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STRUCTURAL PROPERTIES OF NONLINEAR MONOTONE OPERATORS WITH VALUES IN L (X, Y)

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Let X be a locally convex space and Y — an ordered locally convex space. A multivalued operator $T: X \to \mathbf{L}(X, Y)$ is called *monotone* if $(A_1-A_2)(x_1-x_2) \geqq 0$ for all $x_i \in D(T)$ and $A_i \in T(x_i)$, $i=1, 2$. In the scalar case $Y=\mathbf{R}$, this definition coincides with the well known definition of monotone operator of Browder (1965). Monotone operators with values in $\mathbf{L}(X, Y)$ have been studied by many authors Hadjisavvas et al. (1989), Jouak et al. (1985), Kirov (1983, 1985) and Kusraev (1978). This note contains sufficient and necessary conditions for a set $K \subset \mathbf{L}(X, Y)$ to be the image $T(x)$ of a monotone (resp. maximal monotone) operator T at an internal point x of its domain. In the case $Y=\mathbf{R}$, we show that K should be bounded (resp. w^*-closed, convex and bounded). For general Y, the situation is more complicated: K should be globally bounded (resp. globally bounded and a hereditary order interval).

1. Preliminaries. Throughout this note X will be a real locally convex Hausdorff space and Y a real locally convex Hausdorff space which is also an ordered linear space with closed positive cone Y_+. We denote by $\mathbf{L}(X, Y)$ the space of all linear and continuous operators from X into Y and by $\mathbf{L}_s(X, Y)$ the space $\mathbf{L}(X, Y)$ endowed with the topology of simple convergence. Let $F: X \to Y$ be a convex operator, i. e., such that $F(\lambda x+(1-\lambda)y) \leqq \lambda F(x)+(1-\lambda)F(y)$ for all $x, y \in X$ and $0 \leqq \lambda \leqq 1$. The epigraph of F is defined by epi $F=\{(x,y): y \geqq F(x), x \in X\}$. The subdifferential of F at x_0 is the set $\partial F(x_0)=\{A \in \mathbf{L}(X, Y): A(x-x_0) \leqq F(x)-F(x_0)$ for all $x \in X\}$. The convex operator F is said to be sublinear if it is positively homogeneous, i. e., $F(\lambda x)=\lambda F(x)$ for every $x \in X$ and $\lambda \geqq 0$. Whenever Y_+ is normal [10], a sublinear operator $F: X \to Y$ which is continuous at 0 is also continuous at every $x \in X$. This follows from the obvious inequality: $-F(x-y) \leqq F(y)-F(x) \leqq F(y-x)$.

A subset K of $\mathbf{L}(X, Y)$ is called weakly order bounded if the set $Kx=\{Ax: A \in K\}$ is order bounded for all $x \in X$. Whenever Y is order complete [4], to each weakly order bounded set K we associate the sublinear operator $P_K(x)=\sup\{Ax: A \in K\}$.

We call the multivalued operator $T: X \to \mathbf{L}(X, Y)$ with domain $D(T)$ monotone if $(A_1-A_2)(x_1-x_2) \geqq 0$ for all $x_i \in D(T)$ and $A_i \in T(x_i)$ $i=1, 2$. A monotone operator T is said to be maximal if its graph $G(T)=\{(x, A): x \in D(T), A \in T(x)\}$ is not properly contained in the graph of any other monotone operator. It is known that the subdifferential ∂F of a convex operator $F: X \to Y$ is monotone (and even maximal monotone, under suitable assumptions on X and Y [5]).

The cor$N$ for a subset N of X is the set of all internal points of N. We write $\langle x^*, x \rangle$ in place of $x^*(x)$ for $x \in X$ and $x^* \in X^*$. For more details on ordered topological vector spaces see [4, 10].

2. Main results. Theorem 1 [3]. *Let X be a Frechet space, Y a normed space with normal positive cone and $T: X \to \mathbf{L}(X, Y)$ be monotone. If $x_0 \in corD(T)$, then there exists such a neighborhood U of x_0 that the set $T(U)$ is equicontinuous.*

We shall also need a notion stronger than the weak order boundedness:

Definition 1. Let Y be a normed space which is order complete. A weakly order bounded subset K of $\mathbf{L}(X, Y)$ is called *globally bounded,* if there exists a neighborhood U of 0 in X such that $\sup_{x \in U} \|\sup_{A \in K} Ax\| < \infty$.

