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EXISTENCE OF SOLUTIONS OF FIRST ORDER PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS VIA THE METHOD OF LINES

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In this paper we prove a theorem on the existence of a solution for a non-linear first order partial differential-functional equation with Cauchy data. The proof of existence is constructive and it is based on the method of lines. By using a discretization in the spatial variable, the original problem is replaced by a sequence of initial problems for ordinary differential-functional equations. We investigate the question of under what conditions the solutions of ordinary differential-functional equations tend to a solution of the original problem when the step size tends to zero.

1. Introduction. Denote by $C(X, Y)$ the class of continuous mappings from X into Y where X and Y are metric spaces. Suppose that $\tau_0 \in R_+$, $R_+=[0, +\infty)$ and $\tau=(\tau_1, \ldots, \tau_n) \in R^n_+$. We define $D=[-\tau_0, 0]\times[-\tau, \tau]$, $E=[0, a]\times R^n$ where $a>0$ and $E_0=[-\tau_0, 0]\times R^n$. If $z: E_0 \cup E \to R$ is a function of variables $(x, y)=(x, y_1, \ldots, y_n)$ and there exist derivatives $D_{y_i} z$, $i=1, \ldots, n$, then we write $D_y z=(D_{y_1} z, \ldots, D_{y_n} z)$. For the above z and $(x, y) \in E$ we denote by $z_{xy}: D \to R$ the function given by $z_{xy}(t, s)=z(x+t, y+s)$, $(t, s)=(t, s_1, \ldots, s_n) \in D$. Let $\Omega=E\times C(D, R)\times R^n$. Suppose that $f: \Omega \to R$ and $\varphi: E_0 \to R$ are given functions. We consider the differential-functional problem

$$(1)\qquad \begin{aligned} &D_x z(x, y)=f(x, y, z_{xy}, D_y z(x, y)),\\ &z(x, y)=\varphi(x, y) \quad \text{for} \quad (x, y) \in E_0. \end{aligned}$$

In this note we prove a theorem on the existence of solutions of (1). The proof is constructive and it is based on the method of lines. We consider also approximate solutions of (1).

The method of lines for partial differential or differential-functional equations consists in replacing derivatives with respect to spatial variables by difference operators. Then the initial (or initial-boundary) value problem is replaced by a sequence of initial problems for ordinary differential or differential-functional equations. The method of lines can be considered as a method of approximate solving of partial differential equations. The main problem in these investigations is to find such a differential-difference approximation which satisfies some consistency conditions with respect to an original problem and it is stable. The method of lines can be considered as a tool for proving existence theorems for initial or initial-boundary value problems for partial differential equations.

Both of the above aspects of the method of lines will be considered in the paper. We prove an existence of theorem for (1) and we give an estimation of the existence domain of a solution. We prove also an error estimate implying the convergence of the method,

The method of lines for first order hyperbolic systems in two independent variables is considered in [8]. The author gives in [8] a convergence theorem and an existence theorem based on the method of lines for the Cauchy problem with respect to non-linear hyperbolic systems. Difference methods for first order partial differential-functional equations have been considered in [4], [6], [10], [20], (see also [1], [11]). Existence and uniqueness of solutions of initial problems for first order partial differential-functional equations have been studied in [2], [9], [11]. Differential-functional problems considered in [2], [11] have the following property. The right hand sides of systems are superpositions of a function $F=(F_1, \dots, F_m)$ defined on a set in R^k with some operators $V=(V_1, \dots, V_l)$ of the Volterra type. In our paper we omit this assumption. The papers [12], [13] contain sufficient conditions for the existence and uniqueness of solutions of generalized Cauchy problem for differential-functional systems of the Fredholm type.

Differential equations with a retarded argument and integro-differential problems can be obtained from (1) by specializing the function f. It is easy to see that the problems considered in [2], [11] can be formulated in the form (1).

For further references concerning the method of lines see the monograph by Walter [19] and the papers [17], [18].

If $z \in C(E_0 \cup E, R)$, $x \in [-\tau_0, a]$, then we define $\| z \|_x = \sup\{ |z(t, s)| : t \in [-\tau_0, x], y \in R^n\}$. For $w \in C(D, R)$ we write $\| w \|_{C(D, R)} = \max\{ |w(t, s)| : (t, s) \in D\}$. We will denote a function η of the variable t for $t \in [-\tau_0, a]$ by $\eta(\cdot)$ or $(\eta(t))_{[-\tau_0, a]}$. If $\eta \in C([-\tau_0, a], R)$ and $t \in (-\tau_0, a]$, then $D_-\eta(t)$ $(D^-\eta(t))$ is the left hand lower (upper) Dini derivative of η at the point t. If X and Y are Banach spaces, then $CL(X, Y)$ denotes the set of all linear continuous operators defined on X and taking values in Y. If $X=C(D, R)$, $Y=R$, then $\| \cdot \|_{CL}$ is the norm in $CL(C(D, R), R)$. We shall use vector inequalities, with the understanding that the same inequalities hold between their corresponding components.

