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THE NIELSEN NUMBER WITH RESPECT TO A SUBSET ON MANIFOLDS

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In this paper we will investigate $\mathfrak{N}(f, B)$ — the Nielsen number with respect to a subset. It is a homotopy invariant which gives a lower bound of the number of points in $f^{-1}(B)$ where $f: X \to Y$ is continuous and $B \subseteq Y$. We will assume that X, Y, B are closed oriented and smooth manifolds, B is a submanifold of Y and $\dim X = \dim Y - \dim B$.

1. Introduction. The Nielsen number with respect to a subset was defined by R. Dobrenko and Z. Kucharski in [1]. It is defined for continuous maps $f: X \to Y$ where X is compact, locally path connected and $B \subseteq Y$ is such that there is an open neighbourhood $W \subseteq Y$ of B which can be deformed to B in Y. First, let us quote some of the definitions given in [1].

We say that the points $x_0, x_1 \in f^{-1}(B)$ are in the Nielsen relation with respect to B iff there exists a path $\omega: 0 \to X$ such that $\omega(0) = x_0$, $\omega(1) = x_1$ and the path $f \circ \omega: 0 \to Y$ is homotopic rel $\{0, 1\}$ to some path $\eta: 0 \to Y$ such that $\eta(0) \subseteq B$. This is an equivalence relation in $f^{-1}(B)$. The classes of this relation, (which are called Nielsen classes) are open in the compact set $f^{-1}(B)$, so there can be only finite number of them. We say that a given Nielsen class of f is essential if it cannot be deformed to the empty set during any continuous deformation of f. The number of essential Nielsen classes is called the Nielsen number with respect to B and is denoted by $\mathfrak{N}(f, B)$. Thus the number $\mathfrak{N}(f, B)$ is a homotopy invariant and gives a lower bound of $\# f^{-1}(B)$ (we denote by $\# S$ the cardinality of the set S). In the quoted paper [1] the following theorem is proved:

Theorem 1.1 If

(*a*) *X, Y are oriented smooth and closed manifolds,*

(*b*) *B is an oriented smooth and closed submanifold of Y,*

(*c*) $\dim X = \dim Y - \dim B \geq 3$,

then the Nielsen number $\mathfrak{N}(f, B)$ is the best homotopy invariant which is a lower bound of $\# f^{-1}(B)$.

In order words for any continuous map $f: X \to Y$ there is a map $g: X \to Y$ homotopic to f such that $\# g^{-1}(B) = \mathfrak{N}(g, B) = \mathfrak{N}(f, B)$.

In the present paper we investigate the case when $\dim X = \dim Y - \dim B = 2$. In this case the theory is analogous to the fixed point theory of surface maps (see [2] [4] [5]). As in the fixed point theory, there exists a map which cannot be homotopically deformed to a map such that the cardinality of an inverse image of a given set coincides with the Nielsen number. One of the simplest examples of such a map is given in the Appendix.

The main purpose of this paper is to prove theorem (2.6). It generalizes theorem (1.1) to the case $\dim X = \dim Y - \dim B = k = 2$ under the additional assumption about the position of B in Y. This additional assumption is automatically satisfied when $k \geq 3$.

In our paper we will need one more construction from [1]. This is the local index $I(f, B, U)$ of the continuous map f with respect to B and to an open set $U \subseteq X$

(the set U should satisfy $\partial U \cap f^{-1}(B) = \emptyset$). When f is transversal to the submanifold B, the index is defined by the formula:

$$I(f, B, U) = \sum_{x \in f^{-1}(B) \cap U} I(f, B, x) \quad \text{where}$$

$$I(f, B, x) = \operatorname{sgn}_{T_{f(x)}} B \, [df_x T_x X \oplus T_{f(x)} Y].$$

The index is useful when computing $\mathfrak{N}(f, B)$. Because if the index of a Nielsen class (i. e. index of f with respect to a neighbourhood U of $\mathcal{N}$ such that $\mathcal{N} = U \cap f^{-1}(B) = \bar{U} \cap f^{-1}(B)$) is not equal to 0, then the class is essential. If $k \geq 3$ or $k = 2$ and the inclusion $j\colon Y \setminus B \to Y$ is locally trivial (see definition 2.4), then a Nielsen class is essential iff its index is not equal to 0.

