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RELAXATIONS OF ORTHOGONAL PROJECTORS

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Linear operators T reducing the norm of elements with the exception of fixed points are investigated in Hilbert spaces. Procedures are given to construct such so-called relaxations by combination of other linear operators. Applications and connections to the theory of linear iterative methods are mentioned.

1. Relaxations and iterative methods. Let X and Y be Hilbert spaces. The space of all linear continuous operators on X into Y is denoted by $L(X, Y)$. Iterative methods of the general form

$$x_{n+1}=(I-D_nA)x_n+D_nb=x_n+D_n(b-Ax_n) \tag{1.1}$$

with given elements $x_0\in X$, $b\in Y$ and operators $A\in L(X, Y)$, $D_n\in L(Y, X)$ can be used to determine solutions or generalized solutions of the linear operator equation

$$Ax=b. \tag{1.2}$$

The rests or defects $r_n=b-Ax_n$ of (1.2) related to x_n fulfil the iterative relation

$$r_{n+1}=(I-AD_n)r_n. \tag{1.3}$$

In papers [7] and [9] the operators D_n have been chosen so as the corresponding operators $T_n=I-D_nA$ in (1.1) or $S_n=I-AD_n$ in (1.3) to represent the so-called relaxations of orthogonal projectors (orthoprojectors). The relaxations T of orthoprojectors T' are introduced in [7] as operators which are norm-reducing outside the range $R(T')$ of T'. This concept in [9] is reduced to the special case of operators

$$T=(1-\lambda)I+\lambda T', \quad |1-\lambda|<1. \tag{1.4}$$

In this paper such operators are said to be scalar relaxations since they are characterized by the scalar λ. A wide class of relaxations T is constituted by each orthoprojector T'. They belong to a certain neighbourhood of T' and have some analogous properties. Thus convergence results for the iterative methods (1.1) are preserved when the orthoprojectors T_n or S_n are replaced by the corresponding relaxations (cf. [7], [9] with [5], [6]).

Now we will explain the origin of the name "relaxation". If we choose the operators $D'_n\in L(Y, X)$ and put

$$D_n=\lambda_nD'_n, \quad |1-\lambda_n|<1.$$

in (1.1) and (1.3), then we get

$$x_{n+1}=(I-\lambda_nD'_nA)x_n+\lambda_nD'_nb=x_n+\lambda_nD'_n(b-Ax_n), \tag{1.1'}$$

$$r_{n+1}=(I-\lambda_nAD'_n)r_n. \tag{1.3'}$$

By using the notations $T'_n=I-D'_nA$ and $S'_n=I-AD'_n$ we have

$$T_n=(1-\lambda_n)I+\lambda_nT'_n, \quad S_n=(1-\lambda_n)I+\lambda_nS'_n.$$

If T_n' or S_n' are orthoprojectors, the corresponding T_n or S_n turn out to be scalar relaxations (see (1.4)). Usually the scalars λ_n in (1.1′) and (1.3′) are called relaxation parameters. For real λ_n the condition $|1-\lambda_n|<1$ means $0<\lambda_n<2$. It is convenient to speak of underrelaxation for $0<\lambda_n<1$ and of overrelaxation for $1<\lambda_n<2$ (see e. g. [12]). In [1] iterative methods (1.1) with

$$D_n=(E_nA)^+\Lambda_nE_n$$

are considered, assuming finite dimensional spaces X and Y, where the matrices E_n generate blocks of A and $\boldsymbol{b}$ respectively and the matrices Λ_n being quadratic of block size satisfy

$$\|(E_nA)^+(I-\Lambda_n)E_nA\|<1.$$

Thereby $(E_nA)^+$ denotes the orthogonal generalized inverse (or Moore-Penrose inverse) of E_nA (see e. g. [3]). For this choice of D_n the relaxation parameter λ_n is replaced by a relaxation matrix Λ_n. In section 4 it will be proved that the matrices

$$T_n=I-(E_nA)^+\Lambda_nE_nA$$

represent relaxations of the orthoprojection matrices

$$T_n'=I-(E_nA)^+E_nA$$

in our sense under the mentioned conditions for Λ_n.

In this paper a general concept of relaxations is proposed. We study the global properties of such operators with the exception of convergence results and the phenomenon of underrelaxation or overrelaxation which will be the topic of other papers. Besides, we show that there is a close connection with the concept of projection kernels playing also an important part in the convergence theory of iterative methods (1.1) (see [4], [7]).

2. Projection kernels of an operator set. This section contains some results from [7] without proving them. Let H be a Hilbert space and $\{T_n:\ n\in\mathbb{N}\}$ be a set of operators $T_n\in L(H)=L(H, H)$. Null spaces and ranges of T_n will be denoted by $\mathsf{N}(T_n)$ and $\mathsf{R}(T_n)$ respectively.

