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MAPPING THEOREMS FOR Z_p-ACTIONS WITH FIXED POINTS

SIMEON T. STEFANOV

Some theorems of the Borsuk—Ulam type for arbitrary Z_p-actions are obtained (p is prime). The case of free Z_p-actions has been studied by many authors. We give a generalization of some of their results for n-spheres. Furthermore, we get mapping theorems for Z_p-actions in n-balls.

The various generalizations of the Borsuk—Ulam theorem usually concern fixed point free periodic transformations of a prime period p, i. e. free Z_p-actions. However, when $p>2$ many spaces do not admit a free Z_p-action (for example the even dimensional spheres). It is natural to see what happens in this case. In the present paper we deal with arbitrary periodic transformations of a prime period acting in some compact space X. We define some homological invariant of X in terms of its equivariant homologies and we apply it so as to obtain a coincidence point result for spheres (theorem 2). This theorem is a partial generalization of the corresponding results of A. Schwarz [14], H. Munkholm [6] and A. Necochea [7] for free Z_p-actions. Finally, we get a mapping theorem for Z_p-actions in balls (theorem 3) which in the case of involutions is proved by the author [10].

The homological invariant, we shall construct, is the so-called "index" introduced by P. Smith [9] for free Z_p-actions in homology spheres (p—prime), further defined and investigated by C. Yang [11] for fixed point free involutions in arbitrary spaces. Here we shall give definition of the "index" for arbitrary Z_p-actions in some compact topological space.

1. The index. Let X be a compact Hausdorff space and $T: X\to X$ be a periodic map of a prime period p. A subset F of X is called *invariant*, if $TF=F$. We denote by X_T the fixed points set of T, $X_T=\{x\in X \mid Tx=x\}$. Since p is prime, each orbit $x=\{x, Tx, \dots, T^{p-1}x\}$ consists either of 1 or of p points.

In order to define the index of X with respect to T we make use of Čech homologies, therefore we shall assume first X being a simplicial complex and $T: X\to X$ a simplicial transformation of a prime period p such that for any simplex σ the points of $\sigma\cap T\sigma$ are fixed points of T. The last condition is always satisfied for the barycentric subdivision of a periodic simplicial transformation (see G. Bredon [3]). Then the set X_T is a subcomplex of X.

Recall now the definition of the equivariant homology groups of X. Consider the operators $\delta=1+T+\dots+T^{p-1}$, $\tau=1-T$, acting in the group $C_n(X)$ of the n-chains with coefficients in Z_p. As usual, by ρ we mean one of δ and τ, then $\bar\rho$ means the other one. Evidently $\rho\bar\rho=0$. The chains, which vanish under ρ are called ρ-chains. We denote

$$C_n^\rho(X;\ T)=\{\varkappa\in C_n(X) \mid \rho\varkappa=0\}.$$

Furthermore, ρ — cycles, ρ — boundaries and equivariant ρ-homology groups are defined as follows:

$$Z_n^\rho(X;\ T)=\{z \in C_n^\rho(X;\ T) \mid \partial z=0\}$$
$$B_n^\rho(X;\ T)=\partial C_{n+1}^\rho(X;\ T)$$
$$H_n^\rho(X;\ T)=Z_n^\rho(X;\ T)/B_n^\rho(X;\ T).$$

The relation between ρ and $\bar\rho$-homologies is illustrated by the following elementary proposition (P. Smith [9]).

Proposition 1. *$\varkappa$ is a ρ-chain if and only if $\varkappa=\bar\rho c+\varkappa_T$, where the chain $\varkappa_T$ consists of all simplexes of $\varkappa$ lying in X_T (taken with the corresponding coefficients). Provided that $\varkappa$ is a ρ-cycle, $\varkappa_T$ is also a cycle.*

Assume first, that $X_T \neq \emptyset$. For $n \geq r$ we shall define by recurrence the index homomorphisms $\nu_{n,r}: Z_n^\rho(X;T) \to \tilde H_r(X_T)$ of the n-dimensional ρ-cycles group into the reduced (ordinary) r-homology group of X_T.

1) $n=r$. Let z be an r-dimensional ρ-cycle. Then by proposition 1 $z=\bar\rho c+z_T$. Put $\nu_{r,r}(z)=\{z_T\}$, where $\{z_T\}$ is the class of the cycle z_T in $\tilde H_r(X_T)$.

