# Serdica
## Bulgariacae mathematicae publicationes

# Сердика
## Българско математическо списание

# FINDING THE ZEROS OF EXPONENTIAL QUASI-POLYNOMIALS

TATIANA I. ENCHEVA, ANATOLY YU. LEVIN

An algoriyhm for determining the sign-change points of an exponential quasi-polinomial is considered. Its computing complexity is discussed. An estimate for the number of odd-multiple zeros obtained by the help of the discussed algorithm is also represented.

**1. Introduction.** Consider the question of finding the sign-change points of the real quasi-polynomial

$$(1) \qquad f(t) = \sum_{j=1}^{n} c_j \exp(\lambda_j t)$$

in the interval $(a, b)$. Among $c_j, \lambda_j$ $(j=1, 2, \ldots, n)$ could be found non-real numbers acting as complex-conjugate couples. Without loss of generality let us assume that all the coefficients $c_j$ are not equal to zero and that all the coefficients $\lambda_j$ are different. Furthermore, since for $n=2$ the problem is trivial, let us suppose that $n \geq 3$.

The importance of the given problem might be determined, for example, by the fact that at each iteration of the algorithm for synthesizing time-optimal control in a linear system proposed in [1] one has to solve such problems. Evidently the problem considered is of some interest itself.

Taking into account the availability of some well known and quite fast procedures for finding the localized zeros of a smooth function (see e. g. [2]), it is easy to understand that the problem discussed is actually reduced to the localization of the zeros of $f(t)$. By the localization of the zero $t_0$ of the function $f(t)$ we, as usual, mean the determination of the interval $(t_0', t_0'')$, at the ends of which $f(t)$ is opposite in sign and $f(t)$ vanishes only at the point $t_0$ within this interval. It is clear, that the question of localization is meaningful only for the zeros of odd multiplicity. (Besides, we are interested only in them for the linear time-optimal control problem mentioned above, because the even-multiple zero is not accompanied by a sign change of $f(t)$ and hence it is not a switching point of the control.)

There is a vast literature on the question of the determination of the zeros of a smooth function. The methods for "refining" the zero, i. e. for calculating approximately the zero for which an initial approach is known in advance have been covered entirely. A classical method of such type is the Newton method (see e. g. [3, 4]). Its advantages are the quite general assumptions for the domain of definition of a function (that could be any Banach space) and the fast convergence. However, this method can be used only if we have a qualitative enough initial approach of the searched zero. Computational procedures related to the Newton method have been considered by many authors. From the recent papers we may mention [5]. We do not dwell on the commonly known computational schemes for the scalar case, such as the chord method, the chord and tangent method, the bisection method, etc. since they also find a localized (in some sense) zero and do not concern the question about how to localize the zero itself.

On the other hand, some methods for finding zeros of polynomials without preliminary localization are known. Here we can mention the classical Lobachevsky method [6], the parabola method [7], etc. Note that these methods find not only real, but also complex zeros. The convergence of some algorithms of this type has been confirmed only empirically.

The specificity of our situation is characterized by the following conditions: 1) $f(t)$ is a quasi-polynomial; 2) we are interested only in real zeros (of odd multiplicity) of $f(t)$; 3) no initial approaches to the searched zeros are given in advance and moreover their number is unknown. Apparently, in this respect the question of finding (or, which is actually the same, of localizing) the zeros of $f(t)$ has not been studied.

**2. Algorithm.** The approach proposed in [8] uses the specific properties of quasi-polynomials; moreover, no preliminary information about the number or disposition of the zeros is required. For the sake of simplicity we suppose that $a=0$, $b=T$, where $T$ is a finite positive number (as for the case $T=\infty$ see [8].)

**2.1. Real case.** Let $\lambda_j$, $c_j$ $(j=1, 2, \ldots, n)$ be real, $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and let us suppose that not all the coefficients $c_j$ have one and the same sign. As it is known (see e. g. [9], p. 117) the quasi-polynomial (1) has at most $n-1$ zeros. More precisely, according to Descartes generalized rule (see [10], p. 60) the number of zeros of $f$ does not exceed the number of the sign changes in the sequence $c_1, c_2, \ldots, c_n$.