Definition 2.1. *$P\in L(H)$ is called a* projection kernel *of $\{T_n\}$ if the equations*

$$P^2=P=T_nP=PT_n$$

hold for all n. A projection kernel P of $\{T_n\}$ is said to be orthogonal *if it is selfadjoint ($P=P^*$).*

Projection kernels can still be characterized by other properties referring to subspaces of the operators involved.

Theorem 2.2. *For a projector $P\in L(H)$ the following conditions are equivalent:*

a) *P is a projection kernel of $\{T_n\}$.*

b) *$\mathsf{R}(P)\subseteq\bigcap_n\mathsf{N}(I-T_n)$ and $\mathsf{N}(P)\supseteq\operatorname{span}\bigcup_n\mathsf{R}(I-T_n)$ hold.*

c) *$T_n|\mathsf{R}(P)=I|\mathsf{R}(P)$ and $T_n\mathsf{N}(P)\subseteq\mathsf{N}(P)$ hold for all n.*

The property b) of projection kernels where span stands for the closed linear hull suggests a further notion.

Definition 2.3. *A projection kernel P of $\{T_n\}$ is called optimal if the equations*

$$\mathsf{R}(P)=\bigcap_n\mathsf{N}(I-T_n),\quad \mathsf{N}(P)=\operatorname{span}\bigcup_n\mathsf{R}(I-T_n)$$

are fulfilled.

For a set $\{T_n\}$ there can be several projection kernels but at most one optimal projection kernel. Under certain conditions the existence of optimal projection kernels can be guaranteed.

Theorem 2.4. *Let $\{T_n\}$ be a set of operators satisfying* $\mathsf{N}(I-T_n)=\mathsf{N}(I-T_n^*)$. *Then $\{T_n\}$ has an optimal projection kernel, namely the orthoprojector P with*

$$\mathsf{R}(P)=\bigcap_n \mathsf{N}(I-T_n),\quad \mathsf{N}(P)=\operatorname{span}\bigcup_n \mathsf{R}(I-T_n).$$

3. **Nonexpansive linear operators and relaxations of orthoprojectors.** Again, let H denote a Hilbert space. In the next sections the fixed point set $\mathsf{N}(I-T)$ of operator T plays an important role. For convenience we introduce

Definition 3.1. *The subspace $\mathsf{N}(I-T)$ of H is said to be the carrier of $T\in L(H)$.*

Although we are interested first of all in relaxations T, some results more generally hold for nonexpansive linear operators $T(\|T\|\leq 1)$. Relaxations will be defined here analogously as in [7] but without reference to an orthoprojector.

Definition 3.2. *$T\in L(H)$ is said to be a relaxation if $\|T_x\|<\|x\|$ holds for all $x\notin\mathsf{N}(I-T)$.*

Therefore a relaxation is norm-reducing outside its carrier. Evidently relaxations are nonexpansive. Trivial relaxations are the nulloperator 0 with the carrier $\{0\}$ and the identity operator I with the carrier H. Contractive linear operators T ($\|T\|<1$) are relaxations satisfying $\mathsf{N}(I-T)=\{0\}$. An orthoprojector T' is a relaxation with $\mathsf{N}(I-T')=\mathsf{R}(T')$. More general the linear combinations (1.4) of I and T' are relaxations satisfying $\mathsf{N}(I-T)=\mathsf{R}(T')$ (see [7], [9]). But certain non-expansive operators T, for instance isometric operators T ($\|Tx\|=\|x\|$ for all x) with $\mathsf{N}(I-T)\subset H$, are no relaxations. Relaxations are just those nonexpansive operators T for which $\|Tx\|=\|x\|$ holds only on the carrier.

Linear and affine invariant subspaces of an operator $T\in L(H)$ play an important role in the study of its properties. For nonexpansive operators and moreover for relaxations these subspaces have the following property:

Lemma 3.3. *Let $T\in L(H)$ be nonexpansive. Then all invariant affine subspaces of T meet the carrier $\mathsf{N}(I-T)$. Especially the intersection point of such a subspace with the orthogonal complement of its linear part lies in $\mathsf{N}(I-T)$.*

Proof. Let $N_z=z+N$ be an invariant affine subspace of T with the linear part N and the shift element z. Opposite to the second assertion we assume that the only element z' in $N_z\cap N^\perp$ fails to be in $\mathsf{N}(I-T)$. Now z' is the uniquely determined element in N_z satisfying $\|z'\|=\operatorname{dist}(0, N_z)$. Since $z'\notin\mathsf{N}(I-T)$ and $Tz'\in N_z$, then the contradiction $\|Tz'\|>\|z'\|$ resulrs. The first assertion is an immediate consequence of the second.

The last statement can be used to decide the question when linear combinations

$$T=(1-\lambda)I+\lambda P$$

of I and a projector P represent relaxations. First we list some properties of T which can be proved easily.

