Serdica
Bulgariacae mathematicae publicationes

Сердика
Българско математическо списание

MAPPING PROPERTIES AND COMPOSITION STRUCTURE OF CONVOLUTION TRANSFORMS

HANS-JÜRGEN GLAESKE, VU KIM TUAN

The *factorization* of integral transforms into a product of well known investigated factors, as the *Fourier*, the *Laplace* or the *Mellin transform*, is of interest not only from a theoretical point of view. Also the possibility of the use of well tabulated transforms is important for the applications. Such factorizations were given in [5] for a lot of *transforms of the Mellin convolution* type. In [11], [12] such factorizations were used to investigate the mapping properties of integral transforms of this type.

1. Introduction. The aim of this paper is to study such mapping properties for *transformations of the type of the Fourier convolution.* Such a transform is of the form:

$$(Kf)(x)=\int_{-\infty}^{\infty} k(x-y)\,f(y)\,dy, \tag{1.1}$$

where k is a given kernel function and f — an original from a suitable space of ordinary or generalized functions.

In these considerations the Fourier transform

$$\hat{f}(x)=(Ff)(x)=(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ixy} f(y)\,dy \tag{1.2}$$

is very useful. Here the integral has to be understood in different senses: in the sense of L_1- or L_2-convergence, as Cauchy's principal value or in the sense of the bounded convergence, i. e.: if the integral exists and for each $u, v>0$ there exists a constant $C>0$ such that

$$\Big|\int_{-u}^{v} e^{ixy} f(y)\,dy\Big|<C. \tag{1.3}$$

The inversion formula (under assigned conditions) has the form

$$f(x)=(F^{-1}\hat{f})(x)=(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ixy} f(y)\,dy. \tag{1.4}$$

It will be proved that this transformation is an *automorphism*, i. e. a bijective and bicontinuous mapping of the *Wiener ring* L^* (section 3), an *unitary transformation* of L_2 (section 4) and an *automorphism* of the *space of tempered distributions* S', the *Gelfand-Shilov-space of generalized functions* Z' and of the *space of Bessel potentials* L_2^r (section 5). Section 2 contains some preliminary definitions and a lemma and in section 6 some examples are discussed.

2. Preliminaries. Following Akhiezer ([2], 76) we denote by $L^*(R)=L^*$ the Wiener normed ring of all functions f which are Fourier transforms (1.2) of some functions $f\in L_1(R)=L_1$ with the norm

(2.1) $$\|\widehat{f}\|_{L^*}=\|f\|_1,$$

(see also [7]). We would like to remark, that because of the "symmetry" of the formulas (1.2), (1.3) sometimes one also is putting the elements $f\in L^*$ in the form $f=F^{-1}\widehat{f}$, $\widehat{f}\in L_1$.

D e f i n i t i o n 2.1. *Let H be the class of all locally integrable functions $k:R\to C$ such that the Fourier integral (1.2) of k is bounded convergent to the limit $\widehat{k}$. Then k, $h\in H$ are called conjugate kernels of the subclass $H^*\subset H$ iff their Fourier thansforms $\widehat{k}$, $\widehat{h}$ satisfy the equation*

(2.2) $$\widehat{k},\ \widehat{h}=(2\pi)^{-1}.$$

Then we have

L e m m a 2.1. *Let $k\in H$, $f\in L^*$,* i. e. *$f=F^{-1}\widehat{f}$, $\widehat{f}\in L_1$. Then a. e. we have*

(2.3) $$\int_{-\infty}^{\infty} k(y)\,f(x-y)\,dy=\int_{-\infty}^{\infty} e^{-ix-t}\,k(t)\,\widehat{f}(t)\,dt,\ x\in R,$$

i. e.

(2.3′) $$(Kf)(x)=(2\pi)^{1/2}(F^{-1}\,\widehat{k}\,\widehat{f})(x),$$

where the integral on the left hand side of (2.3) converges as an improper integral' while the integral on the right hand side is absolutely convergent.

