corollary 1. *The function F_f defined by* (13) *is constant everywhere if and only if the Fourier coefficients for the function f satisfy the following condition if $m|n$, then $A_n=B_n=0$.*

Remark 2. If f is a L-periodic and continuous function, then $(T_\alpha f)(s)=f(\varphi_\alpha(s))$. Moreover,

(15) $$F_f(s)=f(s)+f(\varphi(s))+\cdots+f(\varphi^{m-1}(s)),$$

where $\varphi(s)=\varphi_\beta(s)$ and $\varphi^l(s)=\underbrace{\varphi(\varphi\ldots\varphi(s)\ldots)}_{l\text{-times}})$ and, $\varphi^0(s)=s$.

Obviously, by (5) $F_f(\varphi(s))=F_f(s)$. In particular, if $k(s)\equiv 1$, then

(16) $$F_f(s)=f(s)+f(s+\frac{2\pi j}{m})+\cdots+f(s+\frac{2\pi j}{m}(m-1)),$$

where $L=2\pi j$.

2. Applications to geometry of plane curves. We will denote by $\mathscr{L}$ the set of all periodic continuous positive functions with the period $L>0$. Let us define the plane curve $s \to r_f(s)=z(s)$ by the formula

$$z(s)=\begin{cases} \int_0^s f(t)\,k(t)\,e^{iK(t)}\,dt, & \text{for} \quad s\geqq 0 \\ -\int_s^0 f(t)\,k(t)\,e^{iK(t)}\,dt, & \text{for} \quad s<0, \end{cases} \tag{17}$$

where $f\in\mathscr{L}$, (formula (16) [1]). Let C denote the set of all continuous functions $k(s)$ satisfying conditions (A), (B), (C). We consider the following class

$$\mathfrak{W}=\bigcup_{k\in C} M(k), \tag{18}$$

where

$$M(k)=\{r_f\colon f\in\mathscr{L}\}. \tag{19}$$

Obviously the curves belonging to $\mathfrak{M}$ are of a class C^1. If $f=\frac{1}{k}$, then the curve r_f is of a class C^2. Moreover, if $j=1$ and the condition of closeness

$$\int_0^L \cos K(t)\,dt = \int_0^L \sin K(t)\,dt=0 \tag{20}$$

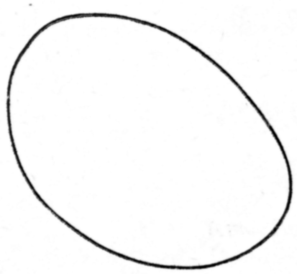

Fig. 1. A simple convex and closed curve $r_f\in\mathfrak{M}$
1^0 $f>0$
2^0 $A_1=B_1=0$
3^0 $j=1$

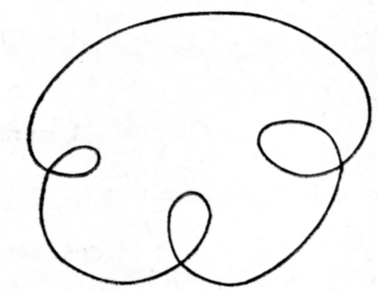

Fig. 2. A locally convex and closed curve $r_f\in\mathfrak{M}$
1^0 $f>0$
2^0 $A_1=B_1=0$
3^0 $j>1$ (in this case $j=4$)

is satisfied, then the curve r_f is an oval, ([5] p. 198). So the set of all ovals is contained in $\mathfrak{W}$. Various types of curves belonging to the class $\mathfrak{W}$ are illustrated in figures 1, 2 and 3. In paper [2] the notion of the equiangular n-polygon has been defined. Now we introduce the m-polygon line described on a curve $r_f\in\mathfrak{W}$.

Definition 1. *Let m be an integer and let $2j<m$. The polygon line (with m vertices) tangent to r_f at the points*

$$z(s),\ z(\varphi(s)),\ \dots,\ z(\varphi^{m-1}(s)),\ z(\varphi^m(s))$$

will be called m-polygon line described on r_f.

We have $\varphi^m(s)=s+L$. Therefore if the curve r_f is closed, then $z(\varphi^m(s))=z(s)$. Then the m-polygon line is closed. If r_f is a simple closed and convex curve, then the m-polygon line described on r_f is equiangular m-polygon in the sense of Definition 1 [2]. Let us point out that the angle between the tangents at the points $z(\varphi^l(s))$, $z(\varphi^{l+1}(s))$, $l=0, 1, \ldots, m-1$, is equal to βj.