Lemma 3.4. a) *The carrier of T is $\mathsf{R}(P)$.* b) *The affine subspaces $z+\mathsf{N}(P)$ with $z\in H$ are invariant under T.* c) *$\|Tx\|=|1-\lambda|\cdot\|x\|$ for all $x\in\mathsf{N}(P)$.*

For $P=I$ or $\lambda=0$ the operator T is the trivial relaxation I. Otherwise T is expansive in the case $|1-\lambda|>1$ and isometric in the case $|1-\lambda|=1$, $\lambda\neq 0$. In both cases T is no relaxation. The other possibilities are covered by the next result.

Theorem 3.5. *Let T be an operator of the form*

$$T=(1-\lambda)I+\lambda P,\quad P^2=P,\quad |1-\lambda|<1.$$

Then the following statements are equivalent: a) *T is a relaxation.* b) *T is nonexpansive.* c) *P is orthogonal.*

Proof. The first implication is obvious. For the second implication we assume T to be nonexpansive. By Lemma 3.4 the affine subspaces $N_z=z+\mathsf{N}(P)$ are invariant under T. Evidently N_z meets both $\mathsf{N}(P)^\perp$ and $\mathsf{R}(P)=\mathsf{N}(I-T)$ at one and only one point. In view of Lemma 3.3 that means $\mathsf{N}(P)^\perp\subseteq\mathsf{R}(P)$. But this leads to $\mathsf{N}(P)^\perp=\mathsf{R}(P)$ and $P=P^*$. The validity of the last implication has already been mentioned (see above and section 1).

In the special case $\lambda=1$ we obtain the equivalence of $P=P^*$ and $\|P\|\leqq 1$ for projectors P. This result can yet be found in [2, p. 84] or [11, p. 84].

Nonexpansive operators possess an outstanding invariant linear subspace, namely the orthogonal complement of their carrier.

Theorem 3.6. *Let $T\in L(H)$ be nonexpansive. Then* $T\,\mathsf{N}(I-T)^\perp\subseteq\mathsf{N}(I-T)^\perp$ *holds.*

Proof. We consider arbitrary elements u in $\mathsf{N}(I-T)$ and v in $\mathsf{N}(I-T)^\perp\setminus\{0\}$. Taking into account the orthogonal decomposition

$$H=\mathsf{N}(I-T)\oplus\mathsf{N}(I-T)^\perp$$

we get $(u,v)=0$, $u+v\notin\mathsf{N}(I-T)$. Furthermore it follows

$$T(u+v)=Tu+Tv=u+Tv$$

and

$$\|T(u+v)\|^2=\|u+Tv\|^2=\|u\|^2+\|Tv\|^2+(u,Tv)+(Tv,u)$$
$$=\|u\|^2+\|Tv\|^2+2\mathrm{Re}\,(u,Tv)\leqq\|u+v\|^2=\|u\|^2+\|v\|^2$$

since T is supposed to be nonexpansive. Hence

$$2\mathrm{Re}\,(u,Tv)\leqq\|v\|^2-\|Tv\|^2.$$

If we replace u consecutively by λu, $-\lambda u$, λiu and $-\lambda iu$, where λ is a positive scalar and i is the imaginary unit, then we obtain the inequalities

$$\pm 2\lambda\,\mathrm{Re}\,(u,Tv)\leqq\|v\|^2-\|Tv\|^2,$$
$$\mp 2\lambda\,\mathrm{Im}\,(u,Tv)\leqq\|v\|^2-\|Tv\|^2$$

due to the relation $\mathrm{Re}\,iz=-\mathrm{Im}\,z$ for an arbitrary complex number z. This means

$$0\leqq|\mathrm{Re}\,(u,Tv)|\leqq\frac{1}{2\lambda}(\|v\|^2-\|Tv\|^2),$$

$$0\leqq|\mathrm{Im}\,(u,Tv)|\leqq\frac{1}{2\lambda}(\|v\|^2-\|Tv\|^2).$$

Since these relations hold for any $\lambda>0$, then

$$\mathrm{Re}\,(u,Tv)=\mathrm{Im}\,(u,Tv)=0$$

and $(u,Tv)=0$. But now we have $Tv\in\mathsf{N}(I-T)^\perp$ and therefore the assertion holds.

The theorem shows that a nonexpansive operator T is completely reduced by the pair $(\mathsf{N}(I-T),\mathsf{N}(I-T)^\perp)$ of orthogonal subspaces (for the notion see [10, p. 268]). Besides the affine subspaces $z+\mathsf{N}(I-T)^\perp$ with $z\in H$ remain invariant under T. (Without loss of generality z can be chosen in $\mathsf{N}(I-T)$.) Beyond it the affine subspaces $z+\mathsf{N}(I-T)$ with $z\in H$ are mapped by T onto the parallel affine subspaces $Tz+\mathsf{N}(I-T)$. (Here we can suppose z to be in $\mathsf{N}(I-T)^\perp$.) Thus T acts in $z+\mathsf{N}(I-T)^\perp$ analogously as in $\mathsf{N}(I-T)^\perp$, only parallelly shifted. These facts allow the following characterization of relaxations.