P r o o f. Pure formally, we have the following chain of equations:

$$\lim_{u,\,v\to\infty}\int_{-u}^{v} k(y)\,f(x-y)\,dy=\lim_{u,\,v\to\infty}(2\pi)^{-1/2}\int_{-u}^{v} k(y)\int_{-\infty}^{\infty} e^{i(y-x)t}\,\widehat{f}(t)\,dtdy$$

$$=\lim_{u,\,v\to\infty}(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ixt}\,\widehat{f}(t)\int_{-u}^{v} e^{ity}\,k(y)\,dydt=\int_{-\infty}^{\infty} e^{-ixt}\,\widehat{f}(t)\lim_{u,\,v\to\infty}(2\pi)^{-1/2}\int_{-\infty}^{v} e^{ity}\,k(y)\,dydt$$

$$=\int_{-\infty}^{\infty} e^{-ixt}\,\widehat{k}(t)\,\widehat{f}(t)dt.$$

The interchanging of the order of integration may be justified by means of the absolute convergence of the double integral on the right hand side ($\widehat{f}\in L_1$!), the interchanging of the integration and the limit by means of the assumption that $k\in H$, such that $\int_{-u}^{v} e^{ity}\,k(y)\,dy$ is uniformly bounded.

From the bounded convergence of the Fourier integral of k we know that $\widehat{k}$ is bounded and therefore $\widehat{k}\,\widehat{f}\in L_1$. By virtue of the *Riemann-Lebesgue Lemma* for the Fourier transform in L_1 we have

C o r o l l a r y 2.1. *Let $k\in H$, $f\in L^*$. Then $Kf\in L^*$ and*

(2.4) $$\lim_{x\to\pm\infty}(Kf)(x)=0.$$

3. Mapping Properties of L^*. Now we are going to prove some mapping properties of the convolution transform in some functional spaces.

T h e o r e m 3.1. *Let k,h be conjugate kernels, k, $h\in H^*$, then the convolution transform (1.1) is an automorphism of the space L^*. Furthermore, we have the inversion formula*

(3.1) $$f(x)=\int_{-\infty}^{\infty} h(x-y)\,g(y)\,dy,$$

i. e. the inversion formula has the same form as the transform (1.1) with the kernel instead of k.

If furthermore a. e. $|\widehat{k}(t)|=(2\pi)^{-1/2}$, then this transform is an isometrical one.

Proof. The statements of this theorem follow easily from (2.3), (2.3').

As an example we are going to consider the *Hilbert transform*

$$g(x)=\pi^{-1}\int_{-\infty}^{\infty} y^{-1} f(x+y)\,dy. \tag{3.2}$$

Obviously $k(x)=x^{-1}$ does not belong to H, but if Fk is understood in the sense of the principal value of Cauchy at the point zero as well as at $\pm\infty$, then Fk is bounded convergent. Dividing (3.1) into two parts (from $-\infty$ to -0 and from $+0$ to $+\infty$) after a substitution of the variable in (3.1), we have the following

Corollary 3.1. *The Hilbert transform*

$$g(x)=\lim_{u,\, v\to\infty} \pi^{-1}\int_{u-1}^{v} y^{-1}\,[f(x+y)-f(x-y)]\,dy$$

is an isometrical automorphism of L^*.

Now we consider a transform with a differentiable kernel:

Theorem 3.2. *Let φ: $R\to R$ be an odd function. Furthermore there exist φ'' and $\lim_{t\to\infty}\varphi'(t)=\pm\infty$, $|\varphi''(t)|\geq\lambda>0$ for some interval (a,∞) and $\varphi''/(\varphi')^2\in L_1(a,\infty)$. Set*

$$h(x)=(2\pi)^{-1}\int_{-\infty}^{\infty} \exp\,[i(xt+\varphi(t))]\,dt \tag{3.3}$$

and assume that $h\in H$, then the integral transform

$$g(x)=(Hf)(x)=\int_{-\infty}^{\infty} h(x+y)\,f(y)\,dy \tag{3.4}$$

(where the integral has to be understood in the sense of the principal value of Cauchy) is an isometrical automorphism of L^* and the inversion formula has a symmetrical form.

Proof. From [10], Th. 11 and from the assumptions of Th. 3.2. it follows that the integral (3.3) is convergent and moreover

$$\int_{-\infty}^{\infty} h(x)\,e^{-ixt}\,dx=e^{-i\varphi(t)}. \tag{3.5}$$

If $f\in L^*$, $h\in H$, then from (3.5) and Lemma 2.1 it follows

$$(Hf)(x)=\int_{-\infty}^{\infty} h(x+y)\,f(y)\,dy=(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-i\varphi(t)}\,\widehat{f}(-t)\,e^{-ixt}\,dt, \tag{3.6}$$

i. e.

$$\widehat{g}(t)=e^{-i\varphi(t)}\,\widehat{f}(-t). \tag{3.7}$$

Therefore $\widehat{f}$ and $\widehat{g}$ simultaneously belong to L_1 or not and hence f and g also simultaneously belong to L^* or not.