2) $n>r$. Let $z=\bar\rho c+z_T$ be an n-dimensional ρ-cycle. Then $\bar\rho\,\partial c=0$, that means ∂c is a $\bar\rho$-cycle of dimensional $n-1$, hence the element $\nu_{n-1,r}(\partial c)$ is defined. Set $\nu_{n,r}(z)=\nu_{n-1,r}(\partial c)$.

Theorem 1. *$\nu_{n,r}$ is a homomorphism such that $\nu_{n,r}B_n^\rho(X;\ T)=0$, consequently it induces a homomorphism*

$$\nu_{n,r}: H_n^\rho(X;\ T)\to\tilde H_r(X_T).$$

Proof. We shall proceed by induction on n.

1. $n=r$. Clearly, $\nu_{r,r}$ is a homomorphism, since for any two ρ-cycles $z=\bar\rho c+z_T$, $z'=\bar\rho c'+z'_T$ we have $z+z'=\bar\rho(c+c')+z_T+z'_T$. We shall verify, that $\nu_{r,r}B_r^\rho(X;\ T)=0$ If $z\in B_r^\rho(X;\ T)$, then $z=\partial\varkappa$ for some $(r+1)$-dimensional ρ-chain $\varkappa$. By Proposition 1 $\varkappa=\bar\rho c+\varkappa_T$, then $z=\partial\varkappa=\bar\rho\partial c+\partial\varkappa_T$, whence $\nu_{r,r}(z)=\{\partial\varkappa_T\}=0$.

2) $n>r$. We have to prove first, that the definition of $\nu_{n,r}$ is correct, i. e. it does not depend on the particular choice of the chain c. Let $z=\bar\rho c+z_T$ and $z=\bar\rho c'+z_T$, then $\bar\rho(c-c')=0$, so that $c-c'$ is a $\bar\rho$-chain. By the induction hypothesis, we have $\nu_{n-1,r}\partial(c-c')=0$, hence $\nu_{n-1,r}\partial c=\nu_{n-1,r}\partial c'$.

Let us show now, that $\nu_{n,r}B_n^\rho(X;\ T)=0$. Let $z\in B_n^\rho(X;\ T)$, i. e. $z=\partial\varkappa$ for some $(n+1)$-dimensional ρ-chain $\varkappa$. Then $\varkappa=\bar\rho c+\varkappa_T$ and $z=\partial\varkappa=\bar\rho\partial c+\partial\varkappa_T$, whereby $\nu_{n,r}(z)=\nu_{n-1,r}(\partial\partial c)=\nu_{n-1,r}(0)=0$. The theorem is proved.

To avoid ambiguity in the definition of the index homomorphism $\nu_{n,r}: H_n^\rho(X;\ T)\to\tilde H_r(X_T)$ we shall assume that

$$\rho=\begin{cases}\delta & \text{if } n-r\equiv 0 \pmod 2\\ \tau & \text{if } n-r\equiv 1 \pmod 2).\end{cases}$$

Definition of the index. Let X be a simplicial complex and T be a simplicial map in X of a prime period p with $X_T\neq\emptyset$. Then $\text{in}(X;\ T)$ is the greatest n such that the homomorphism $\nu_{n,r}$ is nontrivial for some r.

Suppose now $X_T=\emptyset$. Then in the same way we may define $\text{in}(X;\ T)$ — it is enough only to modify 1) from the definition of $\nu_{n,r}$ as follows:

1) $n=0$. Let z be a 0-dimensional ρ-cycle. Then $z=\bar\rho c$ and we put $\nu_0(z)=I(c)\in Z_p$, where $I(c)$ is the index of the 0-chain c modulo p (if $c=\Sigma a_i\sigma_i$, then $I(c)=\Sigma a_i (\text{mod } p)$). Clearly, $\nu_0(z)$ does not depend on the choice of c since if $z=\bar\rho c=\bar\rho c'$, then

$\bar\rho(c-c')=0$, therefore $I(c-c')=0$. In this case we find index homomorphisms $\nu_n: H_n^{\rho}(X;\ T)\to \mathbf{Z}_p$.

Assume as before

$$\rho=\begin{cases}\delta & \text{if} \quad n\equiv 0 \pmod 2\\ \tau & \text{if} \quad n\equiv 1 \pmod 2\end{cases}$$

and define $\operatorname{in}(X;\ T)=\max\{n \mid \nu_n\neq 0\}$.