Corollary 3.7. *$T\in L(H)$ is a relaxation if* $T\,\mathsf{N}(I-T)^\perp\subseteq\mathsf{N}(I-T)^\perp$, $\|Tx\|<\|x\|$ *for all* $x\in\mathsf{N}(I-T)^\perp\setminus\{0\}$. *A corresponding result is true for nonexpansive operators T. Here the strict inequality $<$ has to be replaced by $\leqq$.*

Operators T with the invariant pair $(\mathrm{N}(I-T), \mathrm{N}(I-T)^{\perp})$ can be connected with the orthoprojector T' determined by $\mathrm{R}(T')=\mathrm{N}(I-T)$. In this sense we speak of relaxations T of orthoprojectors T' (see also section 1). Obviously relaxations with the same carrier are relaxatons of the same orthoprojector.

Now we are ready to characterize relaxations by some other conditions.

Theorem 3.8. *Let $T' \in L(H)$ be an orthoprojector. Then the following statements are equivalent:*

a) *T is a relaxation of T'.*

b) $Tx=x$ *for* $x \in \mathrm{R}(T')$, $\|Tx\|<\|x\|$ *for* $x \notin \mathrm{R}(T')$.

c) $T|\mathrm{R}(T')=I|\mathrm{R}(T')$, $T\mathrm{N}(T') \subseteq \mathrm{N}(T')$,
$\|Tx\|<\|x\|$ *for* $x \in \mathrm{N}(T') \setminus \{0\}$.

d) $\mathrm{R}(T') \subseteq \mathrm{N}(I-T)$, $\mathrm{N}(T') \supseteq \mathrm{R}(I-T)$,
$\|Tx\|<\|x\|$ *for* $x \in \mathrm{N}(T') \setminus \{0\}$.

e) $\mathrm{R}(T')=\mathrm{N}(I-T)$, $\mathrm{N}(T')=\overline{\mathrm{R}(I-T)}$,
$\|Tx\|<\|x\|$ *for* $x \in \mathrm{N}(T') \setminus \{0\}$.

f) $T'=T'T=TT'$, $\|Tx-T'x\|<\|x\|$ *for* $x \neq 0$.

Proof. 1. In the cases a) and b) the equation $\mathrm{R}(T')=\mathrm{N}(I-T)$ is fulfilled. Therefore by Definition 3.2 both cases coincide.

2. If we put $P=T'$ and $T_n=T$ for all n in Theorem 2.2 and use Definition 2.1 we get the equivalence of the three conditions

$$T|\mathrm{R}(T')=I|\mathrm{R}(T'),\ T\mathrm{N}(T') \subseteq \mathrm{N}(T'),$$
$$\mathrm{R}(T') \subseteq \mathrm{N}(I-T),\ \mathrm{N}(T') \supseteq \mathrm{R}(I-T),$$
$$T'=TT'=T'T.$$

On the other side each of these conditions ensures the equivalence of the norm relations

$$\|Tx\|<\|x\| \text{ for } x \notin \mathrm{R}(T'),$$
$$\|Tx\|<\|x\| \text{ for } x \in \mathrm{N}(T') \setminus \{0\},$$
$$\|Tx-T'x\|<\|x\| \text{ for } x \neq 0.$$

Obviously the first relation implies the second. The inversion follows if the parallel action of T induced by the pair $(\mathrm{R}(T'), \mathrm{N}(T'))$ is observed. (The direct proof is based on the Pythagorean relation.)

Assuming the second relation we obtain

$$\|Tx-T'x\| = \|T(I-T')x\| < \|(I-T')x\| \leq \|I-T'\| \cdot \|x\| = \|x\|$$

and thus the third. The inversion is evident.

3. The norm conditions in b) — e) cause the equations $\mathrm{R}(T')=\mathrm{N}(I-T)$, $\mathrm{N}(T') =\mathrm{N}(I-T)^{\perp}$ and the inequality $\|T\| \leq 1$. For nonexpansive operators T the decomposition

$$H=\mathrm{N}(I-T) \oplus \overline{\mathrm{R}(I-T)}$$

holds (see [11, p. 214]). The equivalence of the statements b) — f) now it can easily be seen if you take point 2 of the proof and Theorem 3.6 into account.

For nonexpansive operators T analogous equivalent statements cannot be established. But starting with a′) T is a nonexpansive operator connected with T' modified statements b′) — f′) can be derived by changing the relation $<$ to $\leq$.

The last theorem has still some other consequences.

Corollary 3.9. *Let $T \in L(H)$ be a relaxation. Then the following properties hold:*

a) $\mathrm{N}(I-T)^{\perp}=\overline{\mathrm{R}(I-T^{*})}=\overline{\mathrm{R}(I-T)}$, $\mathrm{N}(I-T)=\mathrm{N}(I-T^{*})$.

b) $\mathsf{N}(I-T)=\mathsf{N}((I-T)^2)$, $\overline{\mathsf{R}(I-T)}=\overline{\mathsf{R}((I-T)^2)}$, $(I-T)\,\overline{R(I-T)}=R(I-T)$.

c) *$A=I-T$ has a group inverse $B=(I-T)^{-1}(I-T')$, that is, the relations $ABA=A$ $BAB=B$, $AB=BA$ are fulfilled.*

d) *The corresponding orthoprojector T' is the optimal projection kernel of $\{T\}$.*

Proof. Let T be a relaxation of T'.

