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ON THE CHRISTOFFEL-DARBOUX FORMULA AND ITS APPLICATIONS

R. K. RAINA

Simple operational techniques and summation manipulations are applied to the familiar Christoffel-Darboux formula to obtain a new class of summation relations in order to unify and generalize the various results concerning the finite summations. The usefulness of our main result (2.1) given below is shown by illustrating its applications, thus revealing the relevance to the well-known Srivastava's (1986) recent results.

1. Preliminaries. Motivated by the usefulness of various properties of a well-known triple-series analogue to Appell's double hypergeometric function F_2 [9, p. 33, Eqn. (5)], in particular the production of bremsstrahlung by the interaction of polarized electrons with the Coulomb field of a nucleus [4] and by the calculation of radial matrix elements of the radiative transitions between the states of a relativistic electron in a Coulomb field [5], Srivastava [8] extended his earlier summation formula [6, p. 1088, Eqn. [(1.5)], (See also [1] and [2]) to the generalized triple hypergeometric function $F^{(3)}[x, y, z]$. This function is defined by ([9, p. 69, Eqn. (39)]):

$$(1.1)\qquad F^{(3)}[x, y, z]=F^{(3)}\left[\begin{matrix}(a): & :(b);\,(b');\,(b''):\,(c);\,(c');\,(c''); \\ (e): & :(g);\,(g');\,(g''):\,(h);\,(h');\,(h'');\end{matrix}\; x, y, z\right]$$

$$=\sum_{m,n,p=0}^{\infty}\Lambda(m, n, p)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

where the function $\Lambda(m, n, p)$ is mentioned in [9].

Let a polynomial system be defined by

$$(1.2)\qquad S_n(x)=\sum_{m=0}^{n}\delta(m, n)\,x^m,\quad n\geqq 0.$$

It is well known that if $S_n(x)$ is an orthogonal system, then it will satisfy a three-term recurrence relation [3, p. 271]:

$$(1.3)\qquad S_{n+1}(x)=(A_nx+B_n)S_n(x)-C_nS_{n-1}(x),$$

whree the coefficients A_n, B_n and C_n are easily determinable (see [3]). Further, (1.2) and (1.3) yield the well-known Christoffel-Darboux formula

$$(1.4)\qquad \sum_{k=0}^{n}h_k^{-1}S_k(x)S_k(y)$$

$$=\frac{(A_nh_n)^{-1}}{(x-y)}[S_{n+1}(x)S_n(y)-S_n(x)S_{n+1}(y)],\quad x\neq y,$$

where

$$(1.5)\qquad \begin{cases}A_n=\delta(n+1,\, n+1)/\delta(n,\, n),\\ h_n=(S_n,\, S_n)=\int_a^b w(x)\,S_n^2(x)dx,\end{cases}$$

with $w(x)$ being a weight function which is non-negative over the interval $[a, b]$.

As usual, (a_p) is used to abbreviate the array of p-parameters $a_1, \ldots, a_p (p \geq 1)$, and, $(a)_n = \Gamma(a+n)/\Gamma(a)$.

In this paper we aim at deriving a class of summation relations which are generated by the Christoffel-Darboux formula (1.4) via the application of simple operational techniques and summation manipulations. Our main result (2.1) below provides the unification of various summation formulae and their applications are briefly treated depicting their relevance to some of the recent results due to Srivastava [8].

2. The main result. We shall prove the following summation relation:

$$(2.1) \qquad \sum_{k=0}^{n} h_k^{-1} \sum_{m=0}^{k} \sum_{r=0}^{k} \sum_{s=0}^{\infty} \theta \begin{matrix} (a_p):(\alpha_P) \\ (b_q):(\beta_Q) \end{matrix} [m, r, s]\, \delta(m, k)\, \delta(r, k) \gamma_s \,.\, x^m y^r z^s$$

$$= \frac{(A_n h_n)^{-1}}{(x-y)\, \theta \begin{matrix} (a_p-1):(\alpha_P-1) \\ (b_q-1):(\beta_Q-1) \end{matrix} [1, 0, 0]} \Big[\sum_{m=0}^{n+1} \sum_{r=0}^{n} \sum_{s=0}^{\infty} \theta \begin{matrix} (a_p-1):(\alpha_P-1) \\ (b_q-1):(\beta_Q-1) \end{matrix} [m, r, s]$$