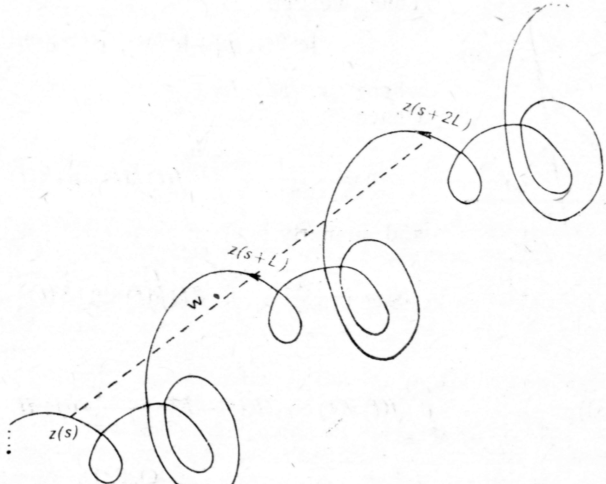

Fig. 3. A locally convex non-closed curve r_f

1^0 $f>0$
2^0 $A_1 \neq 0$ or $B_1 \neq 0$
3^0 $j>1$ (in this case $j=3$)

$w=[\sqrt{\pi j}\,.\,A_1,\ \sqrt{\pi j}\,.\,B_1]=z(s+L)-z(s)$

Theorem 2. *Let $2j<m$ and let $r_f \in \mathfrak{W}$. Then the m-polygon line described on the curve r_f has a constant perimeter if and only if the function F_f defined by* (15) *is a constant function.*

In order to prove this theorem we calculate the sum of the length of the vectors $\overrightarrow{z(s)A}$ and $\overrightarrow{Az(\varphi_\alpha(s))}$ (see Fig. 4.).

Lemma 2. *Let $0<\alpha j<\pi$. The sum $Ł(s, \varphi_\alpha(s))$ of the length of the vectors* $\overrightarrow{z(s)A}$ and $\overrightarrow{Az(\varphi_\alpha(s))}$ *is equal to*

$$Ł(s, \varphi_\alpha(s))=\frac{1}{\cos\frac{1}{2}\alpha j}\int_s^{\varphi_\alpha(s)} f(t)\,k(t)\cos\left(K(t)-K(s)-\frac{1}{2}\alpha\right)dt.$$

Proof. It is easy to prove that the unit vector tangent at the point $z(s)$ to the curve r_f is equal to $e^{iK(s)}$. Hence (see Fig. 4.)

$$Ł(s, \varphi_\alpha(s))=|\overrightarrow{z(s)A}|+|\overrightarrow{Az(\varphi_\alpha(s))}|=\xi-\eta,$$

where

$$z(s)+\xi\, e^{iK(s)}=z(\varphi_\alpha(s))+\eta\, e^{iK(\varphi_\alpha(s))}$$

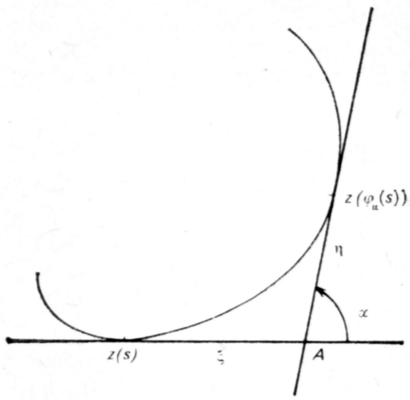

Fig. 4

and $|\mathbf{w}|$ denotes the length of $\mathbf{w}$.
But $K(\varphi_\alpha(s)) = K(s) + \alpha j$ and

$$v = z(\varphi^\alpha(s)) - z(s) = \int_s^{\varphi_\alpha(s)} f(t)\,k(t)\,e^{iK(t)}\,dt.$$

Thus, we get

$$[e^{iK(s)},\ p] + [e^{iK(s)},\ e^{iK(\varphi_\alpha(s))}] = 0,$$

where $[z,\ w] = Im\,\bar{z}\cdot w$.
Hence

$$\eta = \frac{-1}{\sin \alpha j} \int_s^{\varphi_\alpha(s)} f(t)\,k(t) \sin(K(t) - K(s))\,dt$$

and similarly

$$\xi = \frac{-1}{\sin \alpha j} \int_s^{\varphi_\alpha(s)} f(t)\,k(t) \sin(K(t) - K(s) - \alpha j)\,dt.$$

Finally

$$Ł(s,\ \varphi_\alpha(s)) = \frac{1}{\cos \frac{1}{2}\alpha j} \int_s^{\varphi_\alpha(s)} f(t)\,k(t) \cos\left(K(t) - K(s) - \frac{1}{2}\alpha j\right) dt.$$

