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BEST ONESIDED APPROXIMATION WITH ALGEBRAIC POLYNOMIALS

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Hristov (1989) used a locally global norm (see also Wickeren (1989)) for bounded functions and proved that the best onesided approximation (constrained approximation) of a 2π-periodic bounded function with trigonometric polynomials of degree n in the norm L_p is equivalent to the best approximation (without constraints) with trigonometric polynomials of degree n.

In this paper we prove the equivalent proposition with respect to algebraic polynomials.

1. Notations. Let d be an integer and $1\leqq p\leqq\infty$, f be a bounded measurable real valued function defined on Ω, $\Omega=[-1, 1]^d\subset R^d$, R^d is normed by space with elements $x, y, h, x=(x_1, x_2, \dots, x_d)$ and

$$\|x\|=\max\{|x_s|: s=1, 2, \dots, d\}.$$

Let further X be a measurable subset of Ω. We shall use the following notations

$$L_p(X)=\{f: \|f\|_p=\|f\|_{p(X)}=(\int_X |f(x)|^p dx)^{1/p}<\infty\},\quad 1\leqq p\leqq\infty.$$

$$L_\infty(X)=\{f: \|f\|_\infty=\|f\|_{\infty(X)}=\sup\{|f(x)|: x\in X\}<\infty\}.$$

For $u\in[-1, 1]$, $\delta>0$, $\psi(\delta, u):=\Phi(u)+\delta^2$, where $\Phi(u)=\sqrt{1-u^2}$. A δ-neighbourhood of the point $u\in[-1, 1]$ is defined by

$$U(\delta, u):=\{y\in[-1, 1]: |u-y|\leqq\psi(\delta, u)\}.$$

If $\alpha=(\alpha_1, \alpha_2, \dots, \alpha_d)$ is a multi-index, we denote by $D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2}\dots D_d^{\alpha_d}$ the differential operator in R^d ([8, p. 140]), where $D_s^{\alpha_s}=\partial^{\alpha_s}/\partial x_s^{\alpha_s}$, $s=1, 2, \dots, d$.

For $x\in\Omega$, $\delta>0$, $\alpha:\alpha_s=0, 1$, denote

$$\Psi(\delta, x)^\alpha:=\prod_{\{s:\alpha_s=1\}}\psi(\delta, x_s);\quad \Psi(\delta, x):=\prod_{s=1}^{d}\psi(\delta, x_s)$$

and define a δ-neighbourhood of $x\in\Omega$ by

$$U(\delta, x)^\alpha:=\prod_{\{s:\alpha_s=1\}}U(\delta, x_s);\quad U(\delta, x):=U(\delta, x_1)\dots U(\delta, x_d).$$

We also define

(1) $$\|f\|_{\delta, p}=\|f\|_{\delta, p(X)}=\|\,\|f\|_{\infty(U(\delta, 0))}\|_{p(X)}=\|f_\delta\|_{p(X)},\ \text{where}$$

(2) $$f_\delta(x)=\sup\{|f(t)|: t\in U(\delta, x)\}.$$

Let N be a fixed natural number and let us set

$$Z=\{0, 1, \dots, N-1\}^d,\quad Z'=\{0, 1, \dots, N\}^d,$$

$$z_\nu=\cos(\pi-\nu\pi/N),\quad \nu=0, 1, \dots, N,\ z_{-1}=z_0=-1,\quad z_{N+1}=z_N=1.$$

For every $j\in Z$, $j=(j_1, j_2, \dots, j_d)$, we denote

$$\Omega_j=[z_{j_1}, z_{j_1+1}]\times\dots\times[z_{j_d}, z_{j_d+1}]\ \text{and for every}$$

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$j \in Z'$, we denote

$$\Omega'_j = [z_{j_1-1}, z_{j_1+1}] \times \cdots \times [z_{j_d-1}, z_{j_d+1}].$$

By H_n^d we denote the set of all algebraic polynomials in R^d of degree not greater than n.