2. The Minimization Problem. In what follows we assume that X, Y are smooth closed and oriented manifolds, B is a smooth closed and oriented submanifold of Y, $\dim X = \dim Y - \dim B = 2$ and $\dim B = n$.

In order to formulate the main result we need a number of definitions.

For each point $b \in B \subseteq Y$ there is an open neighbourhood $V \subseteq Y$ such that there exists a homeomorphism $\varphi\colon V \to \mathrm{R}^n \times \mathrm{R}^2$ satisfying $\varphi(V \cap B) = \mathrm{R}^n \times \{0\}$. The family of all these open neighbourhoods of b we denote by $\mathcal{V}_b$. For $V \in \mathcal{V}_b$ the homomorphisms induced by inclusions $i_v\colon V \setminus B \to Y \setminus B$, $j_v\colon Y \setminus B \to Y$ we denote by $i_{v,\#}\colon \Pi_1(V \setminus B, y) \to \Pi_1(Y \setminus B, y)$, $j_{v,\#}\colon \Pi_1(Y \setminus B, y) \to \Pi_1(Y, y)$ where y is an arbitrary point in $V \setminus B$.

Definition 2.1. *An inclusion $j\colon Y \setminus B \to Y$ is said to be locally trivial at the point $b \in B$ provided there is $V \in \mathcal{V}_b$ such that $\ker j_{v,\#} \subseteq \operatorname{im} i_{v,\#}$.*

Proposition 2.2 *If $b \in B$, then the following statements are equivalent:*

(*a*) *The inclusion $j\colon Y \setminus B \to Y$ is locally trivial at the point b.*

(*b*) *Each $V \in \mathcal{V}_b$ satisfies $\operatorname{im} i_{v,\#} = \ker j_{v,\#}$.*

(*c*) *For every continuous map $\psi\colon (\mathrm{D}^2, \mathrm{S}^1 \setminus \{x_0\}, \{x_0\}) \to (Y, Y \setminus B, \{b\})$ there exists a map $\tilde{\psi}\colon (\mathrm{D}^2, \mathrm{D}^2 \setminus \{x_0\}, \{x_0\}) \to (Y, Y \setminus B, \{b\})$ such that $\psi^1_{|s} = \tilde{\psi}^1_{|s}$ (x_0 is an arbitrary point in S^1).*

(*d*) *For every path $\alpha\colon 0 \to Y$ being homotopic rel $\{0, 1\}$ to the constant path $\theta(t) = b$ and such that $\alpha^{-1}(B) = \{0, 1\}$, there exists a homotopy $H\colon 0 \times 0 \to Y$ which joins rel $\{0, 1\}$ the paths θ and α and satisfies $H^{-1}(B) = \{0, 1\} \times 0 \cup 0 \times \{0\}$.*

The proof is straight forward and we omit it. □

Remark 2.3. If the inclusion $j\colon Y \setminus B \to Y$ is locally trivial at the point $b \in B$, then it is locally trivial at every point in the same connected component of B.

Proof. It is sufficient to show that the set of points at which the inclusion is locally trivial is open and closed. Openness follows immediately from Definition (2.1). In fact, if the inclusion is locally trivial at $b \in B$, then there is $V \in \mathcal{V}_b$ such that $\ker j_{v,\#} \subseteq \operatorname{im} i_{v,\#}$. But for every point $b' \in V \cap B$ the set V belongs to $\mathcal{V}_{b'}$. Thus by (2.1) the inclusion is trivial at every point $b' \in V \cap B$. Analogously the statement (2.2 b) gives us openness of the set of points at which the inclusion is not locally trivial. This completes the proof. □

Definition 2.4. *The inclusion $j\colon Y \setminus B \to Y$ is said to be locally trivial if it is locally trivial at every point $b \in B$.*