Let $p (1 \leq p \leq n)$ be the minimal index such that there is exactly one sign change in the sequence $c_p, c_{p+1}, \ldots, c_n$. Consider the following function system

$$(2) \qquad f(t) = f_1(t), \ f_2(t), \ldots, f_p(t),$$

where

$$f_k = \frac{df_{k-1}}{dt} - \lambda_{k-1} f_{k-1}$$

$(k=2, 3, \ldots, p)$. By denoting $\gamma_{kj} = (\lambda_j - \lambda_1)(\lambda_j - \lambda_2) \ldots (\lambda_j - \lambda_{k-1})$, $(k=2, 3, \ldots p; j=k, k+1, \ldots, n)$ we can write the system (2) in the form

$$f_1(t) = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t) + \cdots + c_n \exp(\lambda_n t)$$
$$f_2(t) = c_2 \gamma_{22} \exp(\lambda_2 t) + c_3 \gamma_{23} \exp(\lambda_3 t) + \cdots + c_n \gamma_{2n} \exp(\lambda_n t)$$
$$\vdots$$
$$f_p(t) = c_p \gamma_{pp} \exp(\lambda_p t) + c_{p+1} \gamma_{p,p+1} \exp(\lambda_{p+1} t) + \cdots + c_n \gamma_{pn} \exp(\lambda_n t).$$

Since all $\gamma_{kj} > 0$, the number of zeros of each function $f_k$ does not exceed the number of sign changes in the sequence $c_k, c_{k+1}, \ldots, c_n$. Therefore $f_p$ has not more than one zero (in $(-\infty, +\infty)$) and in the case $f_p(0)f_p(T) < 0$ the interval $(0, T)$ is a localization interval for this zero. The latter can be found by means of one of the classical procedures for determining localized zeros.

The remaining part of the algorithm consists of finding the sing-change points of the functions $f_{p-1}, f_{p-2}$, etc. up to $f_1 = f$ consecutively. We describe briefly the general iteration of the algorithm.

Suppose that all the sign-change points $t_1^k, t_2^k, \ldots, t_{m_k}^k$ of the function $f_k$ $(2 \leq k \leq p)$ have already been obtained. Consider the intervals

$$(0, t_1^k) = (t_0^k, t_1^k), \ (t_1^k, t_2^k), \ldots, (t_{m_k}^k, t_{m_k+1}^k) = (t_{m_k}^k, T)$$

(If $m_k = 0$ this system is reduced to the unique interval $(0, T)$.) Compute the values

$$f_{k-1}(0) = f_{k-1}(t_0^k), f_{k-1}(t_1^k), \ldots, f_{k-1}(t_{m_k}^k), f_{k-1}(t_{m_k+1}^k) = f_{k-1}(T).$$

By using the concepts related to the monotonicity and the Rolle theorem we conclude that each of the intervals $(t_l^k, t_{l+1}^k)$, $0 \leq l \leq m_k$, is a localization interval for one of the odd-multiple zeros of $f_{k-1}$ if and only if $f_{k-1}(t_l^k) f_{k-1}(t_{l+1}^k) < 0$.

Having obtained the localization intervals of all the sign-change points of $f_{k-1}$ in $(0, T)$, we use again one of the known methods for the determination of these points. Then similarly we find the sign-change points of $f_{k-2}$, etc. until we reach $f_1 = f$. Thus, the algorithm can be treated as shifting the localization processes by the classical procedures for computing the localized zeros. The following statement refers to the computing complexity of the described algorithm.

Theorem I. *For finding all the sign-change points of the quasi-polynomial* (1) *with real* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *in a finite interval* $(a, b)$ *to within* $\varepsilon$ *(* $\varepsilon < (b-a)/2$ *) the following number of operations is sufficient*

$$Cn^3 \ln((b-a)/\varepsilon),$$

*where C is a constant.*

Proof. It is easy to see that the computing complexity of the algorithm presented is a sum of two components. The first component does not depend on the desired precision $\varepsilon$ and, as it is easy to calculate, it does not exceed $C_1 n^3$, where $C_1$ is a constant. Indeed, the construction of the function system (2) requires arithmetic operations on $n^2$ order. The number of the operations needed for computing the value of the quasi-polynomial $f(t)$ at a single point does not surpass $C_2 n$ (assuming that the exponent is calculated for a finite number of operations). Hence, the values of the functions (2) at the corresponding points are computed for the operations of $n^3$ order. Finally, the number of operations needed for determining the sign changes in the corresponding sequences is of $n^2$ order.