The best approximation of a function $f \in L_p(\Omega)$ with algebraic polynomials from H_n^d in the metric of the space L_p is given by

$$E_n(f)_p = \inf \{\|f-P\|_{p(\Omega)} : P \in H_n^d\},$$

and the best approximation of a function $f \in L_{\delta, p}(\Omega)$ with polynomials from H_n^d in the metric (1) is given by

$$E_n(f)_{\delta, p} = \inf \{\|f-P\|_{\delta, p(\Omega)} : P \in H_n^d\}.$$

The best onesided approximation of a function $f \in L_\infty(\Omega)$ with polynomials from H_n^d in the metrics of the spaces L_p or $L_{\delta, p}$ are respectively given by

$$\widetilde{E}_n(f)_p = \inf \{\|P^+ - P^-\|_{p(\Omega)} : P^\pm \in H_n^d,\ P^-(x) \leqq f(x) \leqq P^+(x),\ x \in \Omega\},$$

$$\widetilde{E}_n(f)_{\delta, p} = \inf \{\|P^+ - P^-\|_{\delta, p(\Omega)} : P^\pm \in H_n^d,\ P^-(x) \leqq f(x) \leqq P^+(x),\ x \in \Omega\}.$$

2. Assertions. Let N be a fixed integer. For $\nu = 0, 1, \ldots, N-1$ set

$$u_\nu = \pi - (2\nu+1)\pi/2N \quad \text{and}$$

$$\Phi_\nu = \Phi_\nu(u) = \sin^4 \frac{\pi}{4N} \left(\frac{\sin^4 N(u-u_\nu)}{\sin^4 (u-u_\nu)/2} + \frac{\sin^4 N(u+u_\nu)}{\sin^4 (u+u_\nu)/2} \right),$$

Φ_ν are even positive trigonometric polynomials of degree $4N-2$ such that $\Phi_\nu(u) \geqq 1$ for $u \in [\pi-(\nu+1)\pi/N, \pi - \nu\pi/N]$. Let $u = \arccos v$, $v \in [-1, 1]$. Then we have

$$F_\nu(v) = F_{\nu, N}(v) = \Phi_\nu(\arccos v).$$

In a multivariate case, we define

$$\Phi_{j,N}(x) = \prod_{s=1}^{d} F_{j_s, N}(x_s).$$

L e m m a 1. [3]. *Let* $j \in Z$, *then*

$$\Phi_{j,N} \in H^d_{d(4N-2)}, \quad \Phi_{j,N} \geqq 0; \tag{3}$$

$$\Phi_{j,N}(x) \geqq 1 \quad \text{for} \quad x \in \Omega_j. \tag{4}$$

L e m m a 2. [3]. *Let* $a_j \geqq 0$, $j \in Z$, *then*

$$\|\sum_{j \in Z} a_j \Phi_{j,N}\|_{p(\Omega)} \leqq C(\sum_{j \in Z} a_j^p \operatorname{meas} \Omega_j)^{1/p}. \tag{5}$$

L e m m a 3. [3]. *Let* $x \in \Omega$, $N = [2\pi/t]+1$ *and* $0 \leqq t \leqq \frac{1}{2}$, *then*

$$\Omega'_j \subset U(t, x) \quad for \quad x \in \Omega'_j. \tag{6}$$

Let $A_0 \supset A_1$ be two quasinormed spaces with norms $\|\cdot\|_{A_0}$ and $\|\cdot\|_{A_1}$. Let us define Peetre K-functional (see e. g. [6] p. 54) for this couple ($f \in A_0$, $t > 0$) as follows

$$K(f, t; A_0, A_1) = \inf \{\|f-g\|_{A_0} + t\|g\|_{A_1} : g \in A_1\}.$$

The following lemma asserts that the space $L_{\delta, p}$ for a fixed δ possesses the interpolation property. Thus an analogue of Riesz—Thorin theorem (see e. g. [6] p. 10) holds for the space $L_{\delta, p}$ (δ-fixed).

L e m m a 4. *Let* $f \in L_\infty(\Omega)$ *and* $\delta > 0$, *then*

$$K(f, t; L_{\delta, p}, L_{\delta, \infty}) = K(f, t; L_p, L_\infty) \sim \left(\int_0^{t^p} (f)^*(s)^p ds\right)^{1/p},$$

with equivalence constants depending only on d and p, where $f_\delta(x)$ *is defined in* (2) *and* g^* *denotes the non-increasing rearrangement of the function g (see* [6] p. 10).

Proof. The proof of this lemma is as given in [2].

