Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

CONVERGENCE OF SUCCESSIVE APPROXIMATION FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH ADDITIVE WHITE NOISE

RALF MANTHEY

The convergence of the successive approximation procedure for a stochastic partial differentia equation with an additive Gaussian space-time white noise is considered. Conditions which ensure this convergence are given. They are weaker than the usual Lipschitz conditions.

1. Introduction. Consider the formal Cauchy problem

$$(C)\qquad \begin{cases} \frac{\partial}{\partial t} u(t, x)=\frac{\partial^2}{\partial x^2}u(t, x)+f(u(t, x))+\sigma\zeta(t, x), & t>0,\ x\in \mathrm{R}. \\ u(0, x)=u_0(x), & x\in \mathrm{R}. \end{cases}$$

Here $f\colon \mathrm{R}\to\mathrm{R}$ is a nonlinear function in general and ζ denotes a space-time Gaussian white noise, i. e. a centered $\mathcal{D}'(\mathrm{R})$ valued Gaussian random variable with covariance functional $\mathrm{E}\zeta(\varphi)\zeta(\psi)=(\varphi, \psi)$, $\varphi, \psi\in\mathcal{D}(\mathrm{R}_+\times\mathrm{R})$. The notation $\mathcal{D}'$ is used for the Schwartz space of distributions, and $(.,.)$ denotes the inner product in $\mathrm{L}_2(\mathrm{R}_+\times\mathrm{R})$. The noise intensity σ represents a nonnegative constant.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space and introduce the class $\mathcal{B}_0(\mathrm{R}_+\times\mathrm{R})$ of such Borel sets A in $\mathrm{R}_+\times\mathrm{R}$ whose Lebesgue measure $\lambda(A)$ is finite. A family of centered Gaussian random variables $(W(A))_{A\in\mathcal{B}_0(\mathrm{R}_+\times\mathrm{R})}$ with $\mathrm{E}W(A)W(B)=\lambda(A\cap B)$ is called Gaussian set-indexed white noise. Introduce the function space

$$\mathrm{C}_e=\{\psi\in\mathrm{C}(\mathrm{R})\colon \sup_{x\in R}|\psi(x)|\exp(-\lambda|x|)<\infty \text{ for some } \lambda>0\},$$

where C denotes the set of continuous functions. The initial condition u_0 is always assumed to be a random process with trajectories in C_e. Finally, introduce the notation $D_t=[0, t]\times\mathrm{R}$ and the $\sigma=$a algebra $\mathcal{F}_t=\sigma\{W(A), u_0(x)\colon A\in\beta_0'([0, t]\times\mathrm{R}), x\in\mathrm{R}\}$, $t\geqq 0$. Fix $T>0$.

The formal problem (C) gets a precise mathematical meaning by the following definition.

Definition. *A pathwise continuous random field $u=(u(t,x))_{(t,x)\in D_t}$ is called solution of* (C) *if it is $\mathcal{F}_t$ measurable for any $t\in[0, T]$ and satisfies the equation*

$$(S)\qquad u(t, x)=\int_R G(t, x, y)u_0(y)dy+\int_0^t\int_R G(t-s, x, y)f(u(s, y))dyds$$

$$+\sigma.\int_0^t\int_R G(t-s, x, y)dW_{sy}=a(t, x)+b(t, x)+\sigma.c(t, x).$$

P a. s. for any $(t, x)\in D_T$. The mapping G is the heat kernel

$$G(t, x)=(4\pi t)^{-1/2}.\exp(-x^2/4t).$$

SERDICA Bulgaricae Mathematicae Publicationes. V. 16, 1990, p. 194-200.

Because of

$$\int_0^T \int_R G^2(t, x)dxdt = (T/2\pi)^{1/2}$$

the stochastic integral in (C) always exists.