1. In view of Theorem 3.8 the equations

$$\mathsf{N}(T')=\mathsf{N}(I-T)^{\perp}=\overline{\mathsf{R}(I-T)}$$

are satisfied. On the other hand, the relations for orthogonal subspaces supply

$$\mathsf{N}(I-T)^{\perp}=\overline{\mathsf{R}(I-T^*)},\ \mathsf{N}(I-T)=\overline{\mathsf{R}(I-T)}^{\perp}=\mathsf{N}(I-T^*)$$

(see e. g. [2, p. 82], [10, p. 250]).

2. $A=I-T$ is a so-called decomposition regular operator, that is $H=\mathsf{N}(A)\oplus\overline{\mathsf{R}(A)}$ (see [8, Definition 1] and Theorem 3.8 e)). By this statements b) and c) are contained in [8, pp. 154-155].

3. Using the notions of Definition 2.3 and Definition 2.1 the last statement follows by Theorem 3.8.

The assertions of Corollary 3.9 hold also for nonexpansive operators T. Finally we deal with relations between T and the adjoint operator T^*. T is nonexpansive iff T^* is. This immediately follows from $\|T\|=\|T^*\|$ (see e. g. [10, p. 249]). For normal operators T $(TT^*=T^*T)$ the property of relaxation evidently is transferred from T to T^* since $\|Tx\|=\|T^*x\|$ holds in this case (see [10, p. 331]). But the assumption of normality is not necessary.

Theorem 3.10. *$T\in L(H)$ is a relaxation iff T^* is a relaxation. Thereby the carriers of T and T^* coincide.*

Proof. Corollary 3.9 supplies $\mathsf{N}(I-T)=\mathsf{N}(I-T^*)$ if T is a relaxation. Obviously we obtain the same equation if T^* is a relaxation.

In each case the subspace

$$\mathcal{N}=\mathsf{N}(I-T)^{\perp}=\mathsf{N}(I-T^*)^{\perp}$$

is invariant under T and T^*. In view of Corollary 3.7 we have to show that the statement

$$-\|Tx\|<\|x\| \text{ for all } x\in\mathcal{N}\setminus\{0\}$$

is fulfilled for T iff the corresponding statement

$$-\|T^*x\|<\|x\| \text{ for all } x\in\mathcal{N}\setminus\{0\}$$

holds for T^*. It suffices to show one direction because of $(T^*)^*=T$. Opposite to the assertion we assume that there is an element x' in $\mathcal{N}\setminus\{0\}$ satisfying $\|T^*x'\|=\|x'\|$. Without loss of generality we can choose $\|x'\|=1$. Now we restrict us to $\mathcal{N}$.

Since TT^* is a nonnegative selfadjoint operator, we find

$$\sup\{(TT^*x,\ x):\ \|x\|=1\}=\|TT^*\|=\|T\|^2\leq 1$$

(see [10, pp. 324-325, pp. 330-331]). Besides we get

$$\|T^*x'\|^2=(T^*x',\ T^*x')=(TT^*x',\ x')=1.$$

Hence it is

$$\sup\{(TT^*x,\ x):\ \|x\|=1\}=(TT^*x',\ x')=1.$$

But then 1 must be an eigenvalue of TT^* since the supremum is attained (see [10, p. 325]). That means $TT^*x'=x'$ and $T^*Ty'=y'$ with $y'=T^*x'$. Thereby x' cannot belong to $\mathsf{N}(T^*)=\mathsf{N}(TT^*)$ in view of $\|x'\|=1$. Thus $y'\neq 0$. Then we can conclude that $z'=y'/\|y'\|$ satisfies

$$T^*Tz'=z',\quad \|z'\|=1.$$

Now it follows

$$\|Tz'\|^2=(Tz',\ Tz')=(T^*Tz',\ z')=(z',\ z')=\|z'\|^2=1$$

in contradiction to the assumption.

4. Composite relaxations and projection kernels. If nothing else is mentioned, the operators act on Hilbert spaces H. The first statement gives a connection between the projection kernels of orthoprojectors T_n' and corresponding relaxations T_n.