Applying a real interpolation method to the couple $(L_{\delta,p_0}, L_{\delta,p_1})$ $1\le p_0\le p_1\le\infty$ and using the reiteration theorem (see [6] p. 66, 144) and lemma 4, we find that an analogue of the Riesz—Thorin theorem holds for the space $L_{\delta,p}$ with different p's.

Next let us give the interpolation property with respect to the space $L_{\delta,p}$.

Theorem A. *Let T be a linear operator from* L_∞ *to* L_∞ *(or to* L_1*) such that for some* δ

$$\|Tf\|_{\delta,1}\le M_1\|f\|_{\delta,1}\quad (or\ \|Tf\|_1\le M_1\|f\|_{\delta,1}),$$

and

$\|Tf\|_\infty\le M_2\|f\|_\infty$. *Then for every* $1\le p\le\infty$, $T: L_{\delta,p}\longrightarrow L_{\delta,p}$ *(or* L_p*) and*

$$\|Tf\|_{\delta,p}\le M_1^{1/p}M_2^{1-1/p}\|f\|_{\delta,p}\ (or\ \|Tf\|_p\le M_1^{1/p}M_2^{1-1/p}\|f\|_{\delta,p}).$$

Lemma 5. [5]. *Let* $\varepsilon\le 1/4$, *then* $\forall t\in[4\varepsilon, \pi-4\varepsilon]$, *we have*

$$[\cos t-\psi(\varepsilon,\cos t),\ \cos t+\psi(\varepsilon,\cos t)]\subset[\cos(t+4\varepsilon),\ \cos(t-4\varepsilon)]. \tag{7}$$

Lemma 6. *Let m and* δ *be numbers such that* $m\delta\le 1/4$. *Then for* $f\in L_\infty(\Omega)$ *we have*

$$\|f\|_{p(\Omega)}\le\|f\|_{\delta,p(\Omega)}\le\|f\|_{\delta,\infty(\Omega)}\le\|f\|_{\infty(\Omega)}, \tag{8}$$

$$\|f\|_{m\delta,p(\Omega)}\le C_d m^{2d/p}\|f\|_{\delta,p(\Omega)}. \tag{9}$$

Proof. The inequality (8) follows immediately from the definitions of the norms. The inequality (9) is obvious (as an equality) for $p=\infty$. Therefore, in view of the interpolation property of the space $L_{\delta,p}$, the validity of (9) for every p will follow from its validity for $p=1$ (see theorem A with T-identity). So we shall prove

$$\|f\|_{m,1(\Omega)}\le C_d m^{2d}\|f\|_{\delta,1(\Omega)}.$$

Now let $d=1$ and

$$[-1,1]=J_1\cup J_2\cup J_3=\{x: x=\cos t,\ t\in[4m\delta, \pi-4m\delta]\}$$
$$\cup\{x: x=\cos t,\ 0\le t\le 4m\delta\}\cup\{x: x=\cos t,\ t\in[\pi-4m\delta,\pi]\}.$$

Then

$$\|f\|_{m\delta,1([-1,1])}=\Big(\int_{J_1}+\int_{J_2}+\int_{J_3}\Big)\|f\|_{\infty(U(m\delta,x))}\,dx=A_1+A_2+A_3. \tag{10}$$

Let x be fixed, $x\in J_1$ and let t be chosen so that $x=\cos t$, $t\in[0,\pi]$. Hence by (7), we obtain

$$U(m\delta,x)=[x-\psi(m\delta,x),\ x+\psi(m\delta,x)]\subset[\cos(t+4m\delta),\ \cos(t-4m\delta)]$$
$$=\bigcup_{k=-4m+1}^{4m}[\cos(t+(2k-1)\delta/2+\delta/2), \cos(t+(2k-1)\delta/2-\delta/2)].$$

Let $\xi_k=\cos(t+(2k-1)\delta/2)=\cos(\arccos x+(2k-1)\delta/2)$, $k=-4m+1,\ -4m+2, \ldots, 4m$.