The aim of this paper is to prove an existence and uniqueness theorem for the solution of (C). It is easy to show the validity of such a theorem in the case of a global Lipschitz condition of f. In this situation the successive approximation is the usual tool for the proof. However, as in the ordinary case this method works also in more general situations, what is described below.

2. The result. Introduce the Banach space $C_\lambda = \{\psi \in C(R) : \sup_{x \in R} |\psi(x)| \exp(-\lambda |x|) < \infty\}$ with the norm $\|\psi\| = \sup_{x \in R} |\psi(x)| \exp(-\lambda |x|)$. For simplicity the mapping $x \to \exp(-\lambda |x|)$ is denoted by ρ. Furthermore, the notation $\|U_s\|_t := \sup_{0 \leq s \leq t} \|U_s\|$ will be used. Finally introduce the set $C_\rho^T = \{u \in C(D_T): \|U_t\|_T < \infty\}$.

Lemma. *If* $u_0 \in C_\lambda$ P *a. s., then* $a + \sigma c = z$ *belongs* P *a. s. to* C_ρ^T.

Proof. See [1].

Fix now $\omega \in \Omega$ such that $z \in C_\rho^T$. Write $F(U)$ for the mapping $x \in R \to f(u(x))$. In view of the above mentioned lemma one can consider (S) as an equation in C_λ:

$$U_t = Z_t + \int_0^t T_{t-s} F(U_s) ds, \qquad \text{(S*)}$$

where the linear operator $T_t: C_\lambda \to C_\lambda$ is given by

$$(T_t\psi)(x) = \int_R G(t, x-y)\psi(y)dy.$$

The main conditions used in the sequel are:

(K) There exists a continuous nondecreasing function k: $[0, \infty] \to [0, \infty)$ with $k(0)=0$ and the property $\int_0^1 \frac{du}{k(u)} = \infty$ such that

$$\|F(U)-F(V)\| \leq k(\|U-V\|), \quad U, V \in C_\lambda,$$

and

(W) There exists a positive constant M such that

$$\|F(U)\| \leq M(1+\|U\|), \quad U \in C_\lambda.$$

The last condition prevents a blow-up and ensures F: $C_\lambda \to C_\lambda$. The main result is the following assertion.

Theorem. *Let* $u_0(\cdot, \omega) \in C_\lambda$, $\omega \in \Omega$. *Under the conditions* (K) *and* (W) *there exists a unique solution of the Cauchy problem* (C) *which can be obtained by successive approximation.*

Remarks. 1. Let H be the fundamental solution (Green's function) of the Dirichlet problem

$$\text{(D)} \quad \begin{cases} \frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + f(u(t, x)) + \sigma\zeta(t, x), & t > 0,\ x \in (0, 1). \\ u(0, x) = u_0(x), & x \in [0, 1], \\ u(t, 0) = u(t, 1) = 0, & t \geq 0, \end{cases}$$

for $\sigma = 0$, $f = 0$. Investigating (D) instead of (C), it remains to deal with $C([0, 1])$ instead of C_λ. The estimate in (K) can be replaced by $|f(u)-f(v)| \leq k(|u-v|)$, $u, v \in R$. Because of $0 \leq H(t, x) \leq G(t, x)$, $t > 0$, $x \in [0, 1]$, all estimations which are used in the proof of the theorem above can be obtained correspondingly in the Dirichlet case. Thus the theorem also holds for this problem.

2. The estimate in (K) can be derived from

$$|f(u)-f(v)| \leq k(|u-v|), \quad u, v \in \mathbf{R}$$

if k is in addition to the other conditions concave else. In fact, choose $U, V \in C_\lambda$, then from $k(0)=0$ follows

$$|f(u(x))-f(v(x))| \leq (\rho(x))^{-1} \cdot k(\rho(x)\,|u(x)-v(x)|),$$

and hence the desired inequality in (K). Finally, the growth condition (W) follows easily from $|f(u)| \leq M(1+|u|)$, $u \in \mathbf{R}$.