The main result will follow from

Lemma 2.5. *Let the inclusion $j\colon Y \setminus B \to Y$ be locally trivial, $f\colon X \to Y$ be a continuous map such that $f^{-1}(B)$ is finite and let the points $x_0, x_1 \in f^{-1}(B)$ belong to the same Nielsen class. Then there is a continuous map $\tilde{f}\colon X \to Y$ homotopic to f which satisfies*

$$\tilde{f}^{-1}(B) = \begin{cases} f^{-1}(B) \setminus \{x_0, x_1\} & \text{if } I(f, B, x_0) + I(f, B, x_1) = 0. \\ f^{-1}(B) \setminus \{x_0\} & \text{otherwise.} \end{cases}$$

Proof. Since x_0 and x_1 are in the Nielsen relation with respect to B, then by the definition there are paths $\omega: x_0 \sim\sim x_1$, $\eta: f(x_0) \sim\sim f(x_1)$ such that $f \circ \omega$ and η are homotopic rel $\{0, 1\}$ and $\eta(0) \subseteq B$. Without loss of generality we can assume that ω is an arc (by (3.2) in the homotopy class of ω there is an arc) such that $\omega(0) \cap f^{-1}(B) = \{x_0, x_1\}$ (because the set $f^{-1}(B)$ is finite).

The map $\tilde{f}$ will be constructed in several steps.

Step 1. (*The translation of $f(x_0)$ along the path* η).
We consider a pullback of the tangent bundle of B:

$$\begin{array}{ccc} \eta^*(TB) \simeq 0 \times \mathrm{R}^n & \xrightarrow{\tilde{\eta}} & TB \\ \downarrow & & \downarrow p \\ 0 & \xrightarrow{\eta} & B \end{array}$$

Having $\tilde{\eta}$ we define $\gamma: 0 \times \mathrm{R}^n \to B$ by $\gamma(t, v) = \exp(\tilde{\eta}(t, v))$. Now let $T \subseteq Y$ be an open tubular neighbourhood of the submanifold B and $\zeta: T \to B$ be a normal bundle of B in Y. Then we take a pullback of ζ induced by γ

$$\begin{array}{ccc} \gamma^*(T) \simeq 0 \times \mathrm{R}^n \times \mathrm{R}^2 & \xrightarrow{\alpha} & T \\ \downarrow & & \downarrow \zeta \\ 0 \times \mathrm{R}^n & \xrightarrow{\gamma} & B \end{array}$$

Thus we obtain a map $\alpha: 0 \times \mathrm{R}^n \times \mathrm{R}^2 \to T$ such that

(—) For every $t \in 0$ the map $\alpha_t = \alpha(t, \cdot, \cdot): \mathrm{R}^n \times \mathrm{R}^2 \to Y$ is a local diffeomorphism at 0.

(—) $\alpha(t, 0, 0) = \eta(t)$.

(—) $\alpha^{-1}(B) = 0 \times \mathrm{R}^n \times \{0\}$

Since $f(x_0) = \eta(0) = \alpha_0(0)$ and α_0 is a local diffeomorphism at 0, there is an open neighbourhood $U \subseteq X$ of x_0 such that locally there exists the inverse map $\alpha_0^{-1}: f(\bar{U}) \to \mathrm{R}^{n+2}$. Moreover, since the set $f^{-1}(B)$ is finite we can choose U such that $\bar{U} \cap f^{-1}(B) = \{x_0\}$.

Let us define the map $\lambda: \mathrm{R}^{n+2} \to \mathrm{R}$ by putting

$$\lambda(x) = \begin{cases} 0 & \text{if } \|x\| \geq \varepsilon \\ \dfrac{\varepsilon - \|x\|}{\varepsilon} & \text{if } \|x\| \leq \varepsilon \end{cases}$$

where $\varepsilon = \min\{\|\alpha_0^{-1} \circ f(x)\| : x \in \partial U\} > 0$.
Then the homotopy $G: X \times 0 \to Y$ given by

$$G(x, t) = \begin{cases} f(x) & \text{if } x \notin U \\ \alpha(t^* \lambda \circ \alpha_0^{-1} \circ f(x), \ \alpha_0^{-1} \circ f(x)) & \text{if } x \in U \end{cases}$$

joins f with the map $g = G(\circ, 1)$ satisfying: $g^{-1}(B) = f^{-1}(B)$ and the path $g \circ \omega$ is homotopic to the constant path $\theta(t) = g(x_0) = g(x_1)$.