2. Assumptions. Our basic assumptions are the following:

Assumption H_1. *Suppose that*

1° *the function* $f: \Omega \to R$ *of the variables* (x, y, w, q) *is continuous and bounded on* Ω *and there exist derivatives* $D_y f = (D_{y_1} f, \dots, D_{y_n} f)$, $D_q f = (D_{q_1} f, \dots, D_{q_n} f)$;

2° *for each* $(x, y, w, q) \in \Omega$ *there exists a Frechet derivative* $D_w f(x, y, w, q) \in CL(C(D, R), R)$;

3° *the derivatives* $D_y f$, $D_w f$, $D_q f$ *are continuous on* Ω *and there exists* $A \in R_+$ *such that*

$$|D_{y_i} f(x, y, w, q)| \leq A, \quad \| D_w f(x, y, w, q) \|_{CL} \leq A,$$

$$|D_{q_i} f(x, y, w, q)| \leq A, \quad i=1, \dots, n, \ (x, y, w, q) \in \Omega;$$

4° *there exists* $L>0$ *such that we have*

$$|D_{y_i} f(x, y, w, q) - D_{y_i} f(x, \bar{y}, \bar{w}, \bar{q})| \leq L[\| y-\bar{y} \| + \| w-\bar{w} \|_{C(D, R)} + \| q-\bar{q} \|],$$

$$\| D_w f(x, y, w, q) - D_w f(x, \bar{y}, \bar{w}, \bar{q}) \|_{CL} \leq L[\| y-\bar{y} \| + \| w-\bar{w} \|_{C(D, R)} + \| q-\bar{q} \|],$$

$$|D_{q_i} f(x, y, w, q) - D_{q_i} f(x, \bar{y}, \bar{w}, \bar{q})| \leq L[\| y-\bar{y} \| + \| w-\bar{w} \|_{C(D, R)} + \| q-\bar{q} \|],$$

where $i=1, \dots, n$, *and* $\| y \| = |y_1| + \dots + |y_n|$;

5° *for* $(x, y, w, q) \in \Omega$ *we have*

$$D_{q_i} f(x, y, w, q) \geq 0 \quad \text{for } i=1, \dots, k_0,$$

$$D_{q_i} f(x, y, w, q) \leq 0 \quad \text{for } i=k_0+1, \dots, n,$$

where $1 \leq k_0 \leq n$ *and there exists* $b=(b_1, \dots, b_n)$, $b_i>0$ for $i=1, \dots, n$, *such that* $f(x, y_1, \dots, y_{i-1}, y_i+2b_i, y_{i+1}, \dots, y_n, w, q) = f(x, y, w, q)$, $i=1, \dots, n$, (x, y, w, q).

Assumption H_2. *Suppose that*

1° $\varphi \in C(E_0, R)$, $D_y\varphi(x, y) = (D_{y_1}\varphi(x, y), \ldots, D_{y_n}\varphi(x, y))$ *exists for* $(x, y) \in E_0$ *and* $\varphi(x, y_1, \ldots, y_{i-1}, y_i + 2b_i, y_{i+1}, \ldots, y_n) = \varphi(x, y)$, $i = 1, \ldots, n$, *on* E_0;

2° $D_y\varphi \in C(E_0, R^n)$ *and we have on* Ω

$$|D_{y_i}\varphi(x, y)| \leq A_0, \quad i = 1, \ldots, n,$$

$$|D_{y_i}\varphi(x, y) - D_{y_i}\varphi(x, \bar{y})| \leq L_0 \| y - \bar{y} \|, \quad i = 1, \ldots, n,$$

where $A_0 \geq 0$, $L_0 > 0$.

3. Discretization of the problem (1) and a comparison lemma. For $y = (y_1, \ldots, y_n)$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)$, $y, \bar{y} \in R^n$, we define $y * \bar{y} = (y_1\bar{y}_1, \ldots, y_n\bar{y}_n)$. We introduce a mesh in R^n in the following way. Suppose that for a $h = (h_1, \ldots, h_n)$ there exists $N = (N_1, \ldots, N_n)$ such that N_i are natural numbers and $N * h = b$. Denote by I_0 the set of all h having the above property. Let $J = \{m = (m_1, \ldots, m_n)$: m_i be integers, $i = 1, \ldots, n\}$. For $h \in I_0$ define $y^{(m)} = (y_1^{(m_1)}, \ldots, y_n^{(m)_n}) = m * h$, $m \in J$. Let $E[h] = \{(x, y^{(m)}): x \in [0, a], m \in J\}$, $E_0[h] = \{(x, y^{(m)}): x \in [-\tau_0, 0], m \in J\}$ and $B[h] = E_0[h] \cup E[h]$. For a function $z: B[h] \to R$ we write $z^{(m)}(x) = z(x, y^{(m)})$. Denote by $\mathscr{F}_c(B[h], R)$ the class of all functions $z: B[h] \to R$ such that $z(\cdot, y^{(m)}) \in C([-\tau_0, a], R)$ for $m \in J$. If $z: B[h] \to R$, then we write $\| z(x, \cdot) \|_h = \sup\{|z(x, y^{(m)})|: m \in J\}$. Let $S = \{r = (r_1, \ldots, r_n): r_i \in \{0, 1\}, i = 1, \ldots, n\}$. Suppose that $z: B[h] \to R$ and $y \in R^n$. Then there exists $m \in J$ such that $y^{(m)} \leq y \leq y^{(m+1)}$ where $m + 1 = (m_1 + 1, \ldots, m_n + 1)$. We define for $x \in [-\tau_0, a]$

$$(2) \qquad (T_h z)(x, y) = \sum_{r \in S} z^{(m+r)}(x) \left(\frac{y - y^{(m)}}{h}\right)^r \left(1 - \frac{y - y^{(m)}}{h}\right)^{1-r}, \quad y^{(m)} \leq y \leq y^{(m+1)},$$

where

$$(3) \qquad \begin{gathered} \left(\frac{y - y^{(m)}}{h}\right)^r = \prod_{i=1}^{n} \left(\frac{y_i - y_i^{(m_i)}}{h_i}\right)^{r_i}, \\ \left(1 - \frac{y - y^{(m)}}{h}\right)^{1-r} = \prod_{i=1}^{n} \left(1 - \frac{y_i - y_i^{(m_i)}}{h_i}\right)^{1-r_i}, \quad y^{(m)} \leq y \leq y^{(m+1)}, \end{gathered}$$