3. The uniqueness of a solution always means the pathwise uniqueness in C_λ.

3. Proofs. Define

$$V_t^0 := Z_t$$

and

$$V_t^n := V_t^0 + \int_0^t T_{t-s} F(V_s^{n-1})\,ds$$

for $t \geq 0$. Fix now $\omega \in \Omega$ such that $z \in C_\rho^T$. In order to prove the convergence of $(V^n)_{n \geq 1}$ the technique of T. Yamada [3] will be used. Introduce the notation

$$I_t(V) = \int_0^t T_{t-s} F(V_s)\,ds, \quad v \in C_\rho^T, \quad t \in [0, T].$$

3.1. *Suppose that $V \in C_\rho^T$ and that F satisfies* (W). *Then $I(V) \in C_\rho^T$.*

Proof. It holds

$$|(I_t(V))(x)| \leq M[T + \int_0^t \int_R G(t-s, x-y)\,|v(s, y)|\,dyds]$$

$$\leq M[T + c_0 \cdot \rho(x) \cdot \int_0^t \|V_s\|\,ds] \leq c_1 \cdot \rho(x).$$

Hence $\|I_t(V)\|_T < \infty$. It remains to show the continuity of $I(V)$. One observes

$$|I_t(V)(x_1) - I_t(V)(x_2)| \leq \|F(V)\|_T \cdot \int_0^t \int_{\mathbf{R}} |G(s, x_1-y) - G(s, x_2-y)|\,\rho(y)\,dyds.$$

The Hölder inequality implies that the last expression is smaller than

$$\|F(V)\|_T \left[\int_0^t \int_R [G(s, x_1-y) - G(s, x_2-y)]^2 dyds\right]^{1/4} \cdot Q_1(T, x_1, x_2), \tag{1}$$

where $\sup_{x_1, x_2 \in \mathscr{K}} Q_1(T, x_1, x_2) < \infty$ on every compact set $\mathscr{K}$ in $\mathbf{R}$. By [1], 3.2.1, it is known that the integral in (1) is dominated by $|x_1 - x_2|$. Analogously, it follows

$$|I_{t_1}(V)(x) - I_{t_2}(V)(x)| \leq c_3\,|t_1 - t_2|^{1/8} \cdot \rho(x). \tag{2}$$

Hence $I(V)$ is continuous. This proves 3.1.

3.2. *Suppose that $u_0 \in C_\rho^T$ and that F satisfies* (W). *Then $V^n \in C_\rho^T$. Moreover, it holds $\|V_t^n \leq c_4 \cdot \exp(c_5 t)$, where the constants c_4 and c_5 do not depend on n.*

Proof. This is a consequence of the lemma in Section 2 and

3.1. The stated estimate can be obtained by iterating backwards the inequality

$$\| V_t^m \| \leq \| V_t^0 \| + M[T + c_0 \cdot \int_0^t \| V_s^{m-1} \| ds].$$

3.3. *Suppose* $u_0 \in C_\lambda$ *and assume that F satisfies* (W). *Then* $\| V_s^n - V_s^0 \|_t \leq c_6 t$, *where the constant* c_6 *does not depend on n.*
This follows as in 3.1. The next assertion can be easily derived.

3.4. *Let* $U, V \in C_\rho^T$. *The conditions* (K) *implies*

$$\| I_s(V) - I_s(U) \|_t \leq c_7 \cdot \int_0^t k(\| U_s - V_s \|) ds.$$

Let n be arbitrary but fixed and introduce the following sequences, $t \geq 0$, $i = 0, 1, \ldots$

$$\Gamma_0(t) = c_6 t.$$

$$\Gamma_{i+1}(t) = \int_0^t k(\Gamma_i(s)) ds,$$

$$\Gamma_i^*(t) = \| V_s^{i+n} - V_s^i \|_t.$$

Furthermore, define $T_1 = \{t \in [0, T] : k(c_6 t) \leq c_6$ for all $s \in [0, t]\}$. The continuity of k and $k(0) = 0$ ensure that the set under the supremum is not empty.

