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ON THE NASH-BARGAINING SOLUTION IN STOCHASTIC DIFFERENTIAL GAMES

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The paper deals with two-person cooperative games in which the dynamics is described by Ito stochastic differential equations. The Nash-bargaining solution for such games is introduced. Sufficient conditions for verification of the strategies composing this type of a solution are found. The Pareto-optimality of these strategies is also established.

1. Introduction. It is well-known that Pareto-optimality is one of the basic notions in cooperative differential games. There is a lot of publications on these topics, mainly in the deterministic case (see [8]). In stochastic differential games Pareto-optimal strategies are considered by the author in [3].

Let us mention that Pareto-optimality comes in for criticism at least in two aspects (see [8]). First, the application of Karlin's lemma and the reduction of the problem to a single criterial optimization imply the ambiguity of the strategies. Second, Pareto-optimal strategies can supply some players with values of their cost-functions even greater than the guaranteeing (minimax) strategies can do.

These disadvantages can be overcome if we restrict, in some definite sense, the set of Pareto-optimal strategies. One possibility is to use the Nash-bargaining solution.

The results presented here have been announced without any proof and details in our recent paper [5].

Note that for reduction of calculations we consider games with two participant.

2. Formalization of a stochastic differential game. Consider the system (game)

$$\Gamma=\langle\{1,\ 2\},\ \Sigma,\ \{\mathcal{U}_1,\ \mathcal{U}_2\},\ \{J_1,\ J_2\}\rangle.$$

Here $\{1,\ 2\}$ is the set of players participating in Γ. The evolution of the dynamic system Σ is described by the following stochastic differential equation of Ito type:

$$(*)\qquad dx(t)=f(t,\ x(t),\ u_1,\ u_2)dt+g(t,\ x(t),\ u_1,\ u_2)dw(t),\quad t\in[t_0,\ T]$$

with an initial condition $x(t_0)=x_0\in\mathrm{R}^n$ and $0\leq t_0<T$. The process $w=\{w(t),\ t\in[t_0,\ T]\}$ is a standard m-dimensional Wiener process, defined on some complete probability space $(\Omega,\ \mathcal{F},\ \mathbf{P})$ and adapted to a given family $F=\{\mathcal{F}_t,\ t\in[t_0,\ T]\}$ of nondecreasing sub-σ-algebras of $\mathcal{F}$. The vectors $x(t)\in\mathrm{R}^n$ is the state process and $u_i\in U_i\subset\mathrm{R}^{n_i}$ is the control of the i-th player, $i=1,\ 2$.

Let us make the following assumptions about the functions $f(t,\ x,\ u_1,\ u_2)$ and $g(t,\ x,\ u_1,\ u_2)$. Suppose

$$f:\ [t_0,\ T]\times\mathrm{R}^n\times U_1\times U_2\to\mathrm{R}^n$$

and

$$g:\ [t_0,\ T]\times\mathrm{R}^n\times U_1\times U_2\to\mathrm{R}^n\times\mathrm{R}^m$$

have continuous partial derivatives in $x,\ u_1,\ u_2$. Further, let $C>0$ be a constant such that

$$|f(t,\ 0,\ 0,\ 0)|+|g(t,\ 0,\ 0,\ 0)|\leq C,$$

$$|f_x|+|g_x|+|f_{u_1}|+|g_{u_1}|+|f_{u_2}|+|g_{u_2}|\leq C,$$

where $|\cdot|$ is a general symbol for the norm in the respective space.

We suppose that each player has perfect observations of the vector $x(t)$ at each moment $t\in[t_0,\ T]$ and constructs his strategy in the game Γ as an admissible feedback control (see [2]) of the type

$$u_i=u_i(t,\ x(t)),\quad i=1,\ 2.$$

Here

$$u_i(\cdot\,,\cdot):\ [t_0,\ T]\times R^n\to U_i$$

is a Borel function satisfying the following conditions:

(a) There exists a constant $M_i>0$ such that

$$|u_i(t,\ x)|\leq M_i(1+|x|)\ \text{for all}\ t\in[t_0,\ T],\quad x\in R^n.$$

(b) For each bounded set $B\subset R^n$ and $T^*\in(t_0,\ T)$ there exists a constant $K_i>0$ such that for arbitrary $x,\ y\in B$ and $t\in[t_0,\ T^*]$

$$|u_i(t,\ x)-u_i(t,\ y)|\leq K_i|x-y|.$$

Denote by $\mathcal{U}_i$ the set of strategies of the i-th player, $i=1,\ 2$ and $\mathcal{U}=\mathcal{U}_1\times\mathcal{U}_2$ Let the pair of strategies $u=(u_1,\ u_2)$ be called for brevity just a strategy.