The second component is connected with: 1) the computing complexity $D$ of the procedure used for finding (to within $\varepsilon$) a localized zero and 2) the number of references to this procedure, which as it is easy to check, does not surpass $n^2/2$. In particular, if the bisection method is used and if $l (\leq (b-a))$ is the length of the localization interval, then

$$D \leq C_2 n \log_2(l/\varepsilon) \leq C_2 n \log_2((b-a)/\varepsilon).$$

Naturally, we suppose that $\varepsilon < l/2$. (Otherwise, there is no need to use the procedure.)

Thus, the total computing complexity of the algorithm does not exceed

$$C_1 n^3 + C_3 n^3 \log_2((b-a)/\varepsilon) \leq (C_1 + C_3) n^3 \log_2((b-a)/\varepsilon),$$

what we had to show.

Remark 1. We have considered the interval $(a, b) = (0, T)$. It is clear that the use of any finite interval $(a, b)$ does not influence on the correctness of the estimate. The case of the infinite interval will be reduced to the case of the finite one if we use the upper bound $t_{max}$ for the zeros of the quasi-polynomial [8] and its analogous lower bound $t_{min}$. To obtain $t_{min}$ we may use, for example, the upper estimate $\tilde{t}_{max}$ for the quasi-polynomial $\tilde{f}(t) = f(-t)$.

Remark 2. To prove theorem 1 it has been enough to restrict ourselves to the bisection method, which guarantees a geometrical range of the contraction of the localization interval. Other procedures applicable to refining localized zeros (the Newton method, the chord method, etc.) guarantee still more rapid (superlinear) convergence. We would not consider this question in details for two reasons. First, the notion of the localization interval complicates (some conditions related to the signs of of the $f$ derivatives should be defined). Second, the estimate would be improved, approximately speaking, through replacing $\ln(1/\varepsilon)$ by $\ln\ln(1/\varepsilon)$; in the range of $\varepsilon$ used in practice ($10^{-5} - 10^{-6}$) this fact is not of essential importance.

Now let us compare the algorithm described and the trivial way of determining (to within $\varepsilon$) the sign-change points of the quasi-polynomial $f$ in the interval $(a, b)$ (namely, the consecutive calculation of the values of $f$ at the points $a$, $a+2\varepsilon$, $a+4\varepsilon$, etc. up to $b$) with respect to their computational complexity. The second method requires $C'n(b-a)/\varepsilon$ operations, where $C'$ is a constant. Thus, with the decrease of $\varepsilon$ the computational complexity increases more rapidly than that of the algorithm described above, where the dependence on $\varepsilon$ is of logarithmic nature. On the other hand, in our algorithm with the increase of $n$ the computing complexity increases more rapidly (proportionally to $n^3$ instead of $n$). However, note that in the linear time-optimal control problems mentioned above this fact is not so important, since $n$, as a rule, is not great. It is clear that the computing complexity of the trivial algorithm could be lowered by increasing $\varepsilon$, but then it becomes possible to omit some sign changes of $f$, which is avoided in our algorithm.

**2.2. Non-real case.** For the sake of clarity we assume that $f$ has $m>0$ different couples $\lambda_j=\rho_j+i\omega_j$, $\lambda_{m+j}=\rho_j-i\omega_j$, $\omega_j>0$ $(j=1, 2, \ldots, m)$ of complex conjugate coefficients and $n-2m\geq0$ different $\lambda_j$ $(j=2m+1, \ldots, n)$ real coefficients. (Recall, that in this case $c_j$, $c_{m+j}$ $(j=1, 2, \ldots, m)$ are complex conjugate, $c_j$ $(j=2m+1, \ldots, n)$ are real.) Consider the function system