When $p=2$, we obtain exactly Yang's index [11].

Consider now an arbitrary compact space X together with a periodic transformation $T: X\to X$ of a prime period p. The index of X (with respect to T) is defined in terms of its equivariant homology groups. The group $H_n^{\rho}(X;\ T)$ is by definition the inverse limit

$$H_n^{\rho}(X;\ T)=\varprojlim H_n^{\rho}(N_\omega;\ T),$$

where ω is an open covering of X invariant under T and N_ω is its nerve (see G. Bredon [3] for details). Then the index homomorphisms $\nu_{n,r}: H_n^{\rho}(X\ ;\ T)\to \tilde{H}_r(X_T)$ are obtained as limits of the corresponding index homomorphisms of N_ω. The index $\operatorname{in}(X;\ T)$ is now defined in the same way:

$$\operatorname{in}(X;\ T)=\max\{n \mid \nu_{n,r}\neq 0 \text{ for some } r\}.$$

Evidently, $\operatorname{in}(X;\ T)\leq \dim X$, since the inequality $\dim X\leq k$ implies $H_{k+1}^{\rho}(X;\ T)=0$.

The index is an important invariant, which is convenient for obtaining by induction various coincidence point results.

Definition. *A closed invariant subset F of X is said to be a strong partition in X if it separates the points x and Tx for any $x\in X\setminus F$. Since p is prime, it is clear, that F separates each pair T^ix, T^jx, where $i\neq j \pmod p$. Evidently, F is a strong partition in X iff $X\setminus F=\bigcup_{i=0}^{p-1} T^iU$, where U is an open set such that $T^iU \cap T^jU=\emptyset$ for $i\neq j \pmod p$. Each strong partition contains X_T.*

Proposition 2. *Let F be a strong partition in X. Then there exists a homomorphism $\psi: H_n^{\rho}(X;\ T)\to H_{n-1}^{\bar\rho}(F;\ T)$, such that $\nu_{n-1,r}\psi(\zeta)=\nu_{n,r}(\zeta)$ for any $\zeta\in H_n^{\rho}(X;T)$.*

Proof. For $p=2$ this theorem is proved by Yang. Following his idea, we shall briefly sketch the proof and refer the reader to [11] for details. Analagous property has also the Fadell—Rabinowitz index [5].

Take an arbitrary $\zeta\in H_n^{\rho}(X;\ T)$. Let $\pi_\omega: H_n^{\rho}(N_\omega;\ T)\to H_n^{\rho}(X;\ T)$ be the canonical projection, where ω is some "small" invariant covering of X so that $\pi_\omega(\zeta_\omega)=\zeta$ for some $\zeta_\omega\in H_n^{\rho}(N_\omega;\ T)$. Since F is a strong partition in X, $X\setminus F=\bigcup_{i=0}^{p-1} T^iU$, where $T^iU \cap T^jU=\emptyset$ for $i\neq j \pmod p$. We may assume, that every element of ω nonintersecting F is contained in exactly one of the sets T^iU, $0\leq i\leq p-1$. Set $\omega'=\{V\in\omega \mid \operatorname{St} V\cap F\neq\emptyset\}$. Then the complex $N_{\omega'}$ is a strong partition in N_ω. The ρ-cycle ζ_ω admits the representation $\zeta_\omega=\bar\rho c_\omega+\zeta_\omega^T$, where ∂c_ω is $\bar\rho$-homologous to a cycle z_ω lying in $N_{\omega'}$. Define $\psi_\omega(\zeta_\omega)=z_\omega$. Clearly, $\nu_{n,r}(\zeta_\omega)=\nu_{n-1,r}(\partial c_\omega)=\nu_{n-1,r}(z_\omega)=\nu_{n-1,r}\psi_\omega(\zeta_\omega)$. We have $H_{n-1}^{\bar\rho}(F;\ T)=\varprojlim H_{n-1}^{\bar\rho}(N_{\omega'};\ T)$, thus ψ_ω converges to a homomorphism $\psi: H_n^{\rho}(X;\ T)\to H_{n-1}^{\bar\rho}(F;\ T)$.

Proposition 3. *If $\operatorname{in}(X;\ T)\geq n$ and F is a strong partition in X, then* $\operatorname{in}(F;\ T)\geq n-1$.