3.5. *Under the conditions of the theorem it holds*

$$0 \leq \Gamma_j^*(t) \leq \Gamma_j(t) \leq \Gamma_{j-1}(t) \leq \ldots \leq \Gamma_0(t)$$

for any $t \in [0, T_1]$ *and any* $j = 1, 2, \ldots$

Proof. (i) Let $j = 0$. The assertion 3.3 shows

$$\Gamma_0^*(t) = \| V_s^n - V_s^0 \|_t \leq c_6 t = \Gamma_0(t), \quad t \in [0, T].$$

(ii) Let $j = 1$. By using 3.4 and (i) it follows

$$\Gamma_1^*(t) = \| V_s^{n+1} - V_s^1 \|_t \leq \int_0^t k(\| V_r^n - V_r^0 \|_s) ds = \int_0^t k(\Gamma_0^*(s)) ds \leq \int_0^t k(\Gamma_0(s)) ds = \Gamma_1(t).$$

In addition, one observes

$$\Gamma_1(t) = \int_0^t k(\Gamma_0(s)) ds = \int_0^t k(c_6 s) ds \leq c_6 t = \Gamma_0(t), \quad t \in [0, T_1].$$

(iii) Assume that the assertion is shown for $j = k - 1$. Put $j = k$. For $t \in [0, T_1]$ one gets

$$\Gamma_k^*(t) = \| V_s^{k+n} - V_s^k \|_t \leq \int_0^t k(\Gamma_{k-1}^*(s)) ds \leq \int_0^t k(\Gamma_{k-1}(s)) ds = \Gamma_k(t).$$

Moreover, one observes

$$\Gamma_k(t) \leq \int_0^t k(\Gamma_{k-2}(s)) ds \leq \Gamma_{k-1}(t).$$

This ends the proof of 3.5.

3.6. *It holds* $\lim_{m \to \infty} \| V_s^{m+n} - V_s^m \|_t = 0$ *for any* $n = 0, 1, \ldots$ *and any* $t \in [0, T_1]$.

Proof. From 3.5 one concludes that there exists a limit

$$\Gamma(t) := \lim_{m \to \infty} \Gamma_m(t) \geq \lim_{m \to \infty} \| V_s^{m+n} - V_s^m \|_t \geq 0,$$

for any $n=0,1,\dots$ and $t\in[0, T_1]$. In addition, it follows

$$\Gamma(t)=\lim_{m\to\infty}\Gamma_{m+1}(t)=\int_0^t k(\Gamma(s))\,ds.$$

Hence Γ is continuous on $[0, T_1]$, and $\Gamma(0)=0$. This yields $\Gamma=0$, what proves 3.6.

Let $T_2=\sup\{t\in[0, T]:\lim_{m\to\infty}\|V_s^{m+n}-V_s^m\|_t=0,\ n=0, 1,\dots\}$. Obviously, $T_2\geqq T_1$. Now it suffices to show $T_2=T$.

3.7. *It holds* $\lim_{m\to\infty}\|V_s^{m+n}-V_s^m\|_{T_2}=0,\quad n=0, 1,\dots$

Proof. Let $\varepsilon>0$ and choose $\delta\in(0, T_2\wedge 1)$. Clearly, there exists an m_0 such that $\|V_s^{m+n}-V_s^m\|_{T_2-\delta}<\varepsilon/4$ for all $m\geqq m_0$. On the other side one observes

$$\|V_s^{m+n}-V_s^m\|_{s\in[T_2-\delta,T_2]}\leqq\|I_t(V^{m+n-1})-I_{T_2-\delta}(V^{m+n-1})\|_{t\in[T_2-\delta, T_2]}$$

$$+\|I_{T_2-\delta}(V^{m+n-1})-I_{T_2-\delta}(V^{m-1})\|+\|I_{T_2-\delta}(V^{m-1})-I_t(V^{m-1})\|_{t\in[T_2-\delta, T_2]}.$$

The second expression on the r. h. s. of the last inequality is smaller than $\varepsilon/4$ by construction ($m\geqq m_0$). From (2) one gets the norm continuity of $I(V)$ in time. Consequently, for fixed $m\geqq m_0$ and sufficiently small $\delta>0$ it follows

$$\|V_s^{m+n}-V_s^m\|_{s\in[T_2-\delta, T_2]}\leqq 3/4\varepsilon.$$

This proves 3.7.