$$.\, \delta(m, n+1)\delta(r, n)\gamma_s x^m y^r z^s - \sum_{m=0}^{n} \sum_{r=0}^{n+1} \sum_{s=0}^{\infty} \theta \begin{matrix} (a_p-1):(\alpha_P-1) \\ (b_q-1):(\beta_Q-1) \end{matrix} [m, r, s]\, \delta(m, n)\, \delta(r, n+1)\gamma_s x^m y^r z^s \Big],$$

where, for convenience,

$$(2.2) \qquad \theta \begin{matrix} (a_p):(\alpha_P) \\ (b_q);(\beta_Q) \end{matrix} [m, r, s] = \frac{\prod_{j=1}^{p} (a_j)_{m+r+s} \prod_{j=1}^{P} (\alpha_j)_{m+r}}{\prod_{j=1}^{q} (b_j)_{m+r+s} \prod_{j=1}^{Q} (\beta_j)_{m+r}},$$

provided that $\operatorname{Re}(a_j) > 1$ $(j=1, \ldots, p)$, $\operatorname{Re}(\alpha_j) > 1$ $(j=1, \ldots, P)$, $\operatorname{Re}(b_j) > 1$ $(j=1, \ldots, q)$ and $\operatorname{Re}(\beta_j) > 1$ $(j=1, \ldots, Q)$, $x \neq y$, $|z| < \rho\, (\rho > 0)$, $\{\delta(m, n)\}$ and $\{\gamma_s\}$ are bounded sequences, and A_n and h_n are given by (1.5).

Proof. Let us start with the formula (1.4). First we shall replace x by xt and y by yt, next we shall multiply both sides by $t^{\lambda-1}$, then we shall take the Laplace transform with respect to t, and make use of the Eulerian integral:

$$(2.3) \qquad \int_0^{\infty} e^{-t} t^{z-1} dt = \Gamma(z), \ \operatorname{Re}(z) > 0.$$

If we repeat the same procedure where the multiplying factor now is $t^{\mu-1}$, then (1.4) will give

$$(2.4) \qquad \sum_{k=0}^{n} h_k^{-1} \sum_{m=0}^{k} \sum_{r=0}^{k} (\lambda)_{m+r} (\mu)_{m+r}\, \delta(m, k)\delta(r, k) x^m y^r$$

$$= \frac{(A_n h_n)^{-1}}{(\lambda-1)(\mu-1)(x-y)} \Big[\sum_{m=0}^{n+1} \sum_{r=0}^{n} (\lambda-1)_{m+r} (\mu-1)_{m+r}\, \delta(m, n+1)\delta(r, n) x^m y^r$$

$$- \sum_{m=0}^{n} \sum_{r=0}^{n+1} (\lambda-1)_{m+r} (\mu-1)_{m+r}\, \delta(m, n)\, \delta(r, n+1) x^m y^r \Big],$$

provided that $\operatorname{Re}(\lambda) > 1$, $\operatorname{Re}(\mu) > 1$, and $x \neq y$.

Further, we now replace in (2.4) x by x/t, y by y/t, multiply both sides by t^{-v}, take the inverse Laplace transform, and use the formula

$$\frac{1}{2\lambda i}\int_{\psi-i\infty}^{\psi+i\infty} e^t\, t^{-z}\, dt = \frac{1}{\Gamma(z)}, \quad \psi>0,\ \mathrm{Re}\,(z)>0, \tag{2.5}$$

and again repeat this procedure, where the multiplication is carried out by the factor $t^{-\sigma}$. If in the expression obtained we replace λ by $\lambda+s$, ν by $\nu+s$ and multiply both sides by $(\lambda)_s\,\gamma_s z^s/(\nu)^s$, thus summing from $s=0$ to $s=\infty$, then we have

$$\sum_{k=0}^{n} h_k^{-1} \sum_{m=0}^{k}\sum_{r=0}^{k}\sum_{s=0}^{\infty} \frac{(\lambda)_{m+r+s}(\mu)_{m+r}}{(\nu)_{m+r+s}(\sigma)_{m+r}}\,\delta(m,k)\delta(r,k)\gamma_s x^m y^r z^s \tag{2.6}$$

$$= \frac{(A_n h_n)^{-1}(x-1)(\sigma-1)}{(\lambda-1)(\mu-1)(x-y)} \sum_{m=0}^{n+1}\sum_{r=0}^{n}\sum_{s=0}^{\infty} \frac{(\lambda-1)_{m+r+s}(\mu-1)_{m+r}}{(\nu-1)_{m+r+s}(\sigma-1)_{m+r}}\,\delta(m,n+1)\cdot\delta(r,n)\,\gamma_s x^m y^r z^s$$

$$-\sum_{m=0}^{n}\sum_{r=0}^{n+1}\sum_{s=0}^{\infty} \frac{(\lambda-1)_{m+r+s}(\mu-1)_{m+r}}{(\nu-1)_{m+r+s}(\sigma-1)_{m+r}}\,\delta(m,n)\delta(r,n+1)\,\gamma_s x^m y^r z^s,$$

provided that $\mathrm{Re}(\lambda)>1$, $\mathrm{Re}(\mu)>1$, $\mathrm{Re}(\nu)>1$, $\mathrm{Re}(\sigma)>1$, and $x\neq y$.