Furthermore, from

$$\|g\|_{L^*}=\|\widehat{g}\|_1=\|\widehat{f}\|_1=\|f\|_{L^*}$$

we see that the transform is continuous and isometrical. Since from (3.7) (recalling that φ is an odd function) we obtain

(3.8) $$\widehat{f}(t)=e^{-i\varphi(t)}\,\widehat{g}(-t),$$

the inversion formula of (3.4) has a symmetrical form. Thus the theorem is proved.

Theorem 3.3. *Let f_1, $f_2\in L^*$, $\widehat{f}_1\in L_\infty$. and k_1, k_2 be a couple of conjugate kernels. Then for the convolution transforms $(K_1 f_1)$, $(K_2 f_2)$ the Parseval's equality*

(3.9) $$\int_{-\infty}^{\infty}(K_1 f_1)(y)\,(K_2 f_2)\,(x-y)\,dy=\int_{-\infty}^{\infty} f_1(y)\,f_2(x-y)\,dy$$

is valid.

Proof. Obviously it holds

$$\int_{-u}^{u} e^{-ixy} f_1(y)\,dy=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\widehat{f}_1(t)\int_{-u}^{u} e^{-iy(t+x)}\,dy\,dt=(2/\pi)^{-1/2}\int_{-\infty}^{\infty}\widehat{f}_1(t-x)\,t^{-1}\ \sin\ ut\,dt.$$

Since $\widehat{f}_1\in L_1\cap L_\infty$ the latter integral converges boundedly, i. e. $f_1\in H$. From $\widehat{k}_1\widehat{f}_1\in L_1\cap L_\infty$ we conclude analogously that

$$\int_{-\infty}^{\infty} e^{-ixt}\,\widehat{k}_1(t)\,\widehat{f}_1(t)\,dt\in H.$$

Since f_2, $(K_2 f_2)\in L^*$ by means of the result (2.3) we get

$$\int_{-\infty}^{\infty} f_1(y)\,f_2(x-y)\,dy=\int_{-\infty}^{\infty} e^{-ixt}\,\widehat{f}_1(t)\,f_2(t)\,dt$$

and by virtue of (2.2)

$$\int_{-\infty}^{\infty}(K_1 f_1)(y)\,(K_2 f_2)(x-y)\,dt=2\pi\int_{-\infty}^{\infty} e^{-ixt}\,\widehat{f}_1(t)\,\widehat{k}_1(t)\,\widehat{f}_2(t)\,\widehat{k}_2(t)\,dt=\int_{-\infty}^{\infty} e^{-ixt}\,\widehat{f}_1(t)\,\widehat{f}_2(t)\,dt.$$

Comparing these two formulas we have (3.9).

Finally we are going to consider as an application the solution of the integral equation

(3.10) $$f+(2\pi)^{-1/2}\,(Kf)=g.$$

Theorem 3.4. *Lem $g\in L^*$, $k\in L$ such that $\widehat{k}(t)\neq -1$, $t\in R$. Then the integral equation (3.10) has a unique solution $f\in L^*$. This solution is of the form*

(3.11) $$f=g-(2\pi)^{-1/2}\,(\widetilde{K}g),$$

where $\widetilde{K}$ is the transformation (1.1) with the kernel

(3.12) $$\widetilde{k}(x)=(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ixt}\ \frac{\widehat{k}(t)}{\widehat{k}(t)+1}\ dt.$$

Proof. Applying the Fourier transform to (3.10) we have by means of the convolution theorem

$$\widehat{f}+\widehat{k}\,\widehat{f}=\widehat{g},$$

i. e.

(3.13) $$\widehat{f}=\widehat{g}-\frac{\widehat{k}}{\widehat{k}+1}\ \widehat{g}.$$

Now from $k \in L$ we have $\widehat{k} \in L^*$. By virtue of *Wiener-Levi's theorem* (see [2], 75) it follows that $\widehat{k}/(\widehat{k}+1) \in L^*$ iff $\widehat{k}(t) \neq -1$. Applying the inverse Fourier transform to (3.13) we arrive at (3.11), (3,12).

4. Mapping properties of L_2. Now we would like to transfer the Theorem 3.2 to the space $L_2(R)=L_2$. The result is given by the following theorem.