This follows immediately from proposition 2.

Proposition 4. *$\mathrm{in}(S^n; T)=n$ for any periodic map $T: S^n \to S^n$ of a prime period p.*

Proof. As following by the famous theorem of P. Smith [9] the fixed points set of T S^n_T is a homology r-sphere for some $-1 \leqq r \leqq n$. Consider first the case $r \geqq 0$, which means, that $S^n_T \neq \emptyset$. Let z^r be an r-cycle in S^n_T nonhomologous to zero in $\widetilde{H}_r(S^n_T)$. Then by definition $\nu_{r,r}(z^r)=\{z^r\}\neq 0$. Suppose now, that for some $s<n$ we have found an s-dimensional ρ-cycle z^s with $\nu_{s,r}(z^s)\neq 0$. But $z^s \sim 0$ in s^n, i. e. $z^s=\partial\varkappa^{s+1}$ and we set $z^{s+1}=\bar{\rho}\varkappa^{s+1}$. This is a ρ-cycle, since $\partial z^{s+1}=\rho\partial\varkappa^{s+1}=\rho z^s=0$ and therefore $\nu_{s+1,r}(z^{s+1})=\nu_{s+1,r}(\rho\varkappa^{s+1})=\nu_{s,r}(\partial\varkappa^{s+1})=\nu_{s,r}(z^s)\neq 0$. Finally, we get some n-dimensional ρ-cycle z^n with $\nu_{n,r}(z^n)\neq 0$, which implies $\mathrm{in}(S^n; T)\geqq n$. The inverse follows from $\mathrm{in}(S^n; T)\leqq \dim S^n = n$.

Whenever $r=-1$, i. e. $S^n_T=\emptyset$, we have to start from an arbitrary 0-dimensional ρ-cycle z^0 with $\nu_0(z^0)\neq 0$.

In the case of a free Z_p-action, the index is closely associated with the so-called "genus" introduced by A. Schwartz [14], the "B-index" of C. Yang [12] and the "co-index" of Conner and P. Floyd [4] (the last two concepts are introduced for $p=2$). This connection is illustrated by the following.

Proposition 5. *Let $T: X \to X$ be a fixed point free periodic map of a prime period and* $\mathrm{in}(X; T)\geqq n$. *Then for any decomposition $X=\bigcup_{i=1}^{n}\Phi_i$ of X into n closed invariant subsets, some Φ_i contains an invariant continuum K.*

To prove it, one has to carry out induction on n and to make use of Proposition 3. When $X=S^n$ it is proved by M. Krasnosel'skii [13].

We may conclude from Proposition 5, that $\mathrm{in}(X; T)<\infty$ whenever T is fixed point free. Really, one can find a decomposition $X=\bigcup_{i=1}^{n}\Phi_i$ of X into closed invariant subsets, none of which contains an invariant continuum (recall, that X is compact) When T has fixed points this is not always true — for example, if $\widetilde{H}_r(X_T)\neq\{0\}$ for infintely many values of r.

2. Mapping theorems for spheres.

Theorem 2. *Let $T: X \to X$ be a periodic map of a prime period p* and $\mathrm{in}(X; T)\geqq n$. Given a map $f: X \to R^k$ consider the set $A(f)=\{x\in X \mid f(x)=f(Tx)=\ldots=f(T^{p-1}x)\}$. Then $\mathrm{in}(A(f); T)\geqq n-k(p-1)$ and consequently $\dim A(f)\geqq n-k(p-1)$.

Proof. Suppose first $k=1$. Let $f: X\to R^1$. Put $F_s=\{x\in X \mid f$ maps s points of orbit x into a single one$\}$. Clearly $A(f)=F_p\subset F_{p-1}\subset\ldots\subset F_2\subset F_1=X$ and each F_s is a strong partition in F_{s-1}. Then by Proposition 3 $\mathrm{in}(A(f); T)\geqq n-(p-1)$. Let now $f: X\to R^k$ and $f=(f_1,\ldots,f_k)$. Then $\mathrm{in}(A(f_i): T)\geqq n-(p-1)$. Obviously $A(f)=\bigcap_{i=1}^{k}A(f_i)$ and by the same Proposition 3 we get $\mathrm{in}(A(f); T)\geqq n-k(p-1)$.