and we take $0^0 = 1$ in (3). Thus we have $T_h z: E_0 \cup E \to R$. If $z \in \mathscr{F}_c(B[h], R)$, then $T_h z \in C(E_0 \cup E, R)$ ([4]). If $1 \leq i \leq n$, $m \in J$ then we write $i(m) = (m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_n)$ and $-i(m) = (m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_n)$. We define difference operators $(\Delta_1, \ldots, \Delta_n)$ in the following way. If $z: B[h] \to R$, $h \in I_0$, $m \in J$, then

$$(4) \qquad \begin{gathered} \Delta_i z^{(m)}(x) = \frac{1}{h_i}[z^{(i(m))}(x) - z^{(m)}(x)], \quad i = 1, \ldots, k_0, \\ \Delta_i z^{(m)}(x) = \frac{1}{h_i}[z^{(m)}(x) - z^{(-i(m))}(x)], \quad i = k_0 + 1, \ldots, n, \end{gathered}$$

and $\Delta z^{(m)}(x) = (\Delta_1 z^{(m)}(x), \ldots, \Delta_n z^{(m)}(x))$

We consider the following differential-functional problem

$$(5) \qquad \begin{gathered} D_x z^{(m)}(x) = f(x, y^{(m)}, (T_h z)_{xy^{(m)}}, \Delta z^{(m)}(x)), \quad m \in J, \\ z(x, y^{(m)}) = \varphi(x, y^{(m)}) \quad \text{for } (x, y^{(m)}) \in E_0[h]. \end{gathered}$$

Denote by u_h a solution of the line method (5) on $[-\tau_0, a]$ and write $U_h = T_h u_h$. We give sufficient conditions for the following requirements to be satisfied: (i) there exists $u = \lim_{h \to 0} U_h$, (ii) u is a solution of (1).

In the sequel we will use the following lemma.

L e m m a 1 ([4]). *Suppose that*

1° $\bar{z} \in C(E_0 \cup E, R)$ *and* $\bar{z}_h$ *is the restriction of* $\bar{z}$ *to the set* $B[h]$;

2° *the derivatives* $(D_{y_1}\bar{z}(x, \cdot), \dots, D_{y_n}\bar{z}(x, \cdot)) = D_y\bar{z}(x,)$ *exist on* R^n *and* $D_y\bar{z}(x, \cdot) \in C(R^n, R^n)$ *for* $x \in [-\tau_0, a]$;

3° *there exists* $C_0 \in R_+$ *such that* $|D_{y_i}\bar{z}(x, y)| \leq C_0$ *for* $(x, y) \in E_0 \cup E$, $i = 1, \dots, n$.

Then $\| T_h\bar{z}_h - \bar{z} \|_x \leq C_0 \| h \|$, $x \in [-\tau_0, a]$.

Let $S_0 = \{r = (r_1, \dots, r_n) : r_i \in \{-1, 0, 1\},\ i = 1, \dots, n\}$ and $S_0' = S_0 \setminus \{\Theta\}$ where $\Theta = (0, \dots, 0) \in R^n$. We will denote by $\tilde{\Delta} = (\tilde{\Delta}_1, \dots, \tilde{\Delta}_n)$ the difference operator given by

$$\tilde{\Delta}_i z^{(m)}(x) = \frac{1}{h_i} \sum_{r \in S_0} c_r^{(i)} z^{(m+r)}(x), \quad i = 1, \dots, n, \tag{6}$$

where $c_r^{(i)} \in R$, $z : B[h] \to R$ and $\tilde{\Delta}_z^{(m)}(x) = (\tilde{\Delta}_1 z^{(m)}(x), \dots, \tilde{\Delta}_n z^{(m)}(x))$. Now we prove a comparison lemma which enables us to estimate a function satisfying differential-difference inequalities by the maximum solution of an initial value problem for ordinary differential-functional system. It will be a modification of Theorem 5 from [3].

A s s u m p t i o n H_3. *Suppose that*

1° *the function* $g = (g_1, \dots, g_k) : [0, a] \times R_+^k \times C([-\tau_0, a], R_+^k \to R_+^k$ *of the variables* (x, ξ, η), $\xi = (\xi_1, \dots, \xi_k)$, $\eta = (\eta_1, \dots, \eta_k)$ *is non-decreasing with respect to the functional argument and satisfies the following Volterra condition: if* $\eta, \bar{\eta} \in C([-\tau_0, a], R_+^k)$, $(x, \xi) \in [0, a] \times R_+^k$ *and* $\eta(t) = \bar{\eta}(t)$ *for* $t \in [-\tau_0, x]$, *then* $g(x, \xi, \eta) = g(x, \xi, \bar{\eta})$;

2° *g possesses the following quasi-monotone property: for each* i, $1 \leq i \leq k$, g_i *is non-decreasing in* ξ_j, $j = 1, \dots, k$, $j \neq i$;

3° *g is continuous and for each* $\eta_0 \in C([-\tau_0, 0], R_+^k)$ *there exists on* $[-\tau_0, a]$ *the right hand maximum solution of the problem*

$$\eta'(x) = g(x, \eta(x), \eta), \quad \eta(x) = \eta_0(x) \quad \text{for } x \in [-\tau_0, 0]; \tag{7}$$

4° $\psi_j = (\psi_{1j}, \dots, \psi_{nj}) : E[h] \times \mathscr{F}_c(B[h], R^k) \to R^n$, $j = 1, \dots, k$, *and*

$$\psi_{ij}(x, y, z) \geq 0 \quad \text{for } i = 1, \dots, k_0, \quad j = 1, \dots, k,$$

$$\psi_{ij}(x, y, z) \leq 0 \quad \text{for } i = k_0 + 1, \dots, n,\ j = 1, \dots, k,$$

where $(x, y, z) \in E[h] \times \mathscr{F}_c(B[h], R^k)$;