The assumptions mentioned above imply the existence and sample path uniqueness (see [2]) of the solution $X=\{x(t),\ t\in[t_0,\ T]\}$ of Ito equation (*) corresponding to the control $u=(u_1,\ u_2)$. Moreover, X is an a. s. continuous Markov process and if $\mathcal{A}(u)$ denotes its infinitesimal operator (see [1]), then

$$\mathcal{A}(u)W(t,\ x)=f'(t,\ x,\ u_1,\ u_2)Wx(t,\ x)+\frac{1}{2}\operatorname{tr}[g(t,\ x,\ u_1,\ u_2)\,g'(t,\ x,\ u_1,\ u_2)\,Wxx(t,\ x)].$$

Here prime denotes vector or matrix transpose and $W(t,\ x)$ is a real-valued function with continuous partial derivatives up to second order for all $t\in[t_0,\ T],\ x\in R^n$.

Let us consider the continuous functions Q_i satisfying the growth condition

$$|Q_i(t,\ x)|\leq C_i(1+|x|^k),$$

where C_i, k are positive constants. Introduce now the cost-function $J_i(u)$ of the i-th player of a terminal type

$$J_i(u)=\mathbf{E}_{t_0,\ x_0}\{Q_i(T,\ x(T))\},\quad i=1,\ 2$$

with respect to the initial situation $x(t_0)=x_0$.

Every stochastic differential game develops in the following way. Each player, e. g. the i-th one, chooses his strategy $u_i\in\mathcal{U}_i$ according to some principle of optimality. Thus we have the pair of strategies $u=(u_1,\ u_2)$. Further, the solution X of Ito equation (*) is found. Finally, X and u determine the value of $J_i(u)$, $i=1,\ 2$. The object of each player in the game Γ is to minimize his cost-function.

3. Definition and properties. Let us recall the following notion of an optimal strategy in a stochastic differential game (see [4]).

Definition. *The strategy $u^g=(u_1^g,\ u_2^g)$ is a guaranteeing (minimax) strategy in the game Γ if*

$$\min_{u_1}\max_{u_2}J_1(u_1,\ u_2)=\max_{u_2}J_1(u_1^g,\ u_2)=J_1^g$$

and

$$\min_{u_2}\max_{u_1}J_2(u_1,\ u_2)=\max_{u_1}J_2(u_1,\ u_2^g)=J_2^g,$$

Now introduce the functional

$$I_{nb}(u)=[J_1^g-J_1(u)]\,[J_2^g-J_2(u)].$$

Definition. *The strategy* $u^{nb}=(u_1^{nb}, u_2^{nb})$ *is called a Nash-bargaining solution in the game* Γ, *if for each* $u=(u_1, u_2)$ *we have*

$$I_{nb}(u)\leqq I_{nb}(u^{nb}).$$

Note that in deterministic game theory Nash-bargaining solutions are discussed in [7, 8]. This theory requires the consideration of $J_1(u)$ only for strategies $u\in U$ such that $J_i(u)\leqq J_i^g$, $i=1, 2$. Thus we come to the first property of u^{nb}, namely

$$J_i(u^{nb})\leqq J_i^g,\ i=1,\ 2.$$

Proposition. *The Nash-bargaining solution is Pareto-optimal.*

Proof. Let u^{nb} be not Pareto-optimal (see [3]). Then there exists a strategy $\bar{u}=(\bar{u}_1, \bar{u}_2)$ such that the system

$$J_i(\bar{u})\leqq J_i(u^{nb}),\ i=1,\ 2$$

holds, where at least one of these two inequalities is strict. Hence

$$J_i^g-J_i(\bar{u})\geqq J_i^g-J_i(u^{nb})\geqq 0,\ i=1,\ 2$$

where at least one inequality is strict. Therefore

$$[J_1^g-J_1(\bar{u})]\,[J_2^g-J_2(\bar{u})]>[J_1^g-J_1(u^{nb})]\,[J_2^g-J_2(u^{nb})],$$

i. e.

$$I_{nb}(\bar{u})>I_{nb}(u^{nb}).$$

Obviously this relation contradicts the definition of u^{nb}. Thus we get the Pareto-optimality of u^{nb}.