Theorem 4.1. *Let T_n be relaxations of orthoprojectors T_n'. Then each projection kernel P of $\{T'_n\}$* is also a projection kernel of *$\{T_n\}$ and vice versa. Moreover, $\{T_n'\}$ and $\{T_n\}$ possess a common orthogonal projection kernel T' being optimal for both $\{T_n'\}$ and $\{T_n\}$.*

Proof. 1. In view of Theorem 3.8 we obtain

$$\mathrm{N}(I-T_n)=\mathrm{R}(T_n')=\mathrm{N}(I-T_n'),$$

$$\overline{\mathrm{R}(I-T_n)}=\mathrm{N}(T_n')=\mathrm{R}(I-T_n')$$

for all n. This means

$$\bigcap_n \mathrm{N}(I-T_n)=\bigcap_n \mathrm{N}(I-T_n'),$$

$$\operatorname{span}\bigcup_n \mathrm{R}(I-T_n)=\operatorname{span}\bigcup_n \mathrm{R}(I-T_n').$$

Now the first assertion is a consequence of Theorem 2.2.

2. By Theorem 2.4 the set $\{T_n'\}$ has an orthogonal optimal projection kernel T' satisfying

$$\mathrm{R}(T')=\bigcap_n \mathrm{N}(I-T_n),\ \mathrm{N}(T')=\operatorname{span}\bigcup_n \mathrm{R}(I-T_n').$$

If you consider the relations in the first part of the proof, then you get also

$$\mathrm{R}(T')=\bigcap_n \mathrm{N}(I-T_n),\ \mathrm{N}(T')=\operatorname{span}\bigcup_n \mathrm{R}(I-T_n).$$

In connection with Definition 2.3 this shows the second assertion.

Next we can observe that products of relaxations are relaxations too. Thereby the carrier of the product is the intersection of the carriers of the single factors.

Theorem 4.2. *Let T_i be relaxations of the orthoprojectors T_i' for $i=1,\dots,k$. Then $T=T_k\dots T_1$ is a relaxation of the orthoprojector T' determined by*

$$\mathrm{R}(T')=\bigcap_{i=1}^{k}\mathrm{R}(T_i')=\bigcap_{i=1}^{k}\mathrm{N}(I-T_i).$$

Proof. By Theorem 4.1 the sets $\{T_1',\dots,T_k'\}$ and $\{T_1,\dots,T_k\}$ have a common orthogonal projection kernel T' satisfying the conditions stated in the theorem.

If $x\in\mathrm{R}(T')$, then $x\in\mathrm{N}(I-T_i)$ for $i=1,\dots,k$. Hence $T_ix=x$ for $i=1,\dots,k$ and

$$Tx=T_k\dots T_1x=x.$$

If $x\notin\mathrm{R}(T')$, then there is a natural number $j\le k$ such that

$$x\notin\mathrm{N}(I-T_j),\ x\in\mathrm{N}(I-T_i)\quad\text{for } i=1,\dots,j-1.$$

Therefore we get

$$\|Tx\|=\|T_k\dots T_{j+1}T_jT_{j-1}\dots T_1x\|$$

$$\le\|T_k\|\dots\|T_{j+1}\|\ \|T_jx\|\le\|T_jx\|<\|x\|.$$

Theorem 3.8 supplies the assertion.

By Theorem 4.1, Theorem 3.8 and Corollary 3.9 the orthoprojector T' in Theorem 4.2 is the optimal projection kernel of $\{T_1, \dots, T_k\}$, $\{T_1', \dots, T_k'\}$ and $\{T\}$. Especially it is

$$\mathrm{N}(I-T)=\bigcap_{i=1}^{k}\mathrm{N}(I-T_i),\quad \overline{\mathrm{R}(I-T)}=\operatorname{span}\bigcup_{i=1}^{k}\mathrm{R}(I-T_i).$$

An immediate consequence of Theorem 4.2 is

Corollary 4.3. *The powers T^k of a relaxation T are relaxations with the same carrier as T.*

Besides we get

Corollary 4.4. *Let T be a relaxation. Then TT^* and T^*T are selfadjoint relaxations with the same carrier as T.*

Proof. If T is a relaxation, then T^* is a relaxation with the same carrier in view of Theorem 3.10. Therefore the assertion follows by Theorem 4.2 since TT^* and T^*T are selfadjoint.

The assumptions of Theorem 4. 2 can be weakened if we suppose operators with a common carrier. Somewhat more general we obtain

Theorem 4.5. *Let T' be an orthoprojector and T_i $(i=1,\dots,k)$ be nonexpansive operators with the property*

$$\bigcap_{i=1}^{k}\mathrm{N}(I-T_i)=\mathrm{R}(T').$$

Then $T=T_k\dots T_1$ is a relaxation of T' if one of the T_i, say T_j, is a relaxation of T'.