5° *the operator* $(\tilde{\Delta}_1, \dots, \tilde{\Delta}_n)$ *given by* (6) *satisfies the conditions:*

$$c_r^{(i)} \geq 0 \text{ for } i = 1, \dots, k_0, \quad r \in S_0',$$

$$c_r^{(i)} \leq 0 \text{ for } i = k_0 + 1, \dots, n,\ r \in S_0',$$

$$\sum_{r \in S_0} c_r^{(i)} = 0, \quad i = 1, \dots, n.$$

We will denote by $\langle ; \rangle$ the inner product in R^n. If $z = (z_1, \dots, z_k) : B[h] \to R^k$, then we write $\| z(x, \cdot) \|_h = (\| z_1(x, \cdot) \|_h, \dots, \| z_k(x, \cdot) \|_h)$ and $(\| z(t, \cdot) \|_h)_{[-\tau_0, a)} = ((\| z_1(t, \cdot) \|_h)_{[-\tau_0, a]}, \dots, (\| z_k(t, \cdot) \|_h)_{[-\tau_0, a]})$.

L e m m a 2. *Suppose that*

1° *Assumption* H_3 *holds and* $u = (u_1, \dots, u_k) : B[h] \to R^k$ *where* $u_i \in \mathscr{F}_c(B[h], R)$, $i = 1, \dots, k$;

2° *for* $m \in J$, $x \in [-\tau_0, a]$ *and for* $i = 1, \dots, n$ *we have* $u(x, y_1^{(m)}, \dots, y_{i-1}^{(m)}, y_i^{(m+2N_i)}, y_{i+1}^{(m)}, \dots, y_n^{(m)}) = u(x, y^{(m)})$;

3° *the initial inequality* $\|u(x,\cdot)\|_h \le \eta_0(x)$, $x \in [-\tau_0, 0]$, *where* $\eta_0 \in C([-\tau_0, 0], R^k_+)$ *and the differential-difference inequalities*

$$|D^- u_j^{(m)}(x) - \langle \psi_j(x, y^{(m)}, u);\ \tilde{\Delta} u_j^{(m)}(x)\rangle| \le g_j(x, \|u(x,\cdot)\|_h, (\|u(t,\cdot)\|_h)_{[-\tau_0, a]}),\quad x \in (0, a],\ m \in J,\ j = 1, \ldots, k, \tag{8}$$

hold.

Under these assumptions we have

$$\|u(x,\cdot)\|_h \le \omega(x, \eta_0), \quad x \in [0, a], \tag{9}$$

where $\omega(\cdot, \eta_0)$ is the maximum solution of (7).

P r o o f. Let $\bar{w}(x) = (\bar{w}_1(x), \ldots, \bar{w}_k(x)) = \|u(x,\cdot)\|_h$, $x \in [-\tau_0, a]$. We prove that

$$D_- \bar{w}(x) \le g(x, \bar{w}(x), \bar{w}) \quad \text{for } x \in (0, a]. \tag{10}$$

Suppose that $x \in (0, a]$ and $1 \le j \le k$ are fixed. Then there is $m \in J$ such that $\bar{w}_j(x) = |u_j^{(m)}(x)|$. We consider two possibilities: (i) $\bar{w}_j(x) = u_j^{(m)}(x)$ or (ii) $\bar{w}_j(x) = -u_j^{(m)}(x)$. If (i) holds then

$$D^- u_j^{(m)}(x) \le g_j(x, \|u(x,\cdot)\|_h, (\|u(t,\cdot)\|_h)_{[-\tau_0, a]}) + \sum_{i=1}^{n} \psi_{ij}(x, y^{(m)}, u) \frac{1}{h_i} [\sum_{r \in S_0'} c_r^{(i)} u_j^{(m+r)}(x) + c_\Theta^{(i)} u_j^{(m)}(x)] \le g_j(x, \bar{w}(x), \bar{w}). \tag{11}$$

Since $D_- \|u_j(x,\cdot)\|_h \le D^- u_j^{(m)}(x)$, then we have by (11)

$$D_- \bar{w}_j(x) \le g_j(x, \bar{w}(x), \bar{w}).$$

In a similar way we obtain the above inequality if the possibility (ii) holds. Then we have (10). Since $\bar{w}(x) \le \eta_0(x)$ for $x \in [-\tau_0, 0]$ and $\bar{w} \in C([-\tau_0, a], R^k_+)$ it follows from (10) and from the theory of differential-functional inequalities ([5], [7], [14]–[16]) that (9) holds.

4. The convergence of the lines method. At first we prove that solutions of (5) are equibounded.

L e m m a 3. *If Assumptions* H_1, H_2 *are satisfied, then* (i) *for each* $h \in I_0$ *there exists a solution* u_h *of* (5) *on* $[-\tau_0, a]$, (ii) *there exist* $C, C_0 \in R_+$ *such that for* $x \in [0, a]$ *we have*

$$\|u_h(x,\cdot)\|_h \le (C_0 + \frac{C}{A})e^{Ax} - \frac{C}{A} \quad \text{if } A > 0,$$
$$\|u_h(x,\cdot)\|_h \le C_0 + Cx \quad \text{if } A = 0. \tag{12}$$

P r o o f. We see at once that u_h exists on $[-\tau_0, a]$ for $h \in I_0$. Let C and C_0 be such constants that $|f(x, y, 0, \theta)| \le C$ for $(x, y) \in [0, a] \times [-b, b]$ and $|\varphi(x, y)| \le C_0$ for $(x, y) \in [-\tau_0, 0] \times [-b, b]$.