Theorem 4.1. Let φ fulfil the conditions of Theorem 3.2 and h be defined by (3.3). Then the integral transform

$$g(x)=\underset{u,\,v\to\infty}{\text{l. i. m.}} \int_{-u}^{v} h(x+y)\, f(y)\, dy \tag{4.1}$$

is an unitary transform of L_2 and its inversion formula has a symmetrical form. Furthermore we have the factorization

$$g=Fe^{i\varphi}\, Ff. \tag{4.2}$$

Remark. Since the inversion formula is symmetrical we have

$$f=Fe^{i\varphi}\, Fg. \tag{4.2'}$$

Proof of Theorem 4.1. Under the conditions of the theorem the integra (3.3) is uniformly convergent on $[-u, v]$. Therefore

$$g(x)=\underset{u,\,v\to\infty}{\text{l. i. m.}}\, (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ixt+i\varphi(t)} \int_{-u}^{v} e^{iyt} f(y)\, dydt. \tag{4.3}$$

From $\int_{-u}^{v} e^{iyt} f(y)\, dy \in L_2$ we conclude that the integral of the right-hand side of (4.3) converges not only as an improper integral, but in the square mean sense too and both of the limits are identical. Therefore

$$g(x)=\underset{u,\,v\to\infty}{\text{l. i. m.}}\ \underset{w\to\infty}{\text{l. i. m.}}\, (2\pi)^{-1} \int_{-w}^{w} e^{ixt+i\varphi(t)} \int_{-u}^{v} e^{iyt} f(y)\, dydt$$

$$=\underset{u,\,v\to\infty}{\text{l. i. m.}}\, (2\pi)^{-1/2}\, F\Big(e^{i\varphi(t)} \int_{-u}^{v} e^{iyt} f(y)\, dy\Big)(x)=F\Big(e^{i\varphi(t)}\, \underset{u,\,v\to\infty}{\text{l. i. m.}}\, (2\pi)^{-1/2} \int_{-u}^{v} e^{iyt} f(y)\, dy\Big)(x)$$

(because of the continuity of the Fourier transform in L_2), i. e. (4.2). By inversion we get (4.2′):

$$f=F^{-1}\, e^{-i\varphi}\, F^{-1}\, g=Fe^{i\varphi}\, Fg,$$

i. e. the inversion formula has a symmetrical form. Because of the unitarity of the Fourier transform in L_2 from (4.2) we have finally

$$\|g\|_2=\|e^{i\varphi}\, Ff\|_2=\|Ff\|_2=\|f\|_2,$$

hence (4.1) is an unitary transformation of L_2.

5. Mapping Properties in Space of Generalized Functions. As usual, let Θ_M be the linear space of tempered C^∞-functions, i. e. the space of functions of $C^\infty(R)$, which together with their derivatives are increasing as a power of $|x|$ as $|x|\to\infty$. It is well known, that the elements of Θ_M are multipliers in the space S of rapidly decreasing functions. Then we have

Theorem 5.1. *Let $\varphi \in \Theta_M$ fulfil the conditions of Theorem 3.2. Then the transformation (3.4) is an automorphism in the space S of test functions.*

Proof. Since $S \subset L_2$ it follows, that the transform (3.4) has the factorization (4.2). Using the facts that the Fourier transformation gives an automorphism of S and that

$e^{\pm i\varphi}$ is a multiplier in S, we have the statement of the theorem. According to Theorem 5.1 one can define the H-transform in S', the space of tempered distributions, by means of the method of adjoints:

$$\langle Hf, \psi\rangle = \langle f, H, \psi\rangle, \; f \in S', \; \psi \in S. \tag{5.1}$$

Thus we have

Corollary 5.1. *Let φ fulfil the conditions of Theorem 5.1. Then the transform H defined by (5.1), (3.4) is an automorphism of S'.*

Similarly we can consider the case of the Gelfand Shilov-space Z of entire functions. Since $Z \subset L_2$ we have again the factorization (4.2). The Fourier transform of $\psi \in Z$ belongs to the Schwartz space D of finite C^∞-functions (see for example [4], 8.27 — 8.33) and therefore $e^{i\varphi}$ is a multiplier in D. Since the Fourier transform is a homeomorphism of Z onto D, we have