4. Sufficient conditions for the Nash-bargaining solution. First we shall consider the following auxiliary proposition.

Lemma. *Let* X *be the solution of Ito equation* (∗) *with initial condition* $x(t_0)=x_0$. *Then there is a positive constant* A_0 *such that the following estimate holds:*

$$|\mathbf{E}_{t_0,x_0}\{Q_1(T, x(T))Q_2(T, x(T))\}-\mathbf{E}_{t_0,x_0}\{Q_1(T, x(T))\}\,\mathbf{E}_{t_0,x_0}\{Q_2(T, x(T))\}|\leqq A_0,$$

where Q_1, Q_2 *are the functions defining the cost-functions.*

Proof. Taking into account some properties of conditional expectations, Cauchy-Bunyakovskii-Schwarz inequality, the growth conditions of the functions Q_i and a result (see [6], Part 1, § 6, Th. 4), we get

$$|\mathbf{E}_{t_0,x_0}\{Q_1,(T, x(T))\,Q_2(T, x(T))\}-\mathbf{E}_{t_0,x_0}\{Q_1(T, x(T))\}\mathbf{E}_{t_0,x_0}\{Q_2(T, x(T))\}|$$

$$\leqq|\mathbf{E}_{t_0,x_0}\{Q_1(T, x(T))\,Q_2(T, x(T))\}|+|\mathbf{E}_{t_0,x_0}\{Q_1(T, x(T))\}|\,.\,|\mathbf{E}_{t_0,x_0}\{Q_2(T, x(T))\}|$$

$$\leqq 2(\mathbf{E}_{t_0,x_0}\{|Q_1(T, x(T))|^2\}\mathbf{E}_{t_0,x_0}\{|Q_2(T, x(T))|^2\}^{1/2}$$

$$\leqq 2(\mathbf{E}_{t_0,x_0}\{C_1^2(1+|x(T)|^k]\}\mathbf{E}_{t_0,x_0}\{C_2^2[1+|x(T)|^k]^2\})^{1/2}$$

$$=2C_1C_2\mathbf{E}_{t_0,x_0}\{(1+|x(T)|^k)^2\}\leqq 4C_1C_2\mathbf{E}_{t_0,x_0}\{1+|x(T)|^{2k}\}$$

$$=4C_1,\ C_2(1+\mathbf{E}_{t_0,x_0}\{|x(T)^{2k}\})\leqq 4C_1,\ C_2[1+K(1+|x_0|^{2k})]=A_0,$$

where K is a suitably chosen constant.

Remark. Further we shall use the result of the Lemma in the form

$$\mathbf{E}_{t_0,x_0}\{Q_1(T,\ x(T))Q_2(T,\ x(T))\}-\mathbf{E}_{t_0,x_0}\{Q_1(T,\ x(T))\}\mathbf{E}_{t_0,x_0}\{Q_2(T,\ x(T))\}\geqq -A_0.$$

Now we are in position to formulate and prove sufficient conditions, satisfied by the Nash-bargaining solution.

Theorem. *The strategy* $u^{nb}=(u_1^{nb},\ u_2^{nb})$ *is a Nash-bargaining solution in the game* Γ, *if there exist real-valued functions* $V^{(i)}(t,\ x)$ *such that for all* $t\in[t_0,\ T]$, $x\in\mathbf{R}^n$ *and* $i=1,\ 2$ *the following conditions jointly hold*:

(*a*) $V^{(i)},\ V_t^{(i)},\ V_x^{(i)},\ V_{xx}^{(i)}$ *are continuous*;

(*b*) $$[V_t^{(1)}(t,\ x)+\mathscr{A}(u)V^{(1)}(t,\ x)][V^{(2)}(t,x)-J_2^g]+[V_t^{(2)}(t,\ x)+\mathscr{A}(u)V^{(2)}(t,\ x][V^{(1)}(t,\ x)-J_1^g]$$
$$+[V_x^{(1)}(t,\ x)]'g(t,\ x,\ u_1,\ u_2)g'(t,\ x,\ u_1,\ u_2)V_x^{(2)}(t,\ x)\leqq -A$$

for each $u=(u_1,\ u_2)$ *where* $A=A_0/(T-t_0)$;

(c) $V_t^{(i)}(t,\ x)+\mathscr{A}(u^{nb})V^{(i)}(t,\ x)=0$;

(d) $V^{(i)}(T,\ x)=Q_i(T,\ x)$.