Proof. With the abbreviations $y=Sx$, $S=T_{j-1}\dots T_1$ the inequalities

$$\|Tx\|\le\|T_jy\|\le\|y\|\le\|x\|$$

hold. For $x\in\mathrm{R}(T')$ we find at once

$$Tx=T_k\dots T_1x=x.$$

For $x\notin\mathrm{R}(T')$ we must distinguish two cases. If y is not in $\mathrm{R}(T')=\mathrm{N}(I-T_j)$, then $\|T_jy\|<\|y\|$. If y is in $\mathrm{R}(T')$, then $y\in\bigcap_{i=1}^{j-1}\mathrm{N}(I-T_i)\subseteq\mathrm{N}(I-S)$. Besides we get $x\notin\mathrm{N}(I-S)$ since otherwise it would be $x=y\in\mathrm{R}(T')$ in contradiction to the assumption. Due to the remarks after Theorem 3.6 the affine subspace $N_x=x+\mathrm{N}(I-S)^{\perp}$ is invariant under the nonexpansive operator S and meets $\mathrm{N}(I-S)$ in that point of N_x which has the least distance from 0. Therefore it is $\|y\|<\|x\|$. In both cases we have $\|Tx\|<\|x\|$.

Now we give certain conditions under which sums of operators represent relaxations.

Theorem 4.6. *Let A_i $(i=1,\dots,k)$ be operators for which positive numbers λ_i and linear subspaces N_i exist so that*

a) $\lambda_1+\dots+\lambda_k=1$;

b) $A_ix=\lambda_ix$ *for* $x\in N_i$ $(i=1,\dots,k)$;

c) $\|A_ix\|<\lambda_i\|x\|$ *for* $x\notin N_i$ $(i=1,\dots,k)$.

Then $T=A_1+\dots+A_k$ is a relaxation with the carrier $N=\bigcap_{i=1}^{k}N_i$.

Proof. If $x\in N$, then $x\in N_i$ for $i=1,\dots,k$. Hence

$$Tx=(A_1+\dots+A_k)x=(\lambda_1+\dots+\lambda_k)x=x$$

in view of a) and b). Furthermore it is $\|A_ix\|=\lambda_i\|x\|$ for $x\in N_i$. If $x\notin N$, then $x\notin N_i$ for at least one i. Therefore we get $\|A_ix\|\le\lambda_i\|x\|$, where because of c) for at least one i the strict inequality holds. Thus we find

$$\| Tx \| \leqq \| A_1 x \| + \cdots + \| A_k x \| < \lambda_1 \| x \| + \cdots + \lambda_k \| x \| = \| x \|.$$

In Theorem 4.6 the spaces N_i represent eigenspaces and the numbers λ_i the corresponding eigenvalues of the operators A_i. The operators $T_i = \lambda_i^{-1} A_i$ are relaxations with the carriers $N_i = \mathrm{N}(I - T_i)$.

As an immediate consequence of Theorem 4.6 we obtain

Corollary 4.7. *Convex combinations* $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$ *of relaxations* T_i $(i=1, \ldots, k)$ *are relaxations too. Their carriers are* $\bigcap_{i \in I} \mathrm{N}(I - T_i)$ *denoting by* I *the set of those numbers* i *with* $\lambda_i > 0$.

According to Corollary 4.3 the polynomials

$$p(T) = \lambda_0 I + \lambda_1 T + \cdots + \lambda_n T^n, \quad \lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$$

of a relaxation T are relaxations with the same carrier as T. Besides operators

$$U = \lambda T + (1 - \lambda) T^*, \quad 0 < \lambda < 1$$

based on a relaxation T are selfadjoint relaxations with the same carrier as T. This is clear if we remind of Theorem 3.10. Again the assumptions of Theorem 4.6 can be weakened if the subspaces N_i coincide.

Lemma 4.8. *Let* $A_i (i=1, \ldots, k)$ *be operators for which positive numbers* λ_i *and a linear subspace* N *exist so that*

a) $\lambda_1 + \cdots + \lambda_k = 1$,

b) $A_i x = \lambda_i x$ *for* $x \in N$ $(i=1, \ldots, k)$,

c) $\| A_i x \| \leqq \lambda_i \| x \|$ *for* $x \notin N$ $(i=1, \ldots k)$,

$\| A_j x \| < \lambda_j \| x \|$ *for* $x \notin N$ *and some* $j \in \{1, \ldots, k\}$.

Then $T = A_1 + \cdots + A_k$ *is a relaxation with carrier* N.

The proof is simple and quite similar to that of Theorem 4.6. Therefore it is omitted here. Besides there is still another possibility to modify the assumptions of Theorem 4.6. If c) is replaced by

c′) $\| A_i x \| \leqq \lambda_i \| x \|$ for $x \notin N_i$ $(i=1, \ldots, k)$,

$\operatorname{rang}\{A_1 x, \ldots, A_k x\} > 1$ for $x \notin N = \bigcap_{i=1}^{k} N_i$,

then the assertion holds too. Observing the strict convexity of Hilbert spaces we obtain then

$$\| Tx \| < \| A_1 x \| + \cdots + \| A_k x \| \leqq \lambda_1 \| x \| + \ldots + \lambda_k \| x \| = \| x \|$$

for $x \notin N$. In a similar way the assumptions of Corollary 4.7 can be modified starting from nonexpansive operators T_i instead of relaxations.

Finally we present a result of another kind. Here the sum of operators turns out to be a relaxation of one of its summands.