Theorem 5.2. *Let $\varphi \in C^\infty$ fulfil the conditions of Theorem 3.2. Then the transformation (3.4) is an automorphism of Z. Now let Z' be the space of continuous linear functionals on Z. Then by*

$$\langle Hf, \psi\rangle = \langle f, H\psi\rangle, \; f \in Z', \; \psi \in Z \tag{5.2}$$

one can define the transform H on Z' and this yields

Corollary 5.2. *Let φ fulfil the conditions of Theorem 5.2. Then the transformation H defined by (5.2), (3.4) is an automorphism of Z'.*

Finally let us consider the space of Bessel potentials $L_2^r(R) = L_2^r$, i. e.

$$L_2^r = \{f \in S' : (1+x^2)^{r/2} \hat{f} \in L_2\}, \quad r \geqq 0, \tag{5.3}$$

with the norm

$$\|f\|_{L_2^r} = \|(1+x^2)^{r/2} \hat{f}\|^2, \tag{5.4}$$

see [9], V, § 3.

For this space we have

Theorem 5.3. *The H-transform (5.1) is unitary in the space L_2^r.*

Proof. First of all we would like to remark, that factorization (4.2) is valid also for elements of S'. So as $f \in L_2^r$ we have $(1+x^2)^{r/2} \hat{f} \in L_2$ or $e^{i\varphi(x)}(1+x^2)^{r/2} f \in L_2$ and this is equivalent to $g = Fe^{i\varphi} Ff \in L_2^r$. Moreover,

$$\|g\|_{L_2^r} = \|e^{i\varphi(x)} (1+x^2)^{r/2} \hat{f}(x)\|_2 = \|(1+x^2)^{r/2} \hat{f}(x)\|_2 = \|f\|_{L_2^r},$$

i. e., according to (5.1) the transform H is unitary in L_2^r.

6. Examples. Special cases of the transforms (4.1), respectively (5.1), are the transforms with kernels h of the following types:

$$h_1(x) = Ai(x), \tag{6.1}$$

where Ai is the *Airy function*. From [1], 10.4.32 we know that 6.1 has a representation of the form (3.3) with the function

$$\varphi_1(t) = t^3/3. \tag{6.1'}$$

The function φ_1 satisfies all conditions stated for the function φ in Th. 3.2. Moreover, $h_1(x)$ has the asymptotic behaviour (see [1], 10.4.59, 10.4.60)

$$Ai(x)=\frac{1}{2\sqrt{\pi}}\ x^{-1/4}\exp\left(-\frac{2}{3}\ x^{3/2}\right)[1+0(x^{-3/2})],\quad x\to\infty \tag{6.2}$$
$$Ai(-x)=\frac{1}{\sqrt{\pi}}\ x^{-1/4}\left[\sin\left(\frac{2}{3}\ x^{3/2}+\frac{\pi}{4}\right)+0\ (x^{-3/2})\right],\quad x\to\infty.$$

Hence it follows that $\int\limits_{-\infty}^{\infty} Ai(x)\ e^{ixt}\,dx$ is boundedly convergent, i. e. $h_1\in H$.

Further, we are going to consider an integral transform of the index type introduced by Crum [6]. Here

$$h_2(x)=\pi^{-1}\ e^{\pi x/2}\ K_{ix}(a),\quad a>0, \tag{6.3}$$

where K_ν is the *McDonald function* (see [1], 9.6). From [8], 2.4.18, 13. we know that $h_2(x)$ has an integral representation of the form (3.3) with

$$\varphi_2(t)=a\,sh\,t,\quad a>0. \tag{6.3'}$$

The function φ_2 also satisfies the conditions stated for the function φ in Th. 3.2. Moreover, from [3], 7.13.2, (19) we have

$$e^{\pi x/2}\ K_{ix}(a)=\sqrt{\frac{2\pi}{|x|}}\ e^{-\pi|x|}\ [1+0\,(|x|^{-1})],\quad x\to-\infty \tag{6.4}$$
$$e^{\pi x/2}\ K_{ix}(a)=\sqrt{\frac{2\pi}{x}}\ \sin\left(x\log\frac{2x}{x}-x+\pi/4\right)+0\ (x^{-3/2}),\quad x\to+\infty.$$

Hence the integral $\int\limits_{-\infty}^{\infty} e^{\pi x/2}\ K_{ix}(a)\ e^{ixt}\,dx$ is boundedly convergent, i. e. $h_2\in H$.