Step 2. (*Joining x_0 with x_1 in the inverse image of B*). The aim of this step is to construct a map $h: X \to Y$ which is homotopic to g and satisfies $h^{-1}(B) = f^{-1}(B) \cup \omega(0)$, $h \circ \omega(t) = h \circ \omega(0) \in B$ for all $t \in 0$. It can be done by ⟨⟨blowing⟩⟩ the arc ω into the constant path $\theta(t) = g(x_0)$. More precisely, let $U \subseteq X$ be a neighbourhood of the arc ω such that $U \cap g^{-1}(B) = \{x_0, x_1\}$ and there exists a homeomorphism $\varphi: D^2 \to U$ satisfying $\varphi(t, 0) = \omega(\frac{t+1}{2})$ for all $t \in [-1, 1]$. Let $H: 0 \times 0 \to Y$ be a homotopy rel $\{0, 1\}$ joining paths $\theta(t) = g(x_0)$, $g \circ \omega$ and such that $H^{-1}(B) = \{0, 1\} \times 0 \cup 0 \times \{0\}$ (the inclusion $j: Y \setminus B \to Y$ is locally trivial, so by (2.2d) such a homotopy exists). Then the map $h: X \to Y$ is given by

$$h(x) = \begin{cases} g(x) & \text{if } x \notin \operatorname{int} U = \varphi(\operatorname{int} D^2) \\ g \circ \varphi[t, (\rho(t, s) - 1)s] & \text{if } x = \varphi(t, s) \in \operatorname{int} U \text{ and } t^2 + 4s^2 \geq 1 \\ H[t + 1/2, \rho(t, s)] & \text{if } x = \varphi(t, s) \in \operatorname{int} U \text{ and } t^2 + 4s^2 \leq 1, \end{cases}$$

where $\rho: \operatorname{int} D^2 \to [0, 2]$, $\rho(t, s) = \frac{2|s|}{\sqrt{1-t^2}}$.

Step 3. *(Reduction of $\omega(0)$ from $h^{-1}(B)$ to a point or to the empty set).* Since $h \circ (0)$ is a point and $\omega(0)$ is an open subset of $h^{-1}(B)$, then there exist neighborhoods:

(—) $W \subseteq X$ of the arc $\omega: x_0 \sim\sim x_1$, which is homeomorphic to D^2.

(—) $V \subseteq Y$ of the point $h \circ \omega(0)$, which is homeomorphic to $R^n \times R^2$, such that $h(W) \subseteq H$ and $\operatorname{int}(W) \cap h^{-1}(B) = W \cap h^{-1}(B) = \omega(0)$. Then the restriction $h_{|W}: W \to V$ can be considered as a map $(h_1, h_2): W \to R^n \times R^2$. Now, note that $h_2: W \to R^2$, $h_2^{-1}(0) = h^{-1}(B) \cap W = \omega(0)$ and W is homeomorphic to D^2. Thus according to the degree theory we can deform the map h_2 (not changing its value on the border) into a map for which the inverse image of 0 is:

(—) $\emptyset$ if $\deg(h_2, W) = I(f, B, x_0) + I(f, B, x_1) = 0$

(—) $\{x_1\}$ otherwise.