Let us consider the interval

$$[\cos(t+(2k-1)\delta/2+\delta/2),\ \cos(t+(2k-1)\delta/2-\delta/2)]=I_k.$$

We have

$I_k\subset[-1,1]$ where $t\in[4m\delta, \pi-4m\delta]$, $k=0, \pm1, \ldots, \pm4m$,

$$|I_k|=\cos(t+(2k-1)\delta/2-\delta/2)-\cos(t+(2k-1)\delta/2+\delta/2)$$
$$=2\{\sin(t+(2k-1)\delta/2)\}\sin\delta/2\le\delta\sqrt{1-\cos^2(t+(2k-1)\delta/2)}$$
$$\le\delta\sqrt{1-\xi_k^2}\le\psi(\delta,\xi_k).$$

Thus $I_k \subset U(\delta, \xi_k)$.
Therefore

(11)
$$U(m\delta, x) \subset \bigcup_{k=-4m+1}^{4m} I_k \subset \bigcup_{k=-4m+1}^{4m} U(\delta, \xi_k), \text{ where}$$
$$\xi_k = \xi_k(x) = \cos(\arccos x + (2k-1)\delta/2).$$

From (11), we have

$$A_1 = \int_{J_1} \|f\|_{\infty\,(U(m\delta,\,x))} dx \leq \sum_{k=-4m+1}^{4m} \int_{J_1} \|f\|_{\infty\,(U(\delta,\,\xi_k(x)))} dx.$$

Let k be fixed, $(k=-4m+1, -4m+2, \ldots, 4m)$ and let us denote by $y=\xi_k(x)$ $=\xi(x)=\cos(\arccos x+(2k-1)\delta/2)=\xi(t)=\cos(t+(2k-1)\delta/2)$ and let $x=\eta_k(y)=\eta(y)$ be the converse function of $\xi(x)$ such that $\eta_k(\xi_k(x))=x$. So let $x=\eta(y)$.
Thus

$$\int_{J_1} \|f\|_{\infty(U(\delta,\,\xi_k(x)))} dx = \int_{J_1'} \|f\|_{\infty\,(U(\delta,\,y))} \eta'(y)\, dy,$$

where $J_1' \subset [-1, 1]$ is the mapping of J_1 after that. We have

$$\xi'(x) = \frac{d\xi(t)}{dt} \cdot \frac{1}{\frac{dx}{dt}}\Bigg|_{t=\arccos x} = \frac{-\sin(t+(2k-1)\delta/2)}{-\sin t}$$

$$\begin{aligned}\xi'(x) &= (\sin(t+(2k-1)\delta/2))/\sin t \\ &\geq \min\{(\sin(t+(2k-1)\delta/2))/\sin t : 4m\delta \leq t \leq \pi - 4m\delta\} \\ &= (\sin(t+(2k-1)\delta/2))/\sin t\,|_{t=4m\delta} \\ &= (\sin(4m\delta+(2k-1)\delta/2))/\sin(4m\delta) \geq 1/(4m\delta).\end{aligned}$$

Hence

$\eta'(y) = 1/\xi'(x) \leq 1/(1/4m\pi) = 4m\pi$. Using $J_1' \subset [-1, 1]$, we get

$$\int_{J_1'} \|f\|_{\infty\,(U(\delta,\,y))} \eta'(y)\, dy \leq \int_{-1}^{1} \|f\|_{\infty\,(U(\delta,\,y))} 4\pi m\, dy = 4\pi m \|f\|_{\delta,\,1} \text{ and so}$$

(12)
$$A_1 \leq \sum_{k=-4m+1}^{4m} \int_{J_1} \|f\|_{\infty\,(U(\delta,\,\xi_k(x)))} dx \leq \sum_{k=-4m+1}^{4m} 4\pi m \|f\|_{\delta,1} = 32\pi m^2 \|f\|_{\delta,\,1}.$$

Now let us consider

$$A_2 = \int_{J_2} \|f\|_{\infty\,(U(m\delta,\,x))} dx.$$

We have

$$\psi(\delta, y) = \delta\sqrt{1-y^2} + \delta^2 \geq \delta^2, \quad y \in [-1, 1];$$
$$\begin{aligned}\psi(m\delta, x) &= m\delta\sqrt{1-x^2} + (m\delta)^2 = m\delta \sin t + (m\delta)^2 \\ &\leq m\delta 4m\delta + (m\delta)^2 = 5(m\delta)^2,\end{aligned}$$

for every x such that $x = \cos t$, $t \in [0, 4m\delta]$.
Hence.