P r o p o s i t i o n 1. *Let X be a Frechet space and Y an order complete normed lattice. A set $K \subset \mathbf{L}(X,Y)$ is globally bounded if and only if there exists a monotone operator $T: X \rightarrow \mathbf{L}(X,Y)$ and $x_0 \in$ cor $D(T)$ such that $T(x_0)=K$.*

P r o o f. Let T be monotone and $x_0 \in \operatorname{cor} D(T)$. By Theorem 1, there exists a neighborhood U_1 of 0 in X such that $T(x_0+U_1)$ is equicontinuous. Hence, there exists a neighborhood U_2 of 0 in X such that $\|Bx\| \leq 1$ for all $x \in U_2$ and $B \in T(x_0+U_1)$. Now let $x \in U_1 \cap U_2$. Then there exists $\lambda > 0$ such that $x_0 \pm \lambda x \in D(T)$ and $\pm \lambda x \in U = U_1 \cap U_2$. For fixed $B_1 \in T(x_0+\lambda)$ and $B_2 \in T(x_0-\lambda x)$ the monotonicity of T implies $(B_1-A)(x_0+\lambda x-x_0) \geqq 0$ and $(B_2-A)(x_0-\lambda x-x_0) \geqq 0$ for all $A \in T(x_0)$. It then follows that $B_2 x \leqq Ax \leqq B_1 x$ for all $A \in T(x_0)$, so $B_2 x \leqq \sup_{A \in T(x_0)} Ax \leqq B_1 x$. Hence for all $x \in U$ $\| \sup_{A \in T(x_0)} Ax \| \leqq \|B_1 x\| + \|B_2 x\| \leqq 2$, i. e., $T(x_0)$ is globally bounded. Now let K be globally bounded. Then P_K is continuous at 0 and so at any $x \in X$. By [12, Th. 6] we have $\partial P_K(x) \neq \emptyset$ for all $x \in X$. Let $T: X \rightarrow \mathbf{L}(X,Y)$ be an operator defined by

$$T(x)=\begin{cases} K, & \text{if } x=0 \\ \partial P_K(x), & \text{if } x \neq 0. \end{cases}$$

Since $K \subset \partial P_K(0)$, the operator T is monotone as a restriction of the subdifferentia operator ∂P_K.

In some cases weak order boundedness and global boundedness are equivalent as shown by

P r o p o s i t i o n 2. *Let X be a Frechet space and Y — an order complete Banach lattice. Then every weakly order bounded subset K of $\mathbf{L}(X,Y)$ is globally bounded.*

P r o o f. Let $K \subset \mathbf{L}(X,Y)$ be weakly order bounded. The sublinear operator P_K has closed an epigraph, since obviously $\operatorname{epi} P_K = \bigcap_{A \in K} \operatorname{epi} A$ and $\operatorname{epi} A$ is closed by the continuity of A. Hence according to a result of B o r w e i n [1] P_K is continuous. Thus there exists a neighborhood U of 0 in X such that $\|P_K(x)\| \leqq 1$ for all $x \in U$, which means exactly that K is globally bounded. For the study of images of maximal monotone operators we shall also need the following definition.

D e f i n i t i o n 2. *Let Y be order complete. If $K \subset \mathbf{L}(X,Y)$, we define the set $[K]^H$ by $[K]^H = \{B \in \mathbf{L}(X,Y) : Bx \leqq \sup_{A \in K} Ax \ \forall\, x \in X$ for which the sup exists). It is obvious that $[K]^H$ is s-closed and convex.*

We shall call K *hereditary-order interval* (briefly, HOI), if $[K]^H = K$. When K is weakly order bounded, then $[K]^H = \partial P_K(0)$. Hence, in this case K is HOI iff $K = \partial P_K(0)$.

P r o p o s i t i o n 3. *Let Y be order complete and $T: X \rightarrow \mathbf{L}(X,Y)$ be a maximal monotone operator. Then $T(x)$ is* HOI *for all $x \in D(T)$.*

P r o o f. Let $x_0 \in D(T)$ and $A_0 \in [T(x_0)]^H$. For each $(x,B) \in G(T)$ and $A \in T(x_0)$ we have $(B-A)(x-x_0) \geqq 0 \Rightarrow B(x-x_0) \geqq A(x-x_0)$. Therefore, $B(x-x_0) \geqq \sup_{A \in T(x_0)} A(x-x_0) \geqq A_0(x-x_0)$, which implies $(B-A_0)(x-x_0) \geqq 0$ for all $(x,B) \in G(T)$. By the maximality of T we conclude $A_0 \in T(x_0)$.

P r o p o s i t i o n 4. *Let Y be a normed space which is an order complete lattice with normal positive cone. If $K \subset \mathbf{L}(X,Y)$ is* HOI *and globally bounded, then for each $x \in X$ the set Kx is an order interval of Y.*

P r o o f. Since K is HOI, we have $K = \partial P_K(0)$. As in the proof of Proposition 1, P_K is continuous. Then it follows from [12, Th. 6] and [9, p. 187] that Kx is the interval $[-P_K(-x), P_K(x)]$. From Proposition 1 we have that the images of monotone operators at internal points of $D(T)$ are globally bounded. As in the scalar case [11], this does no longer hold, in general, for other points of $D(T)$ as shown by