Corollary 1. *Let $T: S^n\to S^n$ be an arbitrary periodic map of a prime period p. Then for any map $f: S^n\to R^k$ we have* $\dim A(f)\geqq n-k(p-1)$, *where* $A(f)=\{x\in S^n \mid f|_{\text{orbit } x}=\text{const}\}$.

This follows immediately from theorem 2 and Proposition 4. In the case of a fixed point free T it is proved by many authors — C. Yang [11] for $p=2$, A. Schwartz [14], H. Munkholm [6], A. Necochea [7] (last two for maps into a k-manifold). Munkholm showed, that this estimate cannot be strengthened in general.

Corollary 2. *Let* $T:S^n\to S^n$ *and* $\theta:R^k\to R^k$ *be periodic maps of a prime period* p *and* $\varphi:S^n\to R^k$ *be an equivariant map* $(\varphi T=\theta\varphi)$. *Then* $\dim\varphi^{-1}(R^k_\theta)\geqq n-k(p-1)$. *(indeed* $A(\varphi)=\varphi^{-1}(R^k_\theta)$*).*

Corollary 3. *Let* $T:S^n\to S^n$ *be a periodic map prime period p. Given a map* $f:S^n\to R^k$ *consider the set* $B(f)=\{x\in S^n\,|\,f(x)=f(Tx)\}$. *Then* $\dim B(f)\geqq n-(k-1)(p-1)-1$.

Proof. Let $f=(f_1,\ldots,f_k)$. Set

$$A_{k-1}(f)=\{x\in S^n\,|\,f_i|_{\mathrm{orbit}_x}=\text{const for } 1\leqq i\leqq k-1\}$$

$$A_k(f)=\{x\in S^n\,|\,f_k(x)=f_k(T_x)\}.$$

We have by Theorem 2 in $(A_{k-1}(f);T)\geqq n-(k-1)(p-1)$. Evidently, $A_k(f)$ is a strong partition in S^n, thus in $A_{k-1}(f)$ also, whereby in $(A_{k-1}(f)\cap A_k(f);T)\geqq n-(k-1)(p-1)-1$ (Proposition 3). The required inequality follows from the inclusion $B(f)\supset A_{k-1}(f)\cap A_k(f)$. Probably, this result may be reinforced, since the points of $A_{k-1}(f)\cap A_k(f)$ satisfy additional conditions.

It is easy to see, that for free Z_p-actions, the index cannot decrease under equivariant maps. Naturally, for actions with fixed points this is not true. Nevertheless, the following statement is valid:

Proposition 6. *Let* $\varphi:X\to Y$ *be an equivariant map* $(\varphi T_1=T_2\varphi)$ *such that the homomorphism* $\varphi_*:\widetilde{H}_r(X_{T_1})\to\widetilde{H}_r(Y_{T_2})$ *is a monomorphism for every r. Then in* $(X;T_1)\leqq\mathrm{in}\,(Y;T_2)$.

Proof. Let in $(X;T_1)\geqq n$, so that $\nu_{n,r}(\varsigma)\neq 0$ for some $\varsigma\in H^\rho_n(X;T_1)$. Then $\nu_{n,r}\varphi_*(\varsigma)\neq 0$ as following from the commutative diagramm

$$\begin{array}{ccc} H^\rho_n(X;T_1) & \xrightarrow{\nu_{n,r}} & \widetilde{H}_r(X_{T_1}) \\ \varphi^\rho_*\downarrow & & \downarrow\varphi_* \\ H^\rho_n(Y;T_2) & \xrightarrow{\nu_{n,r}} & \widetilde{H}_r(Y_{T_2}) \end{array}$$

Hence in $(Y;T_2)\geqq n$.

3. Mapping theorems for balls. When X is a ball B^n we have $X_T\neq\varnothing$ and $\widetilde{H}_r(X_T)=\{0\}$ for each r (a theorem of Smith [8]). Thus in $(X;T)=0$. In order to obtain a mapping theorem for balls, we ought to consider the index of a pair (X,Y) defined below, which is the natural homological invariant in this case. For $p=2$ this index is introduced and studied by the author [10].