$$\begin{aligned}U(m\delta, x) &= [x-\psi(m\delta, x), x+\psi(m\delta, x)] \cap [-1, 1] \subset [x-5m^2\delta^2, x+5m^2\delta^2] \cap [-1, 1] \\ &\subset \bigcup_{k=-5m^2+1}^{5m^2-1} [x+k\delta^2-\delta^2, x+k\delta^2+\delta^2] \cap [-1, 1]\end{aligned}$$

$$\subset \bigcup_{k=-5m^2+1}^{5m^2-1} [x+k\delta^2-\psi(\delta,\ x+k\delta^2),\ x+k\delta^2+\psi(\delta,\ x+k\delta^2)] \cap [-1,\ 1]$$
$$= \bigcup_{k=-5m^2+1}^{5m^2-1} U(\delta,\ x+k\delta^2).$$

Let us replace $x+k\delta^2$ by y in every integral. Thus we obtain

(13) $$\int_{J_2} \| f \|_{\infty\,(U(m\delta,\ x))} dx \leq \sum_{k=-5m^2+1}^{5m^2-1} \int_{J_2} \| f \|_{\infty (U(\delta,\ x+k\delta^2))}\, dx$$
$$\leq \sum_{k=-5m^2+1}^{5m^2-1} \int_{-1}^{1} \| f \|_{\infty\,(U(\delta,\ y))}\, dy \leq 10m^2 \| f \|_{\delta,\ 1}.$$

Similarly we get

(14) $$A_3 = \int_{J_3} \| f \|_{\infty (U(m\delta,\ x))}\, dx \leq 10m^2 \| f \|_{\delta,\ 1}.$$

Thus from (10), (12), (13) and (14), we have

$$\| f \|_{m\delta,\ 1\,([-1,\ 1])} \leq (32\pi+10+10)\, m^2 \| f \|_{\delta,\ 1([-1,\ 1])} = Cm^2 \| f \|_{\delta,\ 1\,[(-1,\ 1)]}.$$

From the last inequality and theorem A, we obtain (9) for $d=1$. Next we get (9) for $d>1$ by mathematical induction with respect to d. So

$$\| f \|_{m\delta,\ 1\,(\Omega)} \leq C_d m^{2d} \| f \|_{\delta,\ 1\,(\Omega)}.$$

Thus the proof of the lemma is completed.

Lemma 7. *Let $P \in H_n^d$, then*

(15) $$\| P \|_{\delta,\ p\,(\Omega)} \leq C_d (1+\max(n\delta,\ n^2\delta^2))^{d/p} \| P \|_{p(\Omega)},\ 1 \leq p \leq \infty.$$

Proof. For $p=\infty$ this statement is obvious (see (8)). We shall prove it for $p=1$. For $x \in \Omega$ we denote by $\xi_x \in U(\delta,\ x)$ such that $\sup\{| P(y) | : y \in U(\delta,\ x)\} = | P(\xi_x) |$.

Using the representation of the difference $P(\xi_x)-P(x)$ from [7] p. 144 (107) or a similar representation from [9] and using lemmas 3, 4 from [5] we get