(3) $$f(t)=f_1(t), f_2(t), \ldots, f_{m+1}(t),$$

where

(4) $$f_k=(\frac{d^2}{dt^2}+\omega_{k-1}^2)\,g_{k-1}$$

(5) $$g_{k-1}=\exp((\rho_{k-2}-\rho_{k-1})\,t)\,f_{k-1}$$

$(k=2, 3, \ldots, m+1; \rho_0=0)$. System (3) can be written in the form

$$f_1(t) = \sum_{j=1}^{m}(c_j^1\sin\omega_j t+c_{m+j}^1\cos\omega_j t)\exp((\rho_j-\rho_0)\,t) + \sum_{j=2m+1}^{n}c_j^1\exp((\lambda_j-\rho_0)\,t)$$

$$f_2(t) = \sum_{j=2}^{m}(c_j^2\sin\omega_j t+c_{m+j}^2\cos\omega_j t)\exp((\rho_j-\rho_1)\,t) + \sum_{j=2m+1}^{n}c_j^2\exp((\lambda_j-\rho_1)\,t)$$

$$\vdots$$

$$f_m(t)=(c_m^m\sin\omega_m t+c_{2m}^m\cos\omega_m t)\exp((\rho_m-\rho_{m-1})\,t) + \sum_{j=2m+1}^{n}c_j^m\exp((\lambda_j-\rho_{m-1})t)$$

$$f_{m+1}(t) = \sum_{j=2m+1}^{n}c_j^{m+1}\exp(\lambda_j-\rho_m)\,t).$$

Since in the quasi-polynomial $f_{m+1}$ all $c_j^{m+1}$ are real, all its sign changes in $(0, T)$ (evidently their number is not greater than $n-2m-1$) can be determined by the algorithm described in 2.1. (For $n=2m$ there are no real $\lambda_j$ and system (3) consists of $m$ functions. In this case the sign changes of $f_m$ would coincide with the sign changes of a function of the kind $\sin\omega_m(t-t_0)$, i. e. they can be written in an explicit form.)

The remaining part of the algorithm, similar to the algorithm in 2.1, consists of the consecutive determination of the sign-change points of the functions $f_m$, $f_{m-1}$, etc. up to $f_1=f$. Now let us show how if we know all the sign changes of $f_k$ in $(0, T)$, the sign changes of $f_{k-1}$ in $(0, T)$, $2\leq k\leq m+1$, can be localized and computed.

Consider the intervals

(6) $$\Delta_{k-1}^1 = (0, \frac{\pi}{\omega_{k-1}}),\ \Delta_{k-1}^2=(\frac{\pi}{\omega_{k-1}}, \frac{2\pi}{\omega_{k-1}}),\ldots,\Delta_{k-1}^{'k-1} =(\frac{(r_{k-1}-1)\pi}{\omega_{k-1}}, T),$$

where

$$r_{k-1} = \begin{cases} T\omega_{k-1}/\pi, & \text{if} \quad T\omega_{k-1}/\pi \text{ is integer} \\ [T\omega_{k-1}/\pi]+1, & \text{otherwise} \end{cases}$$

$(k = 2, 3, \ldots, m+1)$. The real factorization of the operator $\frac{d^2}{dt^2} + \omega_{k-1}^2$ (see [10], p. 128):

$$(7) \qquad (\frac{d^2}{dt^2} + \omega_{k-1}^2) g_{k-1} = \frac{1}{\sin \omega_{k-1} t} \frac{d}{dt} (\sin^2 \omega_{k-1} t \frac{d}{dt} \frac{g_{k-1}}{\sin \omega_{k-1} t})$$

$(l\pi/\omega_{k-1} < t < (l+1)\pi/\omega_{k-1}, l = 0, 1, \ldots)$ can be realized in each of the intervals (6). Since all the constructions for each of the intervals (6) are performed analogously, it is sufficient to consider only one of them, e. g. $\Delta = \Delta_{k-1}^l$.