Q. E. D.

P r o o f o f t h e o r e m 2. Let the m-polygon line described on the curve r_f be given. From Lemma 2 the perimeter $P(s)$ of the m-polygon line is equal to

$$P(s) = Ł(s,\ \varphi(s)) + Ł(s,\ \varphi^2(s)) + \cdots + Ł(\varphi^{m-1}(s),\ \varphi^m(s))$$

$$= \frac{1}{\cos \pi \frac{j}{m}} \sum_{v=0}^{m-1} \int_{\varphi^v(s)}^{\varphi^{v+1}(s)} f(t)\,k(t) \cos\left(K(t) - K(\varphi^v(s)) - \pi \frac{j}{m}\right) dt.$$

Since $\varphi^v(s) \leq t \leq \varphi^{v+1}(s)$, $v = 1, 2, \ldots, m-1$, setting $t = \varphi^v(u)$, we have

$$P(s) = \frac{1}{\cos \pi \frac{j}{m}} \sum_{v=0}^{m-1} \int_{\varphi(s)}^{\varphi(s)} f(\varphi^v(t))\,k(t) \cos\left(K(t) - K(s) - \pi \frac{j}{m}\right) dt$$

$$= \frac{1}{\cos \pi \frac{j}{m}} \int_s^{\varphi(s)} F_f(t)\,k(t) \cos\left(K(t) - K(s) - \pi \frac{j}{m}\right) dt.$$

If the function F_f is constant, i. e. $F_f(s) \equiv c$, then

$$P(s) = \frac{1}{\cos \pi \frac{1}{m}} \int_s^{\varphi(s)} c \cdot k(t) \cos\left(K(t) - K(s) - \pi \frac{j}{m}\right) dt$$

$$= 2c \cdot \operatorname{tg} \pi \frac{j}{m}.$$

Conversely, if we assume that the perimeter $P(s)$ is constant, then $P'(s) \equiv 0$.

But

$$P'(s)\cos\pi\frac{j}{m}=-k(s)\int_s^{\varphi(s)} F_f(t)\,k(t)\sin(K(t)-K(s)-\pi\frac{j}{m})\,dt$$

and

(21)
$$\left(\frac{1}{k(s)}P'(s)\right)'\cos\pi\frac{j}{m}=2F_f(s)\,k(s)\sin\pi\frac{j}{m}-P(s)\,k(s)\cos\pi\frac{j}{m}\cdot$$

Hence

$$F_f(s)=\frac{1}{2}\,c\,\mathrm{tg}\,(\pi\frac{j}{m})\,P(s)$$

and in consequence $F_f(s)$ is constant.

Q. E. D.

In the case when $r_f\in\mathfrak{W}$ is a simple convex and closed curve, the m-polygon line described on r_f is the equiangular m-polygon. Therefore we have the following corollary which is strictly related to Theorem 4 [2].

Corollary 2. *The perimeter of the equiangular m-polygon described on a simple convex and closed curve r_f is constant if and only if the function F_f defined by* (15) *is constant.*

Now, we are going to examine the perimeter of the star-shaped polygons described on a simple convex and closed curve $r_f\in\mathfrak{W}$.

Definition 2. *Let the equiangular m-polygon described on a simple convex and closed curve r_f be given. Lengthening the sides of the equiangular m-polygon we obtain the equiangular m-star-shaped polygon described on r_f.*

Theorem 3. *Let a simple convex and closed curve $r_f\in\mathfrak{W}$ be given. Then the equiangular m-star-shaped polygons described on the curve r_f have the same perimeter if and only if the function F_f given by* (15) *is constant.*

Proof. Using the formula from Lemma 2, we can express the perimeter $Q(s)$ of the m-star-shaped polygon described on the curve r_f as the following sums

$$Q(s)=\text{Ł}(s,\ \varphi^2(s))+\text{Ł}(\varphi^2(s),\ \varphi^4(s)+\cdots+\text{Ł}(\varphi^{2l-1}(s),\ \varphi^{2l+1}(s))$$
$$+\text{Ł}(\varphi(s)+L,\ \varphi^3(s))+L)+\cdots+\text{Ł}(\varphi^{2l-2}(s)+L,\ s+2L),\ \text{if}\ m=2l+1$$