$$| \| P \|_{\delta,\ 1\,(\Omega)} - \| P \|_{1\,(\Omega)} | \leq \int_{\Omega} | P(\xi_x) - P(x) |\, dx$$
$$\leq \int_{\Omega} \sum_{\substack{|\alpha| \geq 1 \\ \alpha_s = 0,1}} \int_{U(\delta,\ x)^\alpha} | D^\alpha P(x^{(1-\alpha)} + u^{(\alpha)})\, du^{(\alpha)} dx$$
$$\leq \sum_{\substack{|\alpha| \geq 1 \\ \alpha_s = 0,1}} \int_{\Omega} \frac{1}{| U(\delta,\ x)^\alpha |} \int_{U(\delta,\ x)^\alpha} (\delta\Phi(u^{(\alpha)}) + \delta^2)^\alpha$$
$$\cdot | D^\alpha P(x^{(1-\alpha)} + u^{(\alpha)}) |\, du^{(\alpha)} dx$$
$$\leq C_d \sum_{\substack{|\alpha| \geq 1 \\ \alpha_s = 0,1}} \int_{\Omega} | D^\alpha P(x) | (\delta\Phi(x) + \delta^2)^\alpha dx$$
$$= C_d \sum_{\substack{|\alpha| \geq 1 \\ \alpha_s = 0,1}} \| (\delta\Phi + \delta^2)^\alpha D^\alpha P \|_{1\,(\Omega)}$$
$$\leq C_d \sum_{\substack{|\alpha| \geq 1 \\ \alpha_s = 0,1}} (\delta^{|\alpha|} \| \Phi^\alpha D^\alpha P \|_{1\,(\Omega)} + \delta^{2|\alpha|} \| D^\alpha P \|_{1\,(\Omega)})$$
$$\leq C_d \sum_{\substack{|\alpha| \geq 1 \\ \alpha_s = 0,1}} (\max(n\delta,\ n^2\delta^2))^{|\alpha|} \| P \|_{1\,(\Omega)}$$
$$= C_d ((1+\max(n\delta,\ n^2\delta^2))^d - 1) \| P \|_{1\,(\Omega)}.$$

That is

$$\| P \|_{\delta,1\,(\Omega)} \leq C_d (1+\max(n\delta,\ n^2\delta^2))^d \| P \|_{1\,(\Omega)}.$$

Corollary. *For each polynomial* $P\in H_n^d$ *we have*

(16) $$\|P\|_{p(\Omega)}\leq\|P\|_{1/n,\,p(\Omega)}\leq C_d\|P\|_{p(\Omega)},\quad 1\leq p\leq\infty.$$

Proof. The proof follows immediately from (8) and (15).

Now let R_n be the algebraic polynomial of degree n ($R_n\in H_n^d$) such that

$$\|f-R_n\|_{1/n,\,p(\Omega)}=E_n(f)_{1/n,\,p(\Omega)}.$$

We define for $N=[n/4d]$

(17) $$Q_n^{\pm}(f,x)=R_n(f,x)\pm\sum_{j\in Z}\Phi_{j,N}(x)\|f-R_n\|_{\infty(\Omega_j)}.$$

It is clear that $Q_n^{\pm}(f,x)$ are algebraic polynomials of degree not greater than n.

Lemma 8. *For any bounded measurable function* f *in* Ω, *we have*

$$Q_n^-(f,x)\leq f(x)\leq Q_n^+(f,x)\quad\text{for any } x\in\Omega.$$

Proof. Let $x\in\Omega$. Then for some $j\in Z$, $x\in\Omega$ from (4) and (3), we have

$$Q_n^+(f,x)=R_n(f,x)+\sum_{j\in Z_f}\Phi_{N}(x)\|f-R_n\|_{\infty(\Omega_j)}\geq R_n(f,x)+\|f-R_n\|_{\infty(\Omega_j)}$$
$$\geq R_n(f,x)+|f(x)-R_n(f,x)|=f(x).$$

We can prove that $Q_n^-(f,x)\leq f(x)$ in a similar way. Thus the proof of the lemma is completed.

3. Main results. We shall find the relationship between the quantities $\widetilde{E}_n(f)_p$ and $E_n(f)_{1/n,\,p}$ for $f\in L_\infty(\Omega)$.

Theorem 1. *For every* $f\in L_\infty(\Omega)$. *We have*

(18) $$\widetilde{E}_n(f)_p\leq C_dE_n(f)_{1/n,\,p}\leq C_d\widetilde{E}_n(f)_p,\quad 1\leq p\leq\infty.$$

Proof. Let P_n^+, $P_n^-\in H_n^d$ be such that $P_n^-(x)\leq f(x)\leq P_n^+(x)$ for every $x\in\Omega$ and

$$\widetilde{E}_n(f)_p=\|P_n^+-P_n^-\|_{p(\Omega)},$$

then using (16), we get

$$E_n(f)_{1/n,\,p}\leq\widetilde{E}_n(f)_{1/n,\,p}\leq\|P_n^+-P_n^-\|_{1/n,\,p(\Omega)}\leq C_d\|P_n^+-P_n^-\|_{p(\Omega)}=C_d\widetilde{E}_n(f)_p.$$