Proof. Let $x^{nb}(t)$, $t\in[t_0,\ T]$ be the sample path of the solution of Ito equation (*) corresponding to the strategy $u^{nb}=(u_1^{nb},\ u_2^{nb})$. Conditions (c), (d) and Theorem 5 (see [6], part II, ch. 2, § 9) imply the relation

$$V^{(i)}(t_0,\ x_0)=\mathbf{E}_{t_0,x_0}\{Q_i(T,\ x^{nb}(T))\}=J_i(u^{nb}),\ i=1,\ 2.$$

Now let $x(t)$, $t\in[t_0,\ T]$ be the sample path of the solution of Ito equation (*) corresponding to an arbitrary strategy $u=(u_1,\ u_2)$. Write Ito formula for $V^{(i)}(t,\ x)$, $x(t)$ and u (see [2]):

$$dV^{(i)}(t,\ x(t))=[V_t^{(i)}(t,\ x(t))+\mathscr{A}(u)V^{(i)}(t,\ x(t))]\,dt$$
$$+[V_x^{(i)}(t,\ x(t))]'g(t,\ x(t),\ u_1,\ u_2)dw(t),\ i=1,\ 2.$$

Then we have

$$d[V^{(1)}(t,\ x(t))V^{(2)}(t,\ x(t))]=\{[V_t^{(1)}(t,\ x(t))+\mathscr{A}(u)V^{(1)}(t,\ x(t))]V^{(2)}(t,\ x(t))$$
$$+[V_t^{(2)}(t,\ x(t))+\mathscr{A}(u)V^{(2)}(t,\ x(t))]V^{(1)}(t,\ x(t))+[V_x^{(1)}(t,\ x(t))]'g(t,\ x(t),\ u_1,\ u_2)g'(t,\ x(t),$$
$$\times u_1,\ u_2)V_x^{(2)}(t,\ x(t))\}\,dt+\{g'(t,\ x(t),\ u_1,\ u_2)V_x^{(1)}(t,\ x(t))V^{(2)}(t,\ x(t))$$
$$+g'(t,\ x(t),\ u_1,\ u_2)V_x^{(2)}(t,\ x(t))V^{(1)}(t,\ x(t))\}\,dw(t).$$

Hence by integration we get

$$V^{(1)}(T,\ x(T))V^{(2)}(T,\ x(T))-V^{(1)}(t,\ x(t))V^{(2)}(t,\ x(t))$$
$$=\int_t^T\{[V_t^{(1)}(\tau,\ x(\tau))+\mathscr{A}(u)V^{(1)}(\tau,\ x(\tau))]V^{(2)}(\tau,\ x(\tau))+[V_t^{(2)}(\tau,\ x(\tau))$$
$$+\mathscr{A}(u)V^{(2)}(\tau,\ x(\tau))]V^{(1)}(\tau,\ x(\tau))+[V_x^{(1)}(\tau,x(\tau))]'g(\tau,\ x(\tau),\ u_1,\ u_2)g'(\tau,x(\tau),\ u_1,\ u_2)V_x^{(2)}(\tau,\ x(\tau))\}d\tau$$
$$+\int_t^T\{g'(\tau,x(\tau),u_1,u_2)V_x^{(1)}(\tau,x(\tau))V^{(2)}(\tau,x(\tau))+g'(\tau,\ x(\tau),\ u_1,\ u_2)V_x^{(2)}(\tau,\ x(\tau))V^{(1)}(\tau,\ x(\tau))\}dw(\tau).$$

Therefore

$$V^{(1)}(t,\ x)V^{(2)}(t,\ x)=\mathbf{E}_{t,x}\{V^{(1)}(T,\ x(T))V^{(2)}(T,\ x(T))$$

$$-\int_t^T\{[V_t^{(1)}(\tau,\ x(\tau))+\mathscr{A}(u)V^{(1)}(\tau,\ x(\tau))]V^{(2)}(\tau,\ x(\tau))+[V_t^{(2)}(\tau,\ x(\tau))$$

$$+\mathscr{A}(u)V^{(2)}(\tau,x(\tau))]V^{(1)}(\tau,x(\tau))+[V_x^{(1)}(\tau,x(\tau))]g'(\tau,x(\tau),u_1,u_2)g'(\tau,x(\tau),u_1,u_2)V_x^{(2)}(\tau,x(\tau))\}d\tau\}.$$