An easy computation shows that

$$|D_x u_h^{(m)}(x) - \langle \int_0^1 D_q f(Q(x, m, t))dt;\ \Delta u_h^{(m)}(x) \rangle| \le A \|(T_h u_h)_{xy^{(m)}}\|_{C(D, R)} + C, \quad x \in [0, a],\ m \in J, \tag{13}$$

where $Q(x, m, t) = (x, y^{(m)}, (T_h u_h)_{xy^{(m)}}, t\Delta u_h^{(m)}(x))$. It is easily seen that

(14) $$\sum_{r\in S}\left(\frac{y-y^{(m)}}{h}\right)^r\left(1-\frac{y-y^{(m)}}{h}\right)^{1-r}=1 \quad \text{for } y^{(m)}\leqq y\leqq y^{(m+1)}.$$

It follows from (2) that

$$(T_h u_h)_{xy}(t,\ s)=\sum_{r\in S} u_h^{(\bar m+r)}(x+t)\left(\frac{y-y^{(\bar m)}}{h}\right)^r\left(1-\frac{y-y^{(\bar m)}}{h}\right)^{1-r}$$

where $y^{(\bar m)}\leqq y+s\leqq y^{(\bar m+1)}$, $\bar m\in J$. We conclude from (14) that

(15) $$\|(T_h u_h)_{xy^{(m)}}\|_{C(D,\ R)}\leqq \max\{\|u_h(x+t,\cdot)\|_h:\ t\in[-\tau_0,\ 0]\}.$$

Since $\|u_h(x,\cdot)\|_h\leqq C_0$ for $x\in[-\tau_0,\ 0]$, then we obtain (12) from (13), (15) and by applying Lemma 2.

Let us denote by $q_h=(q_{h,1},\ldots,q_{h,n})\colon B[h]\to R^n$ the function given by

$$q_{h,i}^{(m)}(x)=\frac{1}{h_i}[u_h^{(i(m))}(x)-u_h^{(m)}(x)] \quad \text{for } i=1,\ldots,k_0,$$

$$q_{h,i}^{(m)}(x)=\frac{1}{h_i}[u_h^{(m)}(x)-u_h^{(-i(m))}(x)] \quad \text{for } i=k_0+1,\ldots,n,$$

where $x\in[-\tau_0,\ a]$, $m\in J$, $h\in I_0$.

Lemma 4. *If Assumptions H_1, H_2 are satisfied, then*

(16) $$\|q_{h,i}(x,\cdot)\|_h\leqq e^{Ax}(A_0+1)-1,\quad x\in[0,\ a],\ i=1,\ldots,n.$$

Proof. We first prove that

(17) $$\left|D_x q_{h,i}^{(m)}(x)-\left\langle \int_0^1 D_q f(P_i(x,\ m,\ t))dt;\ \ q_{h,i}^{(m)}(x)\right\rangle\right|$$

$$\leqq A+A\max\{\|q_{n,i}(x+t,\cdot)\|_h:\ t\in[-\tau_0,\ 0]\},\quad m\in J,\ i=1,\ldots,n,\ x\in[0,\ a],$$

where $P_i(x,\ m,\ t)=(x,\ y^{(m)},\ (T_h u_h)_{xy^{(m)}},\ \Delta u_h^{(m)}(x)+t[\Delta u_h^{(i(m))}(x)-\Delta u_h^{(m)}(x)])$, for $i=1$ $\ldots,k_0$ and

$$P_i(x,\ m,\ t)=(x,\ y^{(m)},\ (T_h u_h)_{xy^{(m)}},\ \Delta u_h^{(-i(m))}(x)$$
$$+t[\Delta u_h^{(m)}(x)-\Delta u_h^{(-i(m))}(x)]) \quad \text{for } i=k_0+1,\ldots,n.$$

For a fixed i, $1\leqq i\leqq k_0$, we have

(18) $$D_x q_{h,i}^{(m)}(x)=\frac{1}{h_i}[f(x,\ y^{(i(m))},\ (T_h u_h)_{xy^{(i(m))}},\ \Delta u_h^{(i(m))}(x))-f(x,\ y^{(m)},\ (T_h u_h)_{xy^{(m)}},$$

$$\Delta u_h^{(m)}(x))]=\int_0^1 D_{y_i}f(Q_i(x,\ m,\ t))dt+\frac{1}{h_i}\int_0^1 D_w(Q_i(x,\ m,\ t))[(T_h u_h)_{xy^{(i(m))}}-(T_h u_h)_{xy^{(m)}}]dt$$

$$+\sum_{j=1}^n\int_0^1 D_{q_j}f(P_i(x,\ m,\ t))\,dt\,\Delta_j q_{h,i}^{(m)}(x),\quad x\in[0,\ a],$$

where $Q_i(x,\ m,\ t)=(x,\ y^{(m)}+th_i e_i,\ (T_h u_h)_{xy^{(m)}}+t[(T_h u_h)_{xy^{(i(m))}}-(T_h u_h)_{xy^{(m)}}],\ \Delta u_h^{(i(m))}(x))$ and $e_i=(0,\ldots,0,\ 1,\ 0,\ldots,0)\in R^n$, 1 standing on the i-th place. Suppose that $(t,\ s)\in D$, $m\in J$. Then there exists $\bar m=(\bar m_1,\ldots,\bar m_n)\in J$ such that $y^{(\bar m)}\leqq y^{(m)}+s\leqq y^{(\bar m+1)}$. It follows from (2) that

$$\frac{1}{h_i}[(T_hu_h)(x+t,\ y^{(i(m))}+s)-(T_hu_h)(x+t,\ y^{(m)}+s)]$$

$$=\sum_{r\in S}\frac{1}{h_i}[u_h^{(\overline{i(m)}+r)}(x+t)-u_h^{(\overline{m}+r)}(x+t)]\left(\frac{y-y^{(\overline{m})}}{h}\right)^r\left(1-\frac{y-y^{(\overline{m})}}{h}\right)^{1-r},$$

where $y=y^{(m)}+s$. From (14) we conclude that

(19) $$\frac{1}{h_i}\|(T_hu_h)_{xy^{(i(m))}}-(T_hu_h)_{xy^{(m)}}\|_{C(D,\,R)}\leq\max\{\|q_{h,i}(x+t,\cdot)\|_h:\ t\in[-t_0,\ 0]\},\quad x\in 0,\ a\cdot$$

The above estimation and (18) imply (17) for $1\leq i\leq k_0$. In a similar way we prove (17) for $k_0+1\in i\leq n$. Now we obtain (16) from Lemma 2.