Now, we shall prove that

$$\widetilde{E}_n(f)_p\leq C_dE_n(f)_{1/n,\,p}.$$

For this purpose, we shall use the polynomials $Q_n^{\pm}(f,x)$ which are given in (17) and from (5), (6) and (9), we will get

$$\widetilde{E}_n(f)_p\leq\|Q_n^+-Q_n^-\|_{p(\Omega)}=2\|\sum_{j\in Z}\Phi_{j,N}(x)\|f-R_n\|_{\infty(\Omega_j)}\|_{p(\Omega)}$$
$$\leq C(\sum_{j\in Z}\operatorname{meas}\Omega_j\|f-R_n\|^p_{\infty(\Omega_j)})^{1/p}\leq C(\sum_{\in Z}\int_{\Omega_j}\|f-R_n\|^p_{\infty(\Omega_j)}dx)^{1/p}$$
$$\leq C(\sum_{j\in Z}\int_{\Omega_j}\|f-R_n\|^p_{\infty(U(2\pi/(N-1),\,x))}dx)^{1/p}$$
$$\leq C(\int_{\Omega}\|f-R_n\|^p_{\infty(U(2\pi/(N-1),\,x))}dx)^{1/p}$$
$$\leq C_d\|f-R_n\|_{1/n,\,p}=C_dE_n(f)_{1/n,\,p}.$$

Thus the proof of the theorem is completed.

Now let us express the relationship between $E_n(f)_p$ and $\widetilde{E}_n(F)_p$ for $f\in L_\infty(\Omega)$, $F=f$ a. e., F is a continuous function in Ω.

Corollary. *For each polynomial* $P\in H_n^d$ *we have*

(16) $$\|P\|_{p(\Omega)}\leq\|P\|_{1/n,\,p(\Omega)}\leq C_d\|P\|_{p(\Omega)},\ 1\leq p\leq\infty.$$

Proof. The proof follows immediately from (8) and (15).
Now let R_n be the algebraic polynomial of degree n ($R_n\in H_n^d$) such that

$$\|f-R_n\|_{1/n,\,p(\Omega)}=E_n(f)_{1/n,\,p(\Omega)}.$$

We define for $N=[n/4d]$

(17) $$Q_n^{\pm}(f,\,x)=R_n(f,\,x)\pm\sum_{j\in Z}\Phi_{j,\,N}(x)\|f-R_n\|_{\infty(\Omega_j)}.$$

It is clear that $Q_n^{\pm}(f,\,x)$ are algebraic polynomials of degree not greater than n.
Lemma 8. *For any bounded measurable function f in Ω, we have*

$$Q_n^-(f,\,x)\leq f(x)\leq Q_n^+(f,\,x)\quad\text{for any } x\in\Omega.$$

Proof. Let $x\in\Omega$. Then for some $j\in Z$, $x\in\Omega$ from (4) and (3), we have

$$Q_n^+(f,x)=R_n(f,x)+\sum_{j\in Z_f}\Phi_{,N}(x)\|f-R_n\|_{\infty(\Omega_j)}\geq R_n(f,x)+\|f-R_n\|_{\infty(\Omega_j)}$$
$$\geq R_n(f,\,x)+|f(x)-R_n(f,\,x)|=f(x).$$

We can prove that $Q_n^-(f,\,x)\leq f(x)$ in a similar way. Thus the proof of the lemma is completed.

3. Main results. We shall find the relationship between the quantities $\widetilde{E}_n(f)_p$ and $E_n(f)_{1/n,\,p}$ for $f\in L_\infty(\Omega)$.

Theorem 1. *For every $f\in L_\infty(\Omega)$. We have*

(18) $$\widetilde{E}_n(f)_p\leq C_dE_n(f)_{1/n,\,p}\leq C_d\widetilde{E}_n(f)_p,\ 1\leq p\leq\infty.$$

Proof. Let P_n^+, $P_n^-\in H_n^d$ be such that $P_n^-(x)\leq f(x)\leq P_n^+(x)$ for every $x\in\Omega$ and

$$\widetilde{E}_n(f)_p=\|P_n^+-P_n^-\|_{p(\Omega)},$$

then using (16), we get

$$E_n(f)_{1/n,\,p}\leq\widetilde{E}_n(f)_{1/n,\,p}\leq\|P_n^+-P_n^-\|_{1/n,\,p(\Omega)}\leq C_d\|P_n^+-P_n^-\|_{p(\Omega)}=C_d\widetilde{E}_n(f)_p.$$