Taking into consideration condition (d) we have

$$V^{(1)}(t_0,\ x_0)\ V^{(2)}(t_0,\ x_0)=\mathbf{E}_{t_0,x_0}\{Q_1(T,\ x(T))\ Q_2(T,\ x(T))$$

$$-\int_{t_0}^T\{[V_t^{(1)}(t,x(t))+\mathscr{A}(u)V^{(1)}(t,x(t))]V^{(2)}(t,x(t))+[V_t^{(2)}(t,x(t))+\mathscr{A}(u)V^{(2)}(t,x(t))]V^{(1)}(t,x(t))$$

$$+\ [V_x^{(1)}(t,\ x(t))]'g(t,\ x(t),\ u_1,\ u_2)g'(t,\ x(t),\ u_1,\ u_2)V_x^{(2)}(t,\ x(t))\}dt\}.$$

Further, Ito-Dynkin formula (see [2], Ch. 5, Th. 5. 2) gives us

$$V^{(i)}(t_0,\ x_0)=\mathbf{E}_{t_0,x_0}\{Q_i(T,\ x(T))\ -\ \int_{t_0}^T\ [V_t^{(i)}(t,\ x(t))\ +\ \mathscr{A}(u)\ V^{(i)}(t,\ x(t))]\ dt\},\ i=1,\ 2.$$

Thus we obtain the following chain of equalities:

$$J_1(u^{nb})J_2(u^{nb})-J_1^g J_2(u^{nb})-J_2^g J_1(u^{nb})=V^{(1)}(t_0,\ x_0)V^{(2)}(t_0\ x_0)-J_1^g V^{(2)}(t_0,\ x_0)-J_2^g V^{(1)}(t_0,\ x_0)$$

$$=\mathbf{E}_{t_0,x_0}\{Q_1(T,\ x(T))Q_2(T,\ x(T))\ -\ J_1^g Q_2(T,\ x(T))-J_2^g Q_1(T,\ x(T))$$

$$-\ \int_{t_0}^T\ \{[V_t^{(1)}(t,\ x(t))+\mathscr{A}(u)V^{(1)}(t,\ x(t))]\ [V^{(2)}(t,\ x(t))-J_2^g]+[V_t^{(2)}(t,\ x(t))$$

$$+\mathscr{A}(u)V^{(2)}(t,\ x(t))]\ [V^{(1)}(t,\ x(t))\ -J_1^g]+[V_x^{(1)}(t,\ x(t))]'g(t,\ x(t),$$

$$\times\ u_1,\ u_2)g'\ (t,\ x(t),\ u_1,\ u_2)V_x^{(2)}\ (t,\ x(t))\}dt\}.$$

Hence

$$J_1(u^{nb})\ J_2(u^{nb})-J_1^g\ J_2(u^{nb})-J_2^g\ J_1(u^{nb})=J_1(u)J_2(u)-J_1^g\ J_2(u)-J_2^g\ J_1(u)$$

$$+\mathbf{E}_{t_0,x_0}\{Q_1(T,\ x(T))Q_2(T,\ x(T))\}-\mathbf{E}_{t_0,x_0}\{Q_1(T,\ x(T))\}\mathbf{E}_{t_0,x_0}\{Q_2(T,\ x(T))\}$$

$$-\mathbf{E}_{t_0,x_0}\{\ \int_{t_0}^T\ \{[V_t^{(1)}(t,\ x(t)+\mathscr{A}(u)V^{(1)}(t,\ x(t))]\ [V^{(2)}(t,\ x(t))-J_2^g]+[V_t^{(2)}(t,\ x(t))$$

$$+\mathscr{A}(u)V^{(2)}(t,\ x(t))]\ [V^{(1)}(t,\ x(t))-J_1^g]+[V_x^{(1)}(t,\ x(t))]'\ g(t,\ x(t),$$

$$\times u,\ u_2)g'\ (t,\ x(t),\ u_1,\ u_2)V_x^{(2)}\ (t.\ x(t))\}\ dt\}.$$

Now condition (b) and the Remark to the Lemma imply that

$$J_1(u^{nb})\ J_2(u^{nb})-J_1^g\ J_2(u^{nb})-J_2^g\ J_1(u^{nb})\geqq J_1(u)J_2(u)-J_1^g\ J_2(u)-J_2^g\ J_1(u).$$

Therefore for arbitrary $u=(u_1,\ u_2)$

$$[J_1^g-J_1(u^{nb})]\ [J_2^g-J_2(u^{nb})]\geqq[J_1^g-J_1(u)]\ [J_2^g-J_2(u)].$$

The proof of the Theorem is completed.

Remark. The problem of existence of the Nash-bargaining solutions has been considered in a separate paper and it will be published independently.

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