Let $t_1^k, t_2^k, \ldots, t_{n_k}^k$ $(n_k \geq 0)$ be all the sign changes of the function $f_k$ in the interval $\Delta$. Denote by $l_{k-1}$ the number of sign changes of the function $f_{k-1}$ in the interval $\Delta$. With regard to (4) and (5), by virtue of the factorization (7) and the evident modification of the Rolle theorem $l_{k-1} \leq n_k + 2$. (Otherwise the function $f_k$ would have more than $n_k$ sign changes in $\Delta$.) Let

$$(8) \qquad \widetilde{f}_k(t) = \sin^2 \omega_{k-1} t \frac{d}{dt} \frac{g_{k-1}}{\sin \omega_{k-1} t}.$$

By virtue of (4) and (7) we have

$$(9) \qquad \frac{d\widetilde{f}_k}{d\widetilde{f}_k} = \sin \omega_{k-1} t\, f_k(t).$$

It follows from this, that in each of the intervals

$$(0, t_1^k), (t_1^k, t_2^k), \ldots, (t_{n_k}^k, \pi/\omega_{k-1})$$

the function $\widetilde{f}_k$ is strictly monotone. By computing the values

$$\widetilde{f}_k(0), \widetilde{f}_k(t_1^k), \ldots, \widetilde{f}_k(t_{n_k}^k), \widetilde{f}_k(\pi/\omega_{k-1})$$

and fixing the sign changes in this sequence, we find the localization intervals of all the sign-change points of $\widetilde{f}_k$ in $\Delta$. With the help of the classical procedures for the determination of the localized zeros we find the sign-change points of the function $\widetilde{f}_k$ in $\Delta$. Let us denote them by $\widetilde{t}_1^k, \widetilde{t}_2^k, \ldots, \widetilde{t}_{\widetilde{n}_k}^k$ $(0 \leq \widetilde{n}_k \leq n_k + 1)$. By virtue of (8) the intervals

$$(0, \widetilde{t}_1^k) = (\widetilde{t}_0^k, \widetilde{t}_1^k), (\widetilde{t}_1^k, \widetilde{t}_2^k), \ldots, (\widetilde{t}_{\widetilde{n}_k}^k, \widetilde{t}_{\widetilde{n}_k+1}^k) = (\widetilde{t}_{\widetilde{n}_k}^k, \pi/\omega_{k-1})$$

are intervals of strict monotonicity for the function $\frac{g_{k-1}(t)}{\sin \omega_{k-1} t}$ and therefore in each of them $g_{k-1}$ has no more than one sign change. In order to clarify in which of the intervals there exists a sign change, let us compute the values

$$g_{k-1}(\widetilde{t}_0^k), g_{k-1}(\widetilde{t}_1^k), \ldots, g_{k-1}(\widetilde{t}_{\widetilde{n}_k+1}^k).$$

It can be easily seen, that the interval $(\widetilde{t}_l^k, \widetilde{t}_{l+1}^k)$, $0 \leq l \leq \widetilde{n}_k$, is a localization interval for a sign change of the function $g_{k-1}$ if and only if $g_{k-1}(\widetilde{t}_l^k) g_{k-1}(\widetilde{t}_{l+1}^k) < 0$. (In a particular case, when $g_{k-1}(\widetilde{t}_0^k) = 0$ or $g_{k-1}(\widetilde{t}_{\widetilde{n}_{k+1}}^k) = 0$, one should consider the value $g_{k-1}^{(l)}(\widetilde{t}_0^k)$ or $(-1)^l g_{k-1}^{(l)}(\widetilde{t}_{\widetilde{n}_{k+1}}^k)$, where $l$ is the order of the first non-zero derivative of $g_{k-1}$ at a given terminal point of the interval $\Delta$.) After finding all the localization

intervals of the sign-change points of $g_{k-1}$ in $\Delta$ we, naturally, again use the classical procedures for the determination of these points.

As a result we obtain all the sign-change points of $g_{k-1}$ in $\Delta = \Delta_{k-1}^1$ and, evidently, we clarify the signs of $g_{k-1}$ between these points. We repeat the process for the interval $\Delta_{k-1}^2$, etc. up to $\Delta_{k-1}^{r_{k-1}}$. Hence, in view of (5), we find all the sign-change points of $f_{k-1}$ in $(0, T)$.