Thus it is possible to reduce $\omega(0)$ from $h^{-1}(B)$ to a point or to the empty set. Finally we get the required map $\tilde{f}$ as a map homotopic to $h \simeq f$. □

Let us prove the main theorem:

Theorem 2.6. *If X, Y are smooth oriented and closed manifolds, B is a smooth oriented and closed submanifold of Y, $\dim X = \dim Y - \dim B = 2$ and the inclusion $Y \setminus B \subseteq Y$ is locally trivial, then for every continuous map $f: X \to Y$ there is a map $q: X \to Y$ homotopic to f such that $\mathfrak{N}(g, B) = \mathfrak{N}(f, B) = \# g^{-1}(B)$.*

Proof. Without loss of generality we can assume that f is smooth and transversal to the submanifold B. Thus $f^{-1}(B)$ is finite. Then according to lemma (2.5) we can remove every Nielsen class of index zero and every essential class (of index $\neq 0$) can be reduced to a single point. This procedure shows that we can obtain a map with $\mathfrak{N}(f, B)$ points in the inverse image of B, what completes the proof. □

3. Appendix.

Proposition 3.1. *If $f: S^2 \to S^2$ is a map of degree 2 and $B = \{y_0, y_1, y_2\}$, then $\# f^{-1}(B) \geq 4 > 3 = \mathfrak{N}(f, B)$.*

Proof. First, because S^2 is simply connected there are only three Nielsen classes $(f^{-1}(Y_i)$ $i = 0, 1, 2)$. It is clear that they are essential (the indices are equal to 2). Hence $\mathfrak{N}(f, B) = 3$. Suppose that there is a map $f: S^2 \to S^2$ of degree two such that $\# f^{-1}(B) = 3$. Then $f^{-1}(Y_i) = \{x_i\}$ for $i = 0, 1, 2$. Now consider disjoint neighbourhoods C_1, C_2 of points y_1, y_2 which are homeomorphic to $\operatorname{int} D^2$ and such that $y_0 \notin C_i$ for $i = 1, 2$. Let D_1, D_2 be some neighbourhoods of the points x_1, x_2 which are homeomorphic to $\operatorname{int} D^2$ and such that $f(D_i) \subseteq C_i$ for $i = 1, 2$. Then $S^2 \setminus (C_1 \cup C_2) \subseteq f(S^2 \setminus (D_1 \cup D_2)) \subseteq S^2 \setminus \{y_1, y_2\}$.

Note that $S^2 \setminus (C_1 \cup C_2)$ is a retract of $S^2 \setminus \{y_1, y_2\}$ and therefore there is no loss of generality if we assume that $f(\partial D_1) \subseteq \partial C_1$, $f(\partial D_2) \subseteq \partial C_2$. Then the degree of $f_|: \partial D_i \to \partial C_i$ is 2. Now observe that $S^2 \setminus (D_1 \cup D_2)$, $S^2 \setminus (C_1 \cup C_2)$ are homeomorphic to $S^1 \times 0$. Therefore there exist coverings

$$p_1: \mathsf{R}\times 0 \to \mathsf{S}^2\setminus(D_1 \cup D_2), \quad p_2: \mathsf{R}\times 0 \to \mathsf{S}^2\setminus(C_1 \cup C_2)$$

and a lifting

$$\begin{array}{ccc}
(\mathsf{R}\times 0,\ \mathsf{R}\times\{0\},\ \mathsf{R}\times\{1\} & \xrightarrow{\ \widetilde{f}\ } & (\mathsf{R}\times 0,\ \mathsf{R}\times\{0\},\ \mathsf{R}\times\{1\}) \\
\downarrow p_1 & & \downarrow p_2 \\
(\mathsf{S}^2\setminus(D_1 \cup D_2),\ \partial D_1,\ \partial D_2) & \xrightarrow{\ f\ } & (\mathsf{S}^2\setminus(C_1 \cup C_2),\ \partial C_1,\ \partial C_2).
\end{array}$$

It is easy to verify that $\widetilde{f}(\mathsf{R}\times 0) = \mathsf{R}\times 0$ and $\widetilde{f}(t+1, s) = \widetilde{f}(t, s) + (2, 0)$. Now consider a point $\widetilde{y}_0 \in \mathsf{R}\times 0$ such that $p_2(\widetilde{y}_0) = y_0$ and fix two points $\widetilde{x}_1 \in \widetilde{f}^{-1}(\widetilde{y}_0)$, $\widetilde{x}_2 \in \widetilde{f}^{-1} \times(\widetilde{y}_0 + (1, 0))$. Then $fp_1(\widetilde{x}_i) = p_2\widetilde{f}(\widetilde{x}_i) = y_0$ for $i = 1, 2$. As $\# f^{-1}(y_0) = 1$ we have $p_1(\widetilde{x}_1) = p_1(\widetilde{x}_2)$. This implies that there is $n \in \mathsf{Z}$ such that $\widetilde{x}_2 = x_1 + (n, 0)$, which gives $\widetilde{y}_0 + (1, 0) = \widetilde{f}(\widetilde{x}_2) = \widetilde{f}(\widetilde{x}_1 + (n, 0)) = \widetilde{f}(\widetilde{x}_1) + (2n, 0) = \widetilde{y}_0 + (2n, 0)$. Thus we get a contradiction and the proof is completed □