3.8. *It holds* $T_2=T$.

Proof. Assume $0<T_2<T$, that means there exists a $t>0$ such that $T_2+t<T$.
(i) Because of 3.7 one can find a sequence of positive real numbers $(a_m^{(n)})_{m\geqq 1}$ such that $a_m^{(n)}\to 0$ $(n=1, 2,\dots)$ and such that $\|V_s^{m+n}-V_s^m\|_{T_2}\leqq a_m^{(n)}$. One observes

$$\sup_{T_2\leqq s\leqq T_2+t}\|V_s^{m+n}-V_s^m\|\leqq\sup_{T_2\leqq s\leqq T_2+t}\|I_s(V^{m+n-1})-I_{T_2}(V^{m+n-1})\|$$

$$+\|I_{T_2}(V^{m+n-1})-I_{T_2}(V^{m-1})\|+\sup_{T_2\leqq s\leqq T_2+t}\|I_{T_2}(V^{m-1})-I_s(V^{m-1})\|\leqq a_m^{(n)}+c_7t^{1/8},$$

where (2) was used again. Because of 3.1 the constant c_7 does not depend on m.
(ii) Denote $T_2\cdot k(a_j^{(n)})$ by $b_j^{(n)}$ and choose an integer $l\in\{0, 1,\dots\}$ and a real number $\eta\in(0, (T-T_2)\wedge 1)$ such that

$$k(a_l^{(n)}+b_l^{(n)}+c_7t^{1/8})<c_7$$

holds for any $t\in[0, \eta]$. Introduce the sequence of functions $(\psi_{l+m}^{(n)})_{m=0,1,\dots}$ defined by

$$\psi_l^{(n)}(t)=a_l^{(n)}+b_l^{(n)}+c_7t^{1/8}$$

and

$$\psi_{l+m}^{(n)}(t)=b_{l+m}^{(n)}+\int_0^t k(\psi_{l+m-1}^{(n)}(s))ds.$$

Now it will be proved that

$$\psi_l^{(n)}(t)\geq\psi_{l+1}^{(n)}(t)\geq\dots\geq\psi_{l+m}^{(n)}(t)\tag{3}$$

and

$$\sup_{T_2\leqq s\leqq T_2+t}\|V_s^{m+n+1}-V_s^{m+1}\|\leq\psi_{m+1}^{(n)}(t),\quad t\in[0, \eta].$$

Let $m=1$. It holds

$$\sup_{T_2\leq s\leq T_2+t} \| V_s^{l+n+1}-V_s^{n+1}\| \leq b_l^{(n)}+\int_0^t k(\| V_{s+T_2}^{n+l}-V_{s+T_2}^{l}\|)ds$$

$$\leq b_l^{(n)}+\int_0^t k(a_l^{(n)}+c_7 s^{1/8})ds \leq \psi_{l+1}^{(n)}(t).$$

Furthermore, one observes

$$\psi_{l+1}^{(n)}(t)=b_l^{(n)}+\int_0^t k(\psi_l^{(n)}(s))ds \leq b_l^{(n)}+c_7 t \leq a_l^{(n)}+b_l^{(n)}+c_7 t^{1/8} \leq \psi_l^{(n)}(t), \quad t\in[0, \eta].$$

Now, assuming that (3) and (4) are proved for $m=k-1$ these inequalities will be shown now for $m=k$.