Let us denote by $t_h=[t_{h,ij}]_{i,j=1,\dots,n}$, $t_{h,ij}:\ B[h]\to R$, the function given by

(20) $$t_{h,ij}(x,\ y^{(m)})=1/h_j[q_{h,i}^{(j(m))}(x)-q_{n,i}^{(m)}(x)],\quad j=1,\dots,k_0,$$
$$t_{h,ij}(x,\ y^{(m)})=1/h_j[q_{h,i}^{(m)}(x)-q_{h,i}^{(-j(m))}(x)],\quad j=k_0+1,\dots,n,$$

where $i=1,\dots,n$, $x\in[-\tau_0,\ a]$, $m\in J$. We define

$$\overline{A}=e^{Aa}(A_0+1),\ \overline{C}=\overline{A}+\frac{A}{2nL},$$

$$\lambda(x)=[L\overline{C}x(\overline{C}+nL_0)+L_0][1-Lnx(\overline{C}+nL_0)]^{-1},$$

$$a_0=\min(a,\ \varepsilon[Ln(\overline{C}+nL_0)]^{-1},\quad 0<\varepsilon<1.$$

Lemma 5. *If Assumptions H_1, H_2 are satisfied, then*

(21) $$\|t_{h,ij}(x,\cdot)\|_h\leq\lambda(x),\ x\in[0,\ a_0],\quad i,\ j=1,\dots,n.$$

Proof. Suppose that $1\leq i\leq k_0$, $1\leq j\leq k_0$ and $Q(x,\ m,\ t)=(x,\ (1-t)\,y^{(m)}+ty^{(i(m))}$, $(1-t)(T_hu_h)_{xy^{(m)}}+t(T_hu_h)_{xy^{(i(m))}}$, $(1-t)\Delta u_h^{(m)}(x)+t\Delta u_h^{(i(m))}(x))$, $P(x,\ m,\ t)=Q(x,\ j(m),\ t)$, Then we have for $x\in[0,\ a_0]$, $m\in J$

(22) $$D_xt_{h,ij}^{(m)}(x)=\frac{1}{h_i}\int_0^1[D_{y_i}f(P(x,\ m,\ t))-D_{y_i}f(Q(x,\ m,\ t))]\,dt$$
$$+\frac{1}{h_ih_j}\int_0^1D_wf(P(x,\ m,\ t))[(T_hu_h)_{xy^{(j(i(m)))}}-(T_hu_h)_{xy^{(j(m))}}]\,dt$$
$$-\frac{1}{h_ih_j}\int_0^1D_wf(Q(x,\ m,\ t))[(T_hu_h)_{xy^{(i(m))}}-(T_hu_h)_{xy^{(m)}}]\,dt$$
$$+\frac{1}{h_ih_j}\left\langle\int_0^1D_qf(P(x,\ m,\ t))\,dt;\ \Delta u_h^{(j(i(m)))}(x)-\Delta u_h^{(j(m))}(x)\right\rangle$$
$$-\frac{1}{h_ih_j}\left\langle\int_0^1D_qf(Q(x,\ m,\ t))dt;\ \Delta u_h^{(i(m))}(x)-\Delta u_h^{(m)}(x)\right\rangle.$$

It follows from Assumption H_1 and from (19) that

(23) $$\frac{1}{h_i}|D_{y_i}f(P(x,\ m,\ t))-D_{y_i}f(Q(x,\ m,\ t))|\leq L[1+\max\{\|q_{h,i}(x+t,\cdot)\|_h:t\in[-\tau_0,0]\}$$
$$+\sum_{l=1}^{n}\|t_{h,il}(x,\cdot)\|_h],\ x\in[0,\ a_0],\ m\in J,\ t\in[0,\ 1].$$

It is seen at once that the same estimations for

$$\frac{1}{h_j}\,|\,D_{q_l}f(P(x,\ m,\ t))-D_{q_l}f(Q(x,\ m,\ t))\,|,$$

$$\frac{1}{h_j}\,\|\,D_w f(P(x,\ m,\ t))-D_w f(Q(x,\ m,\ t))\,\|_{CL},\quad x\in[0,\ a_0],\ m\in J,\ t\in[0,\ 1],$$

are true. Let

$$(24)\qquad U_{h.ij}=(T_h u_h)_{xy^{(j(i(m)))}}-(T_h u_h)_{xy^{(j(m))}}-(T_h u_h)_{xy^{(i(m))}}+(T_h u_h)_{xy^{(m)}}.$$

We next prove that

$$(25)\qquad \frac{1}{h_i h_j}\,|\,U_{h.ij}\,|_{C(D,\ R)}\leq\max\{\|\,t_{h.ij}(x+t,\cdot)\,\|_h:\ t\in[-\tau_0,\ 0]\},\ x\in[0,\ a_0].$$