Applying once again the Laplace and the inverse Laplace transform techniques following (2.3) and (2.5) (see [9]), then from (2.6) by induction, we obtain (2.1) which completes the proof.

3. Applications. The usefulness of our assertion (2.1) can be illustrated by discussing some of its applications which are of interest. To this end, we set

$$\delta(m,n) = \frac{(-n)_m}{(1+\alpha)_m\, m!}, \quad n\geqq m\geqq 0, \quad \mathrm{Re}\,(\alpha)>-1, \tag{3.1}$$

then (in terms of Laguerre polynomials) from (1.2), we have

$$S_n(x) = \binom{n+\alpha}{n}^{-1} L_n^{(\alpha)}(x), \tag{3.2}$$

and from (1.5) in view of [9, p. 74, Eqn. (19)], we have

$$\begin{cases} A_n = -1/(1+\alpha+n), \\ h_n = \Gamma(1+\alpha)\binom{n+\alpha}{n}^{-1}. \end{cases} \tag{3.3}$$

Using the substitutions (3.1) and (3.3), and putting

$$\gamma_s = \frac{\prod_{j=1}^{u}(\eta_j)_s}{\prod_{j=1}^{v}(\delta_j)_s}, \quad u\geqq 1,\ v\geqq 1, \tag{3.4}$$

then (2.1) by virtue of definition (1.1) gives the three variable summation result:

$$\sum_{k=0}^{n}\binom{k+\alpha}{k} F^{(3)}\left[\begin{matrix}(a_p)::(\alpha_P);-;-:-k;-k;(\eta_u);\\(b_q)::(\beta_Q);-;-:-\alpha+1;\alpha+1;(\delta_v);\end{matrix}\ x,y,z\right] \tag{3.5}$$

$$= \frac{(\alpha+1)_{n+1}}{n!\,(x-y)\theta\begin{matrix}(a_p-1):(\alpha_P-1)\\(b_q-1):(\beta_Q-1)\end{matrix}\left[1,0,0\right]}$$

$$\times F^{(3)}\left[\begin{matrix}(a_p)-1::(\alpha_P)-1;-;-:-n;-n-1;(\eta_u);\\(b_q)-1::(\beta_Q)-1;-;-:\alpha+1;\alpha+1;(\delta_v);\end{matrix}\ x,y,z\right] + x\leftrightarrow y,$$

where $x \leftrightarrow y$ indicates the presence of another term which stems from the first by interchanging x and y.

The formula (3.5) was proved by Srivastava [8, p. 315, Eqn. (2.1)] by appealing to his earlier formula [7, p. 3, Eqn. (2.1)] which in turn originated from the result [6, p. 1088, Eqn. (1.5)].

If next we set

$$\delta(m, n) = \frac{(-n)_m(\alpha+\beta+n+1)_m}{(\beta+1)_m}, \quad n \geq m \geq 0, \quad \operatorname{Re}(\alpha) > -1, \quad \operatorname{Re}(\beta) > -1, \tag{3.6}$$

in (1.2), then in terms of Jacobi polynomials we have

$$S_n\left(\frac{1+x}{2}\right) = (-1)^n \binom{\beta+n}{n}^{-1} P_n^{(\alpha,\beta)}(x), \tag{3.7}$$

and from (1.5), in view of [9, p. 71, Eqn. (1)] we have

$$A_n = \frac{-(\alpha+\beta+n+2)_{n+1}(\beta+1)_n(n+1)}{(\beta+1)_{n+1}(\alpha+\beta+n+1)_n}, \quad \text{and} \tag{3.8}$$

$$h_n = \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)\,(\alpha+\beta+2n+1)}. \tag{3.9}$$

Effecting the above substitutions and using (3.4), it is seen that formula (2.1) yields another known result of Srivastava [8, p. 315, Eqn. (2.2)]. Several other special cases of our result (2.1) can be deduced by giving series summations involving different types of hypergeometric functions.

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Department of Mathematics
College of Technology and Agricultural Engineering
Udaipur-313001, Rajasthan, India

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