P r o p o s i t i o n 5. *Let $T: X \rightarrow \mathbf{L}(X,Y)$ be a maximal monotone operator. Suppose that int $(\overline{co} D(T)) \neq \emptyset$ and x_0 is a point of $D(T)$ not belonging to this interior. Then $T(x_0)$ contains at least one half-line.*

P r o o f. Since int $(\overline{\text{co}}D(T))\neq\emptyset$ and x_0 is a boundary point of $\overline{\text{co}}D(T)$, there exists a supporting hyperplane to $\overline{\text{co}}D(T)$ at x_0. Hence there exists $x^*\in X^*$ such that $\langle x^*,x_0\rangle\geqq\langle x^*,x\rangle$ for all $x\in D(T)$. Define $A_1\in\mathbf{L}(X,Y)$ by $A_1x=\langle x^*,x\rangle y$, where y is an element of $Y_+\setminus\{0\}$. Let A_0 be an element of $T(x_0)$ and $\lambda\geqq0$. Then for any $x\in D(T)$ and $A\in T(x)$ we have $(A_0+\lambda A_1-A)(x_0-x)=(A_0-A)(x_0-x)+\lambda A_1(x_0-x)\geqq0$.

By the maximality of T we get that $A_0+\lambda A_1\in T(x_0)$ for all $\lambda\geqq0$. As it follows from Proposition 1 and 3, the image $T(x)$ of a maximal monotone operator T at an internal point of $D(T)$ is HOI and globally bounded (under suitable conditions for X and Y). The converse also holds:

P r o p o s i t i o n 6. *Let Y be a normed space which is an order complete lattice with normal positive cone and K a globally bounded* HOI *subset of* $\mathbf{L}(X,Y)$. *Then there exists such a maximal monotone operator $T:X\to\mathbf{L}(X,Y)$ that $D(T)=X$ and $T(0)=K$.*

P r o o f. Setting $T=\partial P_K$ we have $T(0)=K$. Since K is globally bounded, we conclude as in the proof of Proposition 1 that P_K is continuous. Hence, by theorem 6 of [12] $\partial P_K(x)\neq\emptyset$, for all $x\in X$. Finally, by [5, Prop. 2. 7], T is maximal monotone.

R e m a r k 2. As an illustration of the above results we consider the case when X is a Banach space and $Y=\mathbf{R}$. Then a subset K of $\mathbf{L}(X,\mathbf{R})=X^*$ is globally bounded iff it is bounded.

Furthermore, a set $K\subset X^*$ is HOI iff it is w^*-closed and convex. Indeed, if K is w^*-closed and convex and $x_0^*\notin K$ then by the Hahn — Banach theorem there exists $x_0\in X$ such that $\sup\{\langle x^*,x_0\rangle : x^*\in K\}<\langle x_0^*, x_0\rangle$ which implies that $x_0^*\notin[K]^H$, i. e. K is HOI. The converse is obvious. Now from the previous propositions we deduce that a set $K\subset X^*$ is bounded (resp. bounded, w^*-closed and convex) if and only if it is the image $T(x)$ of a monotone (resp. maximal monotone) operator T, at an internal point x of its domain.

REFERENCES

1. J. M. B o r w e i n. A Lagrange multiplier theorem and a sandwich theorem for convex relations *Math. Scand.*, **48**, 1981, 189-204.
2. F. E. B r o w d e r. Multivalued monotone mappings and duality mappings in Banach spaces. *Trans. Amer. Math. Soc.*, **118**, 1965, 338-351.
3. N. H a d j i s a v v a s, D. K r a v v a r i t i s, G. P a n t e l i d i s, I. P o l y r a k i s. Nonlinear monotone operators with values in $\mathbf{L}(X,Y)$, *J. Math. Anal. Appl.*, **140**, 1989, 83-94.
4. G. J a m e s o n. Ordered linear spaces. *Lecture Notes*, **141**, 1970.
5. M. J o u a k, L. T h i b a u l t. Monotonie generalisee et sousdifferentiels de fonctions convexes vectorielles. *Optimization*, **16**, 1985, 187-199.
6. N. K. K i r o v. Generalized monotone mappings and differentiability of vector-valued convex mappings. *Serdica*, **9**, 1983, 263-274.
7. N. K. K i r o v. Generic Frechet differentiability of convex operators. *Proc. Amer. Math. Soc.*, **94**. 1985, 97-102.
8. A. G. K u s r a e v. Subdifferential mappings of convex operators. *Optimizacija*, 38, 1978, 36-40,
9. S. S K u t a t e l a d z e. Convex operators. *Russian Math. Surveys*, **34**, 1979, 181-214.
10. A. L. P e r e s s i n i. Ordered topological vector spaces., N. Y., 1967.
11. R. T. R o c k a f e l l a r. Local boundedness of nonlinear monotone operators. *Michigan J. Math.*, **16**, 1969, 397-407.
12. M. V a l a d i e r. Sous-differentiabilite de fonctions convexes à valeurs dans un espace vectoriel ordonne. *Math. Scand.*, **30**,1974, 65-74.

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