Therefore the functions $\varphi_j, h_j, j=1, 2$, satisfy the conditions of Th. 3.2 and Th. 4.1 so that the corresponding transforms H_1 and H_2 are automorphisms of L^*, unitary in L_2 and the inversion formulas have a symmetrical form.

Since $\varphi_1\in\Theta_M$, $\varphi_2\in C^\infty$ one can apply Theorems 5.1 and 5.2 respectively and we obtain that H_1 is an automorphism of S' and of Z' and that H_2 is an automorphism of Z'. By means of simple modifications of the considerations above one can derive similar results for a transform of the index type with a Hankel function in the kernel. We will give these results without proofs.

Theorem 6.1. *The integral transform*

$$g(x)=(H_a^{(1)}f)(x)=\underset{u,\,v\to\infty}{\mathrm{l.\,i.\,m.}}\ \frac{1}{2}\int\limits_{-u}^{v} e^{\pi(y-x)/2}\ H_{i(x-y)}^{(1)}(a)\,f(y)\,dy,\quad a>0 \tag{6.5}$$

is unitary in L_2 and the corresponding inversion formula is

$$f(x)=(H_a^{(1)^{-1}}g)(x)=(H_a^{(2)}g)(x)=\underset{u,\,v\to\infty}{\mathrm{l.\,i.\,m.}}\ \frac{1}{2}\int\limits_{-u}^{v} e^{\pi(x-y)/2}\ H_{i(x-y)}^{2}\,(a)\,g(y)\,dy. \tag{6.5'}$$

Furthermore, we have the factorization

$$(H_a^{(1)}f)(x)=-iF^{-1}\ e^{iach\,x}\ Ff. \tag{6.6}$$

Analogously we have

Theorem 6.2. *The integral transforms*

$$(H_a^{(1)}f)(x)=\frac{1}{2}\int\limits_{-\infty}^{\infty} e^{\pi(y-x)/2}\ H_{i(x-y)}^{(1)}\,(a)\,f(y)\,dy,\quad a>0 \tag{6.7}$$

and

$$(H_a^{(2)} f)(x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{\pi(x-y)/2} H_{i(x-y)}^{(2)}(a) f(y)\, dy, \tag{6.7'}$$

where the integrals have to be understood in the sense of the principal value of Cauchy, are automorphisms in the space L^* and each of them is the inversion of the other.

Furthermore, one can show, that these transforms are automorphisms in the space Z too, so that by the usual definition (5.2) we can conclude, that their generalizations to Z' are automorphisms of Z'.

REFERENCES

1. M. Abramovitz, I. A. Stegun. Handbook of mathematical functions. N. Y., 1964.
2. N. I. Akhiezer. Vorlesungen uber Approximationstheorie, 2nd edition. Berlin, 1967.
3. H. Bateman, A. Erdelyi. Higher transcendental functions. Vol. II. London, 1953.
4. H. Bremermann. Distributions, Complex Variables and Fourier Transforms. Reading, Ma., 1965.
5. Yu. A. Brychkov, H.-J. Glaeske, O. I. Marichev. Factorization of integral transforms of the convolution type. *Itogi Nauki i Techniki. Mat. Analiz.*, VI NITI, Moskva, **21**, 1983, 3—41 (*J. Soviet Math.*, **30**, 1985).
6. M. M. Crum. Some inversion formulae. *Quart. J. Math.*, **11**, 1940, 49—52.
7. M. A. Neumark. Normierte Algebren. Berlin, 1958.
8. A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev. Integrals and Series. Vol. I. Elementary Functions, New York, 1986.
9. E. M. Stein. Singular integrals and differentiability properties of functions. Princeton, 1970.
10. E. C. Titchmarsh. Introduction to the theory of Fourier integrals. Oxford, 1937.
11. Vu Kim Tuan. K teorii obobshchennikh integralnykh preobrazovanii v nekotorom prostranstve funktzhii. *Dokl. Akad. Nauk SSSR*, **286**, 1986, 521—524 (Soviet Math. Dokl., **33**, 1986, 103—106).
12. Vu Kim Tuan, O. I. Marichev, S. B. Jakubovich. Kompositzhionnaya structura integralnykh preobrazovanii. *Dokl. Akad. Nauk SSSR*, **286**, 1986, 786—790 (*Soviet Math. Dokl.*, **33**, 1986, 166—170).

Friedrich-Schiller-Universitat
Sektion Mathematik
UHH
6900 Jena, GDR

Hanoi, Vietnam
106 Tran Hung Dao

Received 20. 06. 1989