It holds

$$\sup_{T_2\leq s\leq T_2+t} \| V_s^{k+n+l+1}-V_s^{n+k+1}\| \leq b_{l+k}^{(n)}+\int_0^t k(\| V_{s+T_2}^{k+n+1}-V_{s+T_2}^{n+k+l-1}\|)ds$$

$$\leq b_{l+k}^{(n)}+\int_0^t k(\psi_{l+k}^{(n)}(s))ds=\psi_{l+k}^{(n)}(t),$$

and from the monotonicity of k follows

$$\psi_{l+k+1}^{(n)}(t)=b_{l+k+1}^{(n)}+\int_0^t k(\psi_{l+k+1}^{(n)}(s))ds \leq b_{l+k}^{(n)}+\int_0^t k(\psi_{l+k}^{(n)}(s))ds=\psi_{l+k}^{(n)}(t).$$

(iii) Obviously,

$$\psi^{(n)}(t):=\lim_{m\to\infty}\psi_{m+l}^{(n)}(t)$$

exists for any $t\in[0, \eta]$. Moreover, one observes

$$\psi^{(n)}(0):=\lim_{m\to\infty} b_{l+m}^{(n)}=0$$

and

$$\psi^{(n)}(t)=\int_0^t k(\psi^{(n)}(s))ds, \quad t\in[0, \eta].$$

Since $\psi^{(n)}$ is continuous on $[0, \eta]$ this yields $\psi^{(n)}=0$, $n=1, 2, \ldots$ Hence

$$0\leq \lim_{m\to\infty}\sup_{T_2\leq s\leq T_2+t}\| V_s^{m+n+1}-V_s^{m+1}\| \leq \lim_{m\to\infty}\psi_{l+m}^{(n)}(t)=0.$$

This contradicts to the definition of T_2. Hence one can conclude that $T=T_2$.

In this way it is proved that the sequence $(V_t^m)_{m\geq 0}$ converges in C_λ uniformly in $t\in[0, T]$. Denote this limit by V.

3.9. *The randon* field $v=(v(t, x))_{(t,x)\in D_T}$ *is a solution of* (C).

P r o o f. Clearly, V_t is $\mathscr{F}_t$-adapted. The continuity of v follows from the construction. Finally, one observes

$$|v(t, x)-v^0(t, x)-\int_0^t\int_R G(t-s, x-y)f(v(s, y))dyds| \leq |v(t, x)-v^n(t, x)|$$

$$+\int_0^t\int_R G(t-s, x-y)|f(v^n(s, y))-f(v(s, y))|dyds \leq \rho(x)[\| V_t-V_s^n\|$$

$$+c_8 \cdot \int_0^t \|F(V_s^n)-F(V_s)\|\, ds] \leqq \rho(x)[\|V_t-V_t^n\|+c_8 \int_0^t k(\|V_s^n-V_s\|)\,ds].$$

The last expression converges to zero uniformly in $t \in [0, T]$.

This proves 3.9.

Proof of the Theorem. It remains to show the uniqueness.

Let u and v be two solutions of (S), then

$$\|U_t-V_t\| \leqq c_8 \int_0^t k(\|U_s-V_s\|)ds, \quad t \in [0, T].$$

This implies $\|U_t-V_t\|_T=0$.

REFERENCES

1. R. Manthey. On the Cauchy Problem for Reaction-Diffusion Equations with White Noise. *Math. Nachr.*, **136**, 1988, 209-228.
2. R. Manthey. Reaktions-Diffussions-Gleichungen mit weißem Rauschen, Diss. B. Friedrich-Schiller-Universität Jena, 1988.
3. T. Yamada. On the successive approximation of solutions of stochastic differential equations. *J. Math. Kyoto Univ.*, **21**, 1981, 13, 501-515.

Friedrich-Schiller Universität
Mathematische Fakultät
Universitätshochhaus, 17. OG
06900 Jena, Germany

Received 4. 11. 1989