Now, we shall prove that

$$\widetilde{E}_n(f)_p\leq C_dE_n(f)_{1/n,\,p}.$$

For this purpose, we shall use the polynomials $Q_n^{\pm}(f,\,x)$ which are given in (17) and from (5), (6) and (9), we will get

$$\widetilde{E}_n(f)_p\leq\|Q_n^+-Q_n^-\|_{p(\Omega)}=2\Big\|\sum_{j\in Z}\Phi_{j,N}(x)\|f-R_n\|_{\infty(\Omega_j)}\Big\|_{p(\Omega)}$$
$$\leq C\Big(\sum_{j\in Z}\text{meas}\,\Omega_j\|f-R_n\|^p_{\infty(\Omega_j)}\Big)^{1/p}\leq C\Big(\sum_{\in Z}\int_{\Omega_j}\|f-R_n\|^p_{\infty(\Omega_j)}dx\Big)^{1/p}$$
$$\leq C\Big(\sum_{j\in Z}\int_{\Omega_j}\|f-R_n\|^p_{\infty(U(2\pi/(N-1),\,x))}dx\Big)^{1/p}$$
$$\leq C\Big(\int_\Omega\|f-R_n\|^p_{\infty(U(2\pi/(N-1),\,x))}dx\Big)^{1/p}$$
$$\leq C_d\|f-R_n\|_{1/n,\,p}=C_dE_n(f)_{1/n,\,p}.$$

Thus the proof of the theorem is completed.

Now let us express the relationship between $E_n(f)_p$ and $\widetilde{E}_n(F)_p$ for $f\in L_\infty(\Omega)$, $F=f$ a. e., F is a continuous function in Ω.

Theorem 2. *If* $f\in L_p(\Omega)$, $\sum_{\nu=1}^{\infty}\nu^{2d/p-1}E_\nu(f)_p<\infty$, *and F is a continuous function in* Ω *such that* $F=f$ a. e., *then*

$$\widetilde{E}_n(F)_p\leq C_d n^{-2d/p}\sum_{\nu=n}^{\infty}\nu^{2d/p-1}E_\nu(f)_p. \tag{19}$$

Proof. Let $Q_\nu\in H_n^d$ be such that $E_\nu(f)_p=\|f-Q_\nu\|_{p(\Omega)}$, $\nu=0, 1.$
Since
$\sum_{\nu=1}^{N}(Q_{n2^\nu}-Q_{n2^{\nu-1}})=Q_{n2^N}-Q_n$ and for $F=f$ a. e., we have $\|F-Q_{n2^N}\|_\infty\to 0$, $(N\to\infty)$' then $F(x)-Q_n(x)=\sum_{\nu=1}^{\infty}(Q_{n2^\nu}-Q_{n2^{\nu-1}})$ for every $x\in R^d$.

Then from (18) and (15), we have

$$\begin{aligned}\widetilde{E}_n(F)_p&\leq C_dE_n(F)_{1/n,\,p}=C_dE_n(F-Q_n)_{1/n,\,p}\leq C_d\|F-Q_n\|_{1/n,\,p}\\
&\leq C_d\sum_{\nu=1}^{\infty}\|Q_{n2^\nu}-Q_{n2^{\nu-1}}\|_{1/n,\,p}\\
&\leq C_d\sum_{\nu=1}^{\infty}C_d(2^\nu)^{2d/p}\|Q_{n2^\nu}-Q_{n2^{\nu-1}}\|_p\\
&\leq C_d\sum_{\nu=1}^{\infty}2^{2\nu d/p}(E_{n2^\nu}(f)_p+E_{n2^{\nu-1}}(f)_p)\\
&\leq C_dn^{-2d/p}\sum_{\nu=1}^{\infty}(2^\nu n)^{2d/p}(E_{n2^\nu}(f)+E_{n2^{\nu-1}}(f)_p)\\
&=C_dn^{-2d/p}\sum_{\nu=n}^{\infty}\nu^{2d/p-1}E_\nu(f)_p.\end{aligned}$$

Thus the proof of the theorem is completed.

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