Let Y be a closed invariant subset of X. The index homomorphisms $\mu_{n,r}$ are the compositions

$$\mu_{n,r}:H^\rho_n(X,Y;T)\xrightarrow{\partial_*}H^\rho_{n-1}(Y;T)\xrightarrow{\nu_{n-1,r}}\widetilde{H}_r(Y_T).$$

The index of the pair (X,Y) is defined by

$$\mathrm{in}\,(X,Y;T)=\max\{n\,|\,\mu_{n,r}\neq 0 \text{ for some } r\}.$$

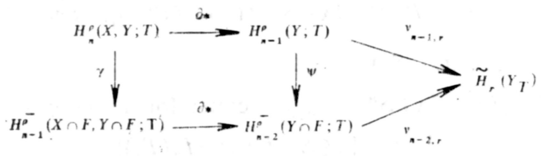
If $Y_T=\varnothing$, consider $\mu_n=\nu_{n-1}\partial_*.\ H^\rho_n(X,Y;T)\to Z_p$.

Obviously, in $(X,Y;T)\leqq\mathrm{in}(Y;T)+1$. The index of a pair has analogous properties to the index of a single space.

Proposition 7. *Let* in $(X,Y;T)\geqq n$ *and F be a strong partition in X. Then* in $(X\cap F,Y\cap F;T)\geqq n-1$. *Moreover, there exists a homomorphism* $\chi:H^\rho_n(X,Y;T)$

$\to H^{\rho}_{n-1}(X\cap F, Y\cap F; T)$ *such that the diagramm is commutative.*

For $p=2$ it is proved in [10] and the proof for arbitrary p is analogous to that of Proposition 2.



Proposition 8. in $(X, Y; T)\leq \dim X$.

Proposition 9. in $(B^n, S^{n-1}; T)=n$ *for any periodic map* $T: B^n\to B^n$ *of a prime period.*

Proof. For $k>0$ we have $H^{\rho}_k(B^n; T)=\{0\}$ (see Bredon [3]). Then by the exact sequence

$$H^{\rho}_n(B^n; T)\to H^{\rho}_n(B^n; S^{n-1}; T)\xrightarrow{\partial_*} H^{\rho}_{n-1}(S_{n-1}; T)\to H^{\rho}_{n-1}(B^n; T)$$

we conclude, that ∂_* is an isomophism. But $\mu_{n,r}=\nu_{n-1,r}\ \partial_*$ and $\nu_{n-1,r}\neq 0$ for some r (Proposition 4), hence $\mu_{n,r}\neq 0$ which means in $(B^n, S^{n-1}; T)\geq n$.

Theorem 3. *Let* in $(X, Y; T)\geq n$, *where* $T: X\to X$ *is a periodic map of a prime period* p. *Given a map* $f: X\to R^k$ *consider the set* $A(f)=\{x\in X \mid f(x)=f(Tx)=\dots=f(T^{p-1}x)\}$. *Then* $\dim A(f)\geq n-k(p-1)$.

The proof is identical with that of theorem 2, we must only refer to Prop. 7 instead of Prop. 3.

Remark. Corollaries 1, 2 and 3 remain valid if we replace S^n by B^n. The following two propositions are proved for $p=2$ in [10]. Their proof for arbitrary p may be obtained by the same reasoning with some insignificant modifications.

Proposition 10. *Let* in $(X, Y; T)\geq n$, $Y_T=\varnothing$ *and* C *be an invariant partition in* X *between* X_T *and* Y. *Then* in $(C; T)\geq n-1$.

In the case $p=2$, $(X, Y)=(B^n, S^{n-1})$ it is proved by D. Bourgin [2].

Proposition 11. *Let* in $(X, Y; T)\geq n$, $Y_T=\varnothing$ *and* C *be a closed invariant subsets of* X *nonintersecting* X_T *and* Y. *Let us have* $\ker i_*\subset \ker \nu_k$ *for some* $k\leq n-1$, where $H^{\rho}_k(X\setminus C; T)\xleftarrow{i_*} H^{\rho}_k(Y; T)\xrightarrow{\nu_k} Z_p$. *Then if* $C=\bigcup_{i=1}^{n-k-1}\Phi_i$ *is the union of* $n-k-1$ *closed invariant sets, some* Φ_i *contains an invariant continuum.*

For $(X, Y)=(B^n, S^{n-1})$, antipodal Z_2-action and $k=0$ the condition simply means, that C separates B^n between O and S^{n-1} and we get a classical theorem due to K. Borsuk [1].

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Suhodolska 1, bl. 13, vh. B, ap. 32
Sofia 1373, Bulgaria

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