Thus, the algorithm for finding the sign-change points of the quasi-polynomial (1) with complex $\lambda_j$ is realized through applying the localization procedures and the computation of the localized zeros consecutively as in the real case. However, in contrast to the real case, here the computing complexity of the algorithm depends not only on the desired accuracy, on the length of the considered interval and $n$, but also on the number of non-real $\lambda_j$ and on the values $\omega_j = Im \lambda_j$. Usually great $|\omega_j|$ correspond to "quickly oscillating" quasi-polynomials, for which the problem complicates. Before proceeding with the computing complexity evaluation of the described algorithm we would like to point out that this algorithm might be used to estimate the number of sign changes of the quasi-polynomial (1) in an arbitrary finite interval $(a, b)$ as well.

**3. Number of zeros of the quasi-polynomial.** The notations $m$, $f_k$, $\omega_k$, $\Delta_{k-1}^j$, $r_{k-1}$ are the same as in 2.2. The following statement holds.

T h e o r e m 2. *The number of sign-change points of the quasi-polynomial* (1) *in the finite interval* $(a, b)$ *is not greater than*

$$(10) \qquad n - 1 + \frac{b-a}{\pi} \sum_{j=1}^{n} |Im \lambda_j|.$$

P r o o f. In order to use the previous notations let us assume that $t - a$ is a new variable. Then the interval $(a, b)$ goes over into the interval $(0, T)$, $T = b - a$.

Let the following condition be fulfilled:

$\mathscr{A}$) Any of the functions $f_1, f_2, \ldots, f_{m+1}$ is different from zero at all terminal points of the intervals of the kind (6).

Denote by $n_k^j (l_{k-1}^j)$ the number of sign changes of $f_k (f_{k-1})$ in the interval $\Delta_{k-1}^j$ $(k = 2, 3, \ldots, m+1; j = 1, 2, \ldots, r_{k-1})$ and by $N_k$ the number of sign changes of $f_k$ in $(0, T)$ $(k = 1, 2, \ldots, m+1)$. Obviously, in view of the condition $\mathscr{A}$ we have

$$N_k = \sum_{j=1}^{r_{k-1}} n_k^j, \quad N_{k-1} = \sum_{j=1}^{r_{k-1}} l_{k-1}^j, \quad k = 2, 3, \ldots, m+1.$$

Further, by virtue of the factorization (7) and the Rolle theorem $l_{k-1}^j \leq n_k^j + 2$ $(k = 2, 3, \ldots, m+1; j = 1, 2, \ldots, r_{k-1})$. From this, in view of the definition of $r_{k-1}$ and the positiveness of $\omega_{k-1} = Im \lambda_{k-1}$ $(k = 2, 3, \ldots, m+1)$, it follows that

$$N_{k-1} = \sum_{j=1}^{r_{k-1}} l_{k-1}^j \leq \sum_{j=1}^{r_{k-1}} (n_k^j + 2) = \sum_{j=1}^{r_{k-1}} n_k^j + 2 r_{k-1}$$
$$= N_k + 2 r_{k-1} \leq N_k + 2 (T \omega_{k-1}/\pi + 1),$$

i. e.

$$(11) \qquad N_{k-1} - N_k \leq 2 (T |Im \lambda_{k-1}|/\pi + 1)$$

$(k = 2, 3, \ldots, m+1)$. Since the quasi-polynomial $f_{m+1}$ has only real coefficients

$$N_{m+1} \leq n - 2m - 1.$$

From this, with regard to (11) it follows that

$$N_1 = (N_1 - N_2) + (N_2 - N_3) + \cdots + (N_m - N_{m+1}) + N_{m+1}$$

$$\leq \sum_{k=2}^{m+1} 2(T\,|\,Im\,\lambda_{k-1}\,|/\pi+1)+n-2m-1$$

$$=\frac{T}{\pi}\sum_{j=1}^{2m}|\,Im\,\lambda_j\,|+n-1.$$

Hence the number of sign changes of $f_1=f$ in $(0,\,T)$ does not exceed

$$(12) \qquad n-1+\frac{T}{\pi}\sum_{j=1}^{n}|\,Im\,\lambda_j\,|.$$

Thus we have found the required estimate under the condition $\mathscr{A}$. Now let us show, that this does not result in restricting the generality, since the condition $\mathscr{A}$ can always be provided by means of a small "shift". In detail, consider the quasi-polynomial $f(t-\varepsilon)$, where $\varepsilon>0$. Denote by $N(\varepsilon)$ the number of sign changes of $f(t-\varepsilon)$ in $(0,\,T)$. Let $T'$ be the greatest sign change of $f=f_1$ in $(0,\,T)$. (If $f$ has no sign changes in $(0,\,T)$, then we can assume that $T'=0$.) Evidently