Proposition 3.2. *Let X be a two dimensional manifold without boundary. Then for every path $\omega: 0 \to X$ with different ends there exists an arc $\widetilde{\omega}: 0 \to X$ homotopic rel $\{0, 1\}$ to ω.*

Proof. According to the general position theorem there is a path $\bar{\omega}: 0 \to X$ homotopic rel $\{0, 1\}$ to ω with a finite number of self-intersection points. Let $\widetilde{\omega}$ be a path with the minimal number of self-intersection points. Suppose $\widetilde{\omega}$ is not an arc. Then denote by $s_0 = \min\{s \in 0: \exists\ t \neq s,\ \widetilde{\omega}(t) = \widetilde{\omega}(s)\}$ the first of self-intersection point of $\widetilde{\omega}$. Let $t_0 \in 0$ be a point such that $t_0 \neq s_0$ and $\widetilde{\omega}(t_0) = \widetilde{\omega}(s_0)$. Now consider an open neighbourhood $V \subseteq X$ of the arc $\widetilde{\omega}_{|[0, s_0]}$ homeomorphic to R^2. Fix $\delta > 0$ satisfying $\omega((t_0 - \delta, t_0 + \delta)) \subseteq V$ and such that there is exactly one self-intersection point t_0 in $(t_0 - \delta, t_0 + \delta)$. It is easy to see that there is an arc γ joining $\widetilde{\omega}(t_0 - \delta)$ with $\widetilde{\omega}(t_0 + \delta)$ and such that $\gamma(0) \subseteq V\setminus\widetilde{\omega}(0\setminus[t_0 - \delta, t_0 + \delta])$. Observe that V is simply connected, so γ is homotopic to $\widetilde{\omega}_{|[t_0 - \delta,\, t_0 + \delta]}$. Thus the path $\alpha = \widetilde{\omega}_{|[0, t_0 - \delta]} * \gamma * \widetilde{\omega}_{|[t_0 + \delta, 1]}$ is homotopic to $\widetilde{\omega}$. Observe that α has less self-intersection points than $\widetilde{\omega}$. It contradicts the definition of $\widetilde{\omega}$ and the proof is completed. □

REFERENCES

1. R. Dobrenko, Z. Kucharski. On the generalization of the Nielsen number, (to appear in Fund. Math.).
2. E. Fadell, S. Husseini. The Nielsen number of surfaces, Topological methods in nonlinear functional analysis. *Contemp. Math.*, **21**, 1983, 59-98.
3. B. Jiang. Estimation of the Nielsen numbers. *Acta. Math. Sinica*, **14**, 1964, 304-312.
4. B. Jiang. Fixed points and braids. *Invent. Math.*, **75**, **1984**, 69-74.
5. B. Jiang. Fixed points and braids. II. *Math. Ann.*, **272**, 1985, 249-256.
6. J. Nielsen. Über die minimalizahl der fixpunkte bei abbildungstypen der ringflachen. *Math. Ann.*, **82**, 1921, 83-93.
7. J. Nielsen. Untersuchungen zun Topologie der geschlossenen zweiseitigen Flachen. I. *Acta. Math.*, **50**, 1927, 189-358, II. 53, 1929, 1-76, III, 58, 1932, 87-167.
8. F. Wechen. Fixpunktklassen. I. *Math. Ann.*, **117**, 1941, 659-671, II. **118**, 1942, 216-234, III. **118**, 1942, 544-577.

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