Theorem 4.9. *Let* T' *be an orthoprojector and* A_i $(i=1, \ldots, k)$ *be operators satisfying*

a) $T' A_i = A_i T' = 0$ *for* $i = 1, \ldots, k$;

b) $\| A_1 x + \cdots + A_k x \| < \| x \|$ for $x \neq 0$.

Then $T = T' + A_1 + \cdots + A_k$ *is a relaxation of* T'.

Proof. By assumption a) we get

$$T'T = T'T' + T'A_1 + \cdots + T'A_k = T',$$
$$TT' = T'T' + A_1 T' + \cdots + A_k T' = T'.$$

Because of b) it is

$$\| Tx - T'x \| = \| A_1 x + \cdots + A_k x \| < \| x \| \text{ for } x \neq 0.$$

Therefore Theorem 3.8 supplies the assertion.

Corollary 4.10. *Let T' be an orthoprojector and Λ be an operator with the properties*

$$T'\Lambda(I-T')=0,$$
$$\|(I-\Lambda)(I-T')x\|<\|x\| \quad for\ x\neq 0.$$

Then $T=I-\Lambda+\Lambda T'$ is a relaxation of T'.

Proof. We put $A_1=(I-\Lambda)(I-T')$. In view of

$$T'(I-T')=(I-T')T'=T'-T'=0$$

we obtain

$$T'A_1=T'(I-\Lambda)(I-T')=T'(I-T')-T'\Lambda(I-T')=0,$$
$$A_1T'=(I-\Lambda)(I-T')T'=0,$$
$$\|A_1x\|=\|(I-\Lambda)(I-T')x\|<\|x\| \text{ for } x\neq 0.$$

Applying Theorem 4.9 for $k=1$ we see that

$$T=T'+A_1=T'+(I-\Lambda)(I-T')=I-\Lambda+\Lambda T'$$

is a relaxation of T'.

For the choice $\Lambda=\lambda I$ Corollary 4.10 shows again that operators $T=(1-\lambda)I+\lambda T'$ with $|1-\lambda|<1$ are relaxations called scalar relaxations (see (1.4) and Theorem 3.5). For $0<\lambda\leq 1$ this result follows from Corollary 4.7. Finally we turn to another application of Corollary 4.10. For that two Hilbert spaces X, Y are considered.

Corollary 4.11. Let Ω be in $L(Y)$. Let $A\in L(X, Y)$ and $B\in L(Y, X)$ be operaors satisfying

$$ABA=A,\ BAB=B,\ (BA)^*=BA.$$

Then $T=I-B\Omega A$ is a relaxation of the orthoprojector

$$T'=I-BA \text{ if } \|B(I-\Omega)Ax\|<\|x\| \text{ holds for } x\neq 0.$$

Proof. For $\Lambda=I-B(I-\Omega)A$ we get

$$(I-\Lambda)(I-T')=B(I-\Omega)ABA=B(I-\Omega)A,$$
$$\Lambda(I-T')=(I-T')-(I-\Lambda)(I-T')=BA-B(I-\Omega)A=B\Omega A,$$
$$T'\Lambda(I-T')=(I-BA)B\Omega A=B\Omega A-BAB\Omega A=0$$

and

$$\|(I-\Lambda)(I-T')x\|=\|B(I-\Omega)Ax\|<\|x\|$$

if $x\neq 0$. In view of Corollary 4.10

$$T=I-\Lambda+\Lambda T'=I-\Lambda(I-T')=I-B\Omega A$$

is a relaxation of $T'=I-BA$.

The assumptions in the above statement about A and B mean that A is a right-orthogonal generalized inverse of B and B is a left-orthogonal generalized inverse of A respectively (see [3]).

The results of this section can be combined in various ways to get relaxations again and again starting from a fixed set of relaxations. Above all this procedure is suitable if we succeed in step by step improving the properties of the relaxations arising. For instance, we can try to reduce the norms of the relaxations T on the orthogonal complements of their carriers $\mathsf{N}(I-T)$. This strategy plays an important part in the convergence theory of iterative methods with relaxations (see [5], [7], [9]). Taking a basic set

$$\mathcal{T}=\{T_{n_1},\ldots,T_{n_k}\}$$

of relaxations we can form segments

$$(T_{n'},\ldots,T_{(n+1)'-1}),\quad 0'=0,\quad (n+1)'>n'$$

with the properties

$$T_i\in\mathcal{T}\quad\text{for } i=n',\ldots,(n+1)'-1,$$

$$\bigcap_{i=n'}^{(n+1)'-1} \mathrm{N}(I-T_i)=\bigcap_{j=1}^{k} \mathrm{N}(I-T_{n_j})=M$$

holding for all n. From these segments we can generate relaxations $T^{(n)}$ with the common carrier M by multiplication, by convex linear combination or also by alternating use of both principles (see Theorem 4.2, Corollary 4.7). These relaxations $T^{(n)}$ can be united in the described manner again to obtain new relaxations with the carrier M (see Theorem 4.5, Lemma 4.8).

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