$$(13) \qquad N_1\leq N(\varepsilon) \quad \text{for} \quad \varepsilon<\varepsilon_0=T-T'.$$

Since the number of terminal points of all the intervals $\Delta^j_{k-1}$, as well as the number of zeros in $(0,\,T)$ of all the functions $f_k$, is finite, the condition $\mathscr{A}$ for $f(t-\varepsilon)$ will be fulfilled for all $\varepsilon\in(0,\,\varepsilon_0)$ except for finitely many. Therefore, one can always choose a "shift" $\varepsilon\in(0,\,\varepsilon_0)$ such that the condition $\mathscr{A}$ is fulfilled for $f(t-\varepsilon)$. By virtue of the statements proved above $N(\varepsilon)$ does not exceed the value (12). Hence, in view of (13) it follows, that the same holds for $N_1$ too. Thus, the theorem is proved.

Remark 1. It is evident, that if $m=0$, then the value (10) is equal to $n-1$, i. e. the estimate coincides with the one known for the sign changes of the quasi-polynomial (1) comprising only real coefficients. In conformity with the number of the optimal control switches theorem 2 can be treated as a generalization (for the complex case) of the Feldbaum theorem (see e. g. [9], p. 116).

Remark 2. Let us consider the accuracy of the estimate (10). For concrete $n$, $m$ the estimate (10) can be improved. For example, for $n=2$, $m=1$ the estimate will be valid if $\pi$ is replaced by $2\pi$. On the other hand, it can be shown, that without additional information on the values of $n$, $m$ the constant $\pi$ appearing in the estimate cannot be improved. (The proof of this assertion is rather bulky and we would not discuss it in this paper.)

Remark 3. Up to now we have treated the quasi-polynomial $f$ as a function of the real argument $t$. However, it is possible to treat it as a function of the complex argument $z$ as well. Since $f(z)$ is an entire function, then the following estimate will be valid (assuming that $f(0)\neq0$) for the number $N(T)$ of zeros in the circle $|z|\leq T$

$$(14) \qquad N(T)\leq\ln\left(\max_{|z|=eT}|f(z)|/|f(0)|\right)$$

(see e. g. [11], p. 26). It is clear, that the right-hand side of (14) is an upper bound for the number of $f$ zeros also in the interval $(0,\,T)$ as well. One of disadvantages of this estimate in comparison with the estimate (10) is that with it the specificity of the real axis is ignored: the right-hand side of (14) is great for great values of $|\lambda_j|$ (which are great not only for great values of $|Im\,\lambda_j|$, but also for great values of $|Re\,\lambda_j|$). In the estimate (10) $|Re\,\lambda_j|$ does not appear at all.

4. Computing complexity of the algorithm. Let us assume that the computation of the exponent (and also of the sine and the cosine) is performed in a finite number of operations. The following statement is valid.

Theorem 3. *For finding all the sign-change points of the quasi-polynomial*
(1) *in a finite interval* $(a, b)$ *to within* $\varepsilon$ $(\varepsilon < (b-a)(2))$ *the following number of ope-*
*rations is sufficient*

$$Cn^3(\omega(b-a)/(\pi+1)\ln((b-a)/\varepsilon,$$

*where C is a constant,* $\omega = \max_{j=\overline{1, n}} Im\, \lambda_j.$

Proof. In order to determine the sign changes of the quasi-polynomial $f_1 = f$ in
the interval $(a, b)$ we find all the sign changes of the functions $f_{m+1}, f_m, \ldots, f_1$,
$\tilde{f}_{m+1}, \tilde{f}_m, \ldots, \tilde{f}_2$ in $(a, b)$. From theorem 2 it follows, that for the number $N_k$ of sign
changes in $(a, b)$ of the quasi-polynomial $f_k$ the following estimate is valid