and

$$Q(s)=\text{Ł}(s,\ \varphi^2(s))+\text{Ł}(\varphi^2(s),\ \varphi^4(s))+\cdots+\text{Ł}(\varphi^{2l-2}(s),\ \varphi^{2l}(s))$$
$$+\text{Ł}(\varphi(s),\ \varphi^3(s))+\text{Ł}(\varphi^3(s),\ \varphi^5(s))+\cdots+\text{Ł}(\varphi^{2l-3}(s),\ \varphi^{2l-1}(s)),$$

if $m=2l$. Hence, for $m=2l$, we have

$$Q(s)\cos2\pi\frac{1}{m}=\sum_{v=0}^{l-1}\int_{\varphi^{2v}(s)}^{\varphi^{2v+2}(s)} f(t)\,k(t)\cos(K(t)-K(\varphi^{2v}(s))-2\pi\frac{1}{m})\,dt$$

$$+\sum_{v=0}^{l-1}\int_{\varphi^{2v+1}(s)}^{\varphi^{2v+3}(s)} f(t)\,k(t)\cos(K(t)-K(\varphi^{2v+1}(s))-2\pi\frac{1}{m})\,dt.$$

Changing the variables in the above integrals, we obtain

$$Q(s)\cos2\pi\frac{1}{m}=\int_s^{\varphi^2(s)}\sum_{v=0}^{l-1} f(\varphi^{2v}(t))\,k(t)\cos(K(t)-K(s)-2\pi\frac{1}{m})\,dt$$

$$+\int_s^{\varphi^2(s)}\sum_{v=0}^{l-1} f(\varphi^{2v+1}(s))\,k(t)\cos(K(t)-K(s)-2\pi\frac{1}{m})\,dt.$$

Since

$$Q(s)\cos 2\pi\frac{1}{m}=\int\limits_{s}^{\varphi^2(s)} F_f(t)\,k(t)\cos\left(K(t)-K(s)-2\pi\frac{1}{m}\right)dt,$$

similarly as in the proof of Theorem 2 it follows that the perimeter $Q(s)$ is constant if and only if the function F_f is constant.

Now we observe that if $f=\frac{1}{k}$, then

$$F_f(s)=\frac{1}{k(s)}+\cdots+\frac{1}{k(\varphi^{m-1}(s))}.$$

Moreover, the curve $r_{\frac{1}{k}}$ is in the arc length parametrization and the curvature of the curve is equal to k. This implies the following

Theorem 4. *Let an oval $r_{\frac{1}{k}}$ be given. Then all equiangular m-star-shaped polygons described on the oval have the same perimeter if and only if the sum of the radii curvatures*

$$\frac{1}{k(s)}+\frac{1}{k(\varphi(s))}+\cdots+\frac{1}{k(\varphi^{m-1}(s))}=C,$$

where $C>0$ is constant.

Theorem 5. *The m-star, shaped polygons described on a simple convex and closed curve $r_f\in\mathfrak{W}$ have the same perimeter if and only if the m-polygons described on this curve have the same perimeter.*

Proof. Let a curve r_f be represented by equation (17). By Theorem 3 and Corollary 2 the m-star-shaped polygons and the m-polygons described on this curve have the same perimeter if and only if F_f is a constant function.

Q. E. D.

Let $f\in\mathscr{L}$ satisfy the condition (22). There exists a subset $\{m_v\}$ of the set of positive integers such that, if $m_v\,|\,n$, then $A_n=B_n=0$. Then by Corollary 1 and Theorem 2 we have the following:

Corollary 3. *For each fixed v the m_v-polygon lines described on r_f have the same perimeter.*

Similarly by Corollary 1 and Theorem 5 it follows

Corollary 4. *Let $r_f\in\mathfrak{W}$ be a simple convex and closed curve such that f satisfies condition (22). Then for each fixed v the equiangular m_v-polygons described on r_f and the equiangular m_v-star-shaped polygons, described on r_f, have the same perimeter.*

REFERENCES

1. S. Góźdź. An antipodal set of a periodic function. *J. Anal. Math. Appl.*, **146**, 1990.
2. W. Cieślak, S. Góźdź. On curves which bound special convex sets. *Serdica*, **13**, 1987, 281-286.
3. W. Cieślak, J. Zając. The Rosettes. *Math. Scand.*, **58**, 1986, 114-118.
4. R. L. Tennison. Smooth curves of constant width. *The Math. Gaz.*, **60**, 1976, 270-272.
5. D. Laugwitz. Differential and Riemannian Geometry. New York, 1965.

Instytut Matematyki UMCS
Pl. M. C. Sklodowskiej 1
20-031 Lublin, Poland

Received 6. 11. 1987
Revised 5. 08. 1989