Suppose that $(t,\ s)\in D,\ m\in J$. Then there exists $\bar m=(\bar m_1,\ldots,\bar m_n),\ \bar m\in J,$ such that $y^{(\bar m)}\leq y^{(m)}+s\leq y^{(\bar m+1)}$ and we have

$$\frac{1}{h_i h_j}U_{h.ij}(t,\ s)=\sum_{r\in S}\frac{1}{h_i h_j}[u_h^{(j(i(\bar m))+r)}(x+t)-u_h^{(j(\bar m)+r)}(x+t)-u_h^{(i(\bar m)+r)}(x+t)$$

$$+u_h^{(\bar m+r)}(x+t)]\times\Big(\frac{y-y^{(\bar m)}}{h}\Big)^r\Big(1-\frac{y-y^{(\bar m)}}{h}\Big)^{1-r},$$

where $y=y^{(m)}+s$. Now we obtain the estimation (25) from (14). Combining (19) (22)—(25) we obtain

$$(26)\qquad |\,D_x t^{(m)}_{h.ij}(x)-\langle\int_0^1 D_q f(P(x,m,t))dt\,;\Delta t^{(m)}_{h.ij}(x)\rangle\,|\leq L[1+\max\{\|\,q_{h.j}(x+t,\cdot)\|_h:t\in[-\tau_0,0]\}$$

$$+\sum_{l=1}^{n}\|\,t_{h.lj}(x,\cdot)\,\|_h][1+\max\{\|\,q_{h.i}(x+t,\cdot)\,\|_h:\ t\in[-\tau_0,\ 0]\}$$

$$+\sum_{l=1}^{n}\|\,t_{h.li}(x,\cdot)\,\|_h]+A\max\{\|\,t_{h.ij}(x+t,\cdot)\,\|_h:\ t\in[-\tau_0,\ 0]\},\ x\in[0,\ a_0],\ m\in J.$$

Note that we have actually proved the differential-difference inequality (26) for $1\leq i\leq k_0,\ 1\leq j\leq k_0$. The same proof of the estimation (26) remains valid for the rest of $i,\ j,\ 1\leq i,\ j\leq n$, the only difference being in the definition of $P(x,\ m,\ t)$. Therefore we omit the details.

The estimates (26) and the inequality $\|\,t_{h.ij}(x,\cdot)\,\|_h\leq L_0,\ i,\ j=1,\ldots,n,\ x\in[-\tau_0,\ 0]$ lead to

$$\|\,t_{h.ij}(x,\cdot)\,\|_h\leq\bar u_{ij}(x),\quad x\in[0,\ a_0],\ i,\ j=1,\ldots,n,$$

where $\bar u=[\bar u_{ij}]_{i,j=1,\ldots,n}$, is a solution of

$$\eta'_{ij}(x)=L[\bar A+\sum_{l=1}^{n}\eta_{lj}(x)][\bar A+\sum_{l=1}^{n}\eta_{li}(x)]+A\max\{\eta_{ij}(x+t):\ t\in[-\tau_0,\ 0]\},\ i,\ j=1,\ldots,n,$$

$$\eta_{ij}(x)=L_0,\ x\in[-\tau_0,\ 0].$$

Since

$$\bar u'_{ij}(x)\leq L[\bar C+nn_{ij}(x)]^2,\quad i,\ j=1,\ldots,n,\ x\in[0,\ a_0],$$

and

$$\lambda'(x)=L[\bar C+n\lambda(x)]^2\quad\text{for } x\in[0,\ a_0],\ \lambda(0)=L_0,$$

we obtain $\overline{u}_{ij}(x) \leqq \lambda(x)$ for $x \in [0, a_0]$, $i, j = 1, \ldots, n$. This completes the proof.

5. The existence theorem. Theorem 1. *If Assumptions H_1, H_2 are satisfied, then there exists a solution u of (1) on $E_0 \cup E^*$ where $E^* = [0, a_0] \times R^n$.*

Proof. Suppose that $U_h: E_0 \cup E^* \to R$, $Q_h = (Q_{h.1}, \ldots, Q_{h.n})$, $Q_h: E_0 \cup E^* \to R^n$ are functions given by

$$U_h = T_h u_h, \quad Q_{h.i} = T_h q_{h.i}, \quad i = 1, \ldots, n. \tag{27}$$

It follows from Lemmas 3—5 that

$$|U_h(x, y)| \leqq \left(C_0 + \frac{C}{A}\right)e^{Aa} - \frac{C}{A}, \quad \text{if } A > 0, \tag{28}$$

$$|U_h(x, y)| \leqq C_0 + Ca_0, \quad \text{if } A = 0,$$

$$\| Q_h(x, y) \| \leqq \overline{A} - 1,$$

$$|U_h(x, y) - U_h(x, \overline{y})| \leqq (\overline{A} - 1) \| y - \overline{y} \|,$$

$$|Q_{h.i}(x, y) - Q_{h.i}(x, \overline{y})| \leqq \lambda(a_0) \| y - \overline{y} \|, \quad i = 1, \ldots, n,$$

on $E_0 \cup E^*$. It is easy to see that there exists $\widetilde{C} \in R_+$ such that

$$|D_x U_h(x, y)| \leqq \widetilde{C}, \quad \| D_x Q_h(x, y) \| \leqq \widetilde{C}, \quad (x, y) \in E_0 \cup E^*.$$

From (27), (28) and from the above estimation it follows that there exists a sequence $\{h^{(k)}\}$, $h^{(k)} \in I_0$, and functions $u \in C(E_0 \cup E^*, R)$, $v = (v_1, \ldots, v_n)$ $C(E_0 \cup E^*, R^n)$ such that $\lim_{k \to \infty} \| h^{(k)} \| = 0$ and

$$u(x, y) = \lim_{k \to \infty} U_{h^{(k)}}(x, y),$$

$$v(x, y) = \lim_{k \to \infty} Q_{h^{(k)}}(x, y),$$

uniformly with respect to $(x, y) \in E_0 \cup E^*$. Now we prove that $D_y u = (D_{y_1} u, \ldots, D_{y_n} u)$ exists on E^* and $D_y u(x, y) = v(x, y)$ for $(x, y) \in E^*$. Let $h \in I_0$, $1 \leqq i \leqq k_0$ and $y_i^{(m)} \geqq (-\widetilde{k} + 1)b_i$, where $\widetilde{k}$ is a natural number. Then we have