$$N_k \leq n - 1 + \frac{2(b-a)\omega}{\pi}(m+1-k)$$

$(k=1, 2, \ldots, m+1)$. In view of (9) the number of sign changes of $\tilde{f}_k$ in $(a, b)$ does not
exceed $N_k + 1$ $(k=2, 3, \ldots, m+1)$. Thus, the number of all sign changes to be found
does not exceed

$$\sum_{k=1}^{m+1}(n-1+\frac{2(b-a)\omega}{\pi}(m+1-k))+\sum_{k=2}^{a+1}(n+\frac{2(b-a)\omega}{\pi}(m+1-k))$$

$$=(m+1)(n-1)+mn+\frac{(b-a)\omega}{\pi}\,2m^2 < 2n^2((b-a)\omega/\pi+1).$$

It is not difficult to see (analogously to the proof of theorem 1), that the con-
struction of all the functions $f_k, \tilde{f}_k$ and the determination of all the localization inter-
vals requires a number of operations of order

$$n^3((b-a)\,\omega/\pi+1).$$

The determination of all the localized sign changes with the help of the bisection
method requires a number of operations of order

$$n^3((b-a)\omega/\pi+1)\ln((b-a)/\varepsilon.$$

Thus, the total computing complexity of the algorithm does not exceed

$$C_1n^3((b-a)\,\omega/\pi+1)+C_2n^3((b-a)\omega/\pi+1)\ln((b-a)/\varepsilon)$$

$$\leq(C_1/\ln 2+C_2)\,n^3\,((b-a)\,\omega/\pi+1)\ln((b-a)/\varepsilon),$$

what was to be proved.
For $\omega = 0$ (i. e. if all the coefficients $\lambda_j$ are real) we obtain the result of theorem 1
as a special case. When discussing theorem 1 we have pointed out an advantage of
our algorithm for $\varepsilon \ll 1$ if compared to the trivial computation of $f$ values at the
points $a, a+2\varepsilon, a+4\varepsilon, \ldots$ The same advantage is valid since $\varepsilon$ is included only loga-
rithmically in the estimate of the computational complexity of the algorithm.

# REFERENCES

1. T. I. E n c h e v a, A. Yu. L e v i n. Central sections in linear time-optimal control problems. *C. R. Acad. Bulg. Sci.*, 43, 1990, 2, 33-37.
2. Дж. Ф о р с а й т, М. М а л ь к о л ь м, К. М о у л е р. Машинные методы математических вычислений. Москва, 1980.
3. А. Н. К о л м о г о р о в, С. В. Ф о м и н. Элементы теории функций и функционального анализа. Москва, 1972.
4. Л. В. К а н т о р о в и ч, Г. П. А к и м о в. Функциональный анализ. Москва, 1977.
5. Хр. И. С е м е р д ж и е в, С. Г. Т а м б у р о в. Метод определения всех нулей обобщенного полинома по произвольной чебышевской системе. *Ж. вычисл. матем. и матем. физ.*, 27, 1987, 1, 16—21.
6. А. Н. К р ы л о в. Лекции о приближенных вычислениях. Москва, 1954.
7. В. В. В о е в о д и н. Численные методы алгебры. Москва, 1966.
8. T. I. E n c h e v a, A. Yu. L e v i n. Zeros of quasi-polynomials and linear time-optimal control problem. *Mathematica Balkanica*, 2, 1988, 4, 296-305.
9. В. Г. Б о л т я н с к и й. Математические методы оптимального управления. Москва, 1968.
10. Д. П о л и а, Г. С е г е. Задачи и теоремы из анализа, Ч. II. Теория функций (специальная часть). Распределение нулей. Полиномы. Определители. Теория чисел. Москва, 1956.
11. Б. Я. Л е в и н. Распределение нулей целых функций. Москва, 1956.

*Institute of Mathematics*
*1090 Sofia P. O. Box 373*

*State University*
*ul. Sovietskaia d. 14*
*150000 Yaroslavl USSR*