$$U_h(x, y_1, \ldots, y_{i-1}, y_i^{(m_i)}, y_{i+1}, \ldots, y_n) = U_h(x, y_1, \ldots, y_{i-1}, -\widetilde{k}b_i, y_{i+1}, \ldots, y_n)$$

$$+ \sum_{j = -\widetilde{k}N_i}^{m_i - 1} h_i Q_{h.i}(x, y_1, \ldots, y_{i-1}, y_i^{(j)}, y_{i+1}, \ldots, y_n),$$

$$(x, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \in [0, a_0] \times R^{n-1},$$

and consequently

$$u(x, y) = u(x, y_1, \ldots, y_{i-1}, -\widetilde{k}b_i, y_{i+1}, \ldots, y_n)$$

$$+ \int_{-\widetilde{k}b_i}^{y} v_i(x, y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_n)dt, \quad (x, y) \in E^*.$$

Therefore we have

$$D_{y_i} u(x, y) = v_i(x, y), \quad (x, y) \in E^*, \tag{29}$$

for $1 \leqq i \leqq k_0$. In a similar way we prove (29) for $k_0 + 1 \leqq i \leqq n$. It follows from (5) that

$$u_h(x, y^{(m)}) = \varphi(0, y^{(m)}) + \int_0^x f(t, y^{(m)}, (T_h u_h)_{ty^{(m)}}, q_h(t, y^{(m)}))dt, \quad m \in J, \ x \in [0, a_0]. \tag{30}$$

It follows from Lemma 1 that

(31) $$\lim_{h\to 0} T_h u_h(x, y) = u(x, y)$$

uniformly with respect to $(x, y) \in E_0 \cup E^*$. For each $y \in R^n$ there exists a sequence of grid points which is convergent to y. We conclude from (29)—(31) that

$$u(x, y) = \varphi(0, y) + \int_0^x f(t, y, u_{ty}, D_y u(t, y))dt, \quad (x, y) \in E^*.$$

Thus we see that u is a solution of (1). This completes the proof.

Remark 1. If Assumptions H_1 and H_2 are satisfied, then the solution u of (1) is unique (see [5]).

6. Error estimation of the method of lines. In this part of the paper we consider the initial problem (1) and we assume that there exists a solution of (1). We investigate the queston of under what conditions the solutions of the problem (5) tend to a solution of the original problem when the step size tends to zero.

Assumption H_4. *Suppose that*

1° $f \in C(\Omega, R)$, *the derivatives* $(D_{q_1} f, \ldots, D_{q_n} f) = D_q f$ *exist on* Ω *and* $D_q f \in C(\Omega, R^n)$;

2° *the condition* 5° *of Assumption* H_1 *holds and there exists* $A \in R_+$ *such that*

$$|f(x, y, w, q) - f(x, y, \bar{w}, q)| \leq A \| w - \bar{w} \|_{C(D, R)} \text{ on } \Omega;$$

3° *the function* φ *satisfies the condition* 1° *of Assumption* H_2;

4° *there exists a solution* u *of* (1) *which is of class* C^1 *on* E *and* $u(x, y_1, \ldots, y_{i-1}, y_i + 2b_i, y_{i+1}, \ldots, y_n) = u(x, y)$ *on* E for $i = 1, \ldots, n$.

Suppose that Assumption H_4 holds. Write

$$\eta_h = \sup\{|f(x, y^{(m)}, u_{xy^{(m)}}, \Delta u^{(m)}(x)) - f(x, y^{(m)}, u_{xy^{(m)}}, D_y u^{(m)}(x))| : x \in [0, a], -N \leq m \leq N\}$$

$$\bar{C} = \sup\{|D_{y_i} u(x, y)| : (x, y) \in [0, a] \times [-b, b], \ i = 1, \ldots, n\}, \ u_h = u|_{B[h]}.$$

Theorem 2. *If Assumption* H_4 *is satisfied, then for each* $h \in I_0$ *there exists a solution* v_h *of* (5) *on* $[-\tau_0, a]$ *and*

(32) $$\| u_h(x, \cdot) - v_h(x, \cdot) \|_h \leq (\bar{C} \| h \| + \eta_h A^{-1})(e^{Ax} - 1) \quad \text{if } A > 0,$$
$$\| u_h(x, \cdot) - v_h(x, \cdot) \|_h \leq \eta_h x \quad \text{if } A = 0,$$

where $x \in [0, a]$. *In particular we have*

$$\lim_{h\to 0} \| u_h(x, \cdot) - v_h(x, \cdot) \|_h = 0 \text{ uniformly with respect to } x \in [0, a].$$

Proof. We see at once that v_h exists on $[-\tau_0, a]$. It follows from Assumption H_4 and from Lemma 1 that

$$\left| D_x u_h^{(m)}(x) - D_x v_h^{(m)}(x) - \langle \int_0^1 D_q f(\tilde{Q}(x, m, t))dt; \ \Delta u_h^{(m)}(x) - \Delta v_h^{(m)}(x) \rangle \right|$$

$$\leq A[\max\{ \| u_h(x+t, \cdot) - v_h(x+t, \cdot) \|_h : t \in [-\tau_0, 0]\} + \bar{C} \| h \|] + \eta_h, \ x \in [0, a], \ m \in J,$$

where $\tilde{Q}(x, m, t) = (x, y^{(m)}, u_{xy^{(m)}}, \Delta v_h^{(m)}(x) + t[\Delta u_h^{(m)}(x) - \Delta v_h^{(m)}(x)])$. Now we obtain (32) from Lemma 2.

Remark 2. The results obtained in this paper can be extended to weakly coupled systems of differential-functional equations.

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