Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

# Serdica

## Bulgariacae mathematicae publicationes

# Сердика

## Българско математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Bulgaricae Mathematicae Publicationes
and its new series Serdica Mathematical Journal
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

# ON ZERO-DIMENSIONALITY IN FUZZY TOPOLOGY

L. PUJATE, A. ŠOSTAK

The topological notion of zero-dimensionality is extended to the category of (Chang) fuzzy topological spaces. Some elementary properties of zero-dimensional fuzzy topological spaces are established. The main result is Theorem 1 characterizing zero-dimensional $W_0$ spaces of a given weight $k (k \geq \aleph_0)$ as subspaces of the universal zero-dimensional space $\mathcal{F}(T)^k$, where $\mathcal{F}(T)$ is the so called fuzzification on the three-point set $T = \{0, 1, 2\}$.

**1. Definition and elementary properties.** Extending the notion of a zero-dimensional topological space (see e. g. [2]) to the category CFT of (Chang) fuzzy topological spaces [1] we arrive naturally to the following

Definition. *A fuzzy topological space $(X, \tau)$ is called zero-dimensional if there exists a base $\beta$ of $\tau$, the elements of which are clopen (=closed and open) fuzzy sets.*

(Recall that a family $\beta$ of fuzzy sets is called a base of a fuzzy topology $\tau$ if $\beta \subset \tau$ and for each $U \in \tau$ there exists a subfamily $\beta_U \subset \beta$ such that $U = V\{V : V \in \beta_U\}$ (see e. g. [18])).

Remark 1. The classic topological definition of zero-dimensionality includes in itself an additional assumption of separatedness. Besides, as it is easy to notice, for a topological space $(X, T)$, whose topology has a base of clopen sets, all separation properties $T_0$, $T_1$, $T_2$, $T_3$, and $T_{3,5}$ are equivalent. The situation differs essentially in the fuzzy setting (see e. g. the discussion in [15]): weaker separation axioms for fuzzy topological spaces when combined with such properties as the existence of a clopen base or compactness type properties do not necessarily imply stronger separation axioms. Besides, in contrast with the situation in general topology, the most important fuzzy topological spaces have very weak separation properties: e. g. the fuzzy real line $\mathcal{F}(\mathrm{R})$ [3], the fuzzy unit interval $\mathcal{F}(I)$ [4] are not even $T_0$-spaces [10]. Therefore here we use as basic the "pure" definition of zero-dimensionality without specifying separation properties. Notice, however, that for the proof of the main results (Section 3) we need to assume the $W_0$ separation axiom [14], [15] which is probably the weakest separation type property considered in fuzzy topology.

In the paper we use the standard terminology and notations accepted in fuzzy topology (see e. g. [8], [9] or [15] for the undefined notions). The term "a fuzzy topological space" is used in the sense of Chang.

The proofs of the next four statements are direct and therefore omitted.

Proposition 1. *The direct sum $(X, \tau) = \oplus_\alpha (X_\alpha, \tau_\alpha)$ of fuzzy topological spaces $(X_\alpha, \tau_\alpha)$, $\alpha \in A$, is zero-dimensional iff all $(X_\alpha, \tau_\alpha)$, $\alpha \in A$ are zero-dimensional.*

Proposition 2. *A subspace $(Y, \tau_Y)$ of a zero-dimensional fuzzy topological space $(X, \tau)$ is zero-dimensional.*

Proposition 3. *The product $(X, \tau) = \Pi_\alpha (X_\alpha, \tau_\alpha)$ of zero-dimensional fuzzy topological spaces $(X_\alpha, \tau_\alpha)$, $\alpha \in A$, is zero-dimensional.*

Notice, however, that (in contrast with classic topology) the converse does not generally hold. This can be illustrated by the next

Example 1. Let $(X, T)$ be a zero-dimensional topological space and let $\omega T$ denote the family of all lower semicontinuous functions defined on $(X, T)$ and with values in $I=[0, 1]$ (see e. g. [6]). Take now $\alpha\in(0, 1)$, $\alpha\neq 1/2$, and let $T'=\{\alpha U: U\in T\}\cup T$. It is easy to notice that $\omega T$ and $T'$ are fuzzy topologies on $X$, $(X, \omega T)$ is zero-dimensional, $(X, T')$ is not zero-dimensional, but nevertheless the product $(X, \omega T)\times(X, T')$ is zero-dimensional.

Proposition 4. *If a fuzzy topological space $(X, \tau)$ has a subbase $\tau$ consisting of clopen fuzzy sets, then $(X, \tau)$ is zero-dimensional.*

**2. Behaviour under functors.** Let us now study the behaviour of zero-dimensionality under some important functors of fuzzy topology. First we recall the definitions of these functors. The iota-functor $\iota$: CFT$\to$Top introduced by Lowen [6] associates with a fuzzy topological space $(X, \tau)$ the topological space $(X, \iota\tau)$ where $\iota\tau=\{U^{-1}(t, 1]: U\in\tau, \tau\in I\}$ (i. e. $\iota\tau$ is the weakest topology on $X$ with respect to which all mappings $U\in\tau$ are lower semicontinuous). The omega-functor $\omega$: Top$\to$CFT considered by Lowen [6] associates with each topological space $(X, T)$ the fuzzy topological space $(X, \omega T)$ where $\omega T$ is defined in the same way as in Example 1. At last let $\lambda$: CFT$\to$CFT be the functor of laminated modification (see e. g. [15], [16]) associating wtth a fuzzy topological space $(X, \tau)$ the fuzzy topological space $(X, \lambda\tau)$ where $\lambda\tau$ is the weakest laminated (=containing all constants; cf [6])) fuzzy topology containing $\tau$. (It is known and easy to notice that $\lambda$ can be characterized as the composition of functors $\iota$ and $\omega$ and that $\omega$ is the restriction of $\lambda$ to the subcategory Top of the category CFT).

Proposition 5. *If $(X, \tau)$ is a zero-dimensional fuzzy topological space, then its laminated modification $(X, \lambda\tau)$ is zero-dimensional, too. In particular, if $(X, T)$ is a zero-dimensional topological space, then the fuzzy topological space $(X, \omega T)$ is also zero-dimensional.*

Proof. Let $\beta$ be a base of $\tau$ consisting of clopen fuzzy sets. It is easy to notice that the family $\sigma=\beta\cup\{c_X: c\in I\}$, where $c_X: X\to I$ is the constant mapping with value $c$, is a subbase of the fuzzy topology $\lambda\tau$ and besides all $U\in\sigma$ are clopen. To complete the proof it is sufficient to apply Proposition 4.

Unlike the functors $\omega$ and $\lambda$, the functor $\iota$ does not preserve zero-dimensionality. This can be shown by the next

Example 2. Let $X=I$. For each $\alpha\in[0, 1]$ define a fuzzy set $S_\alpha: X\to I$ by the equality $S_\alpha(x)=\alpha x$; for each $\alpha\in(1, +\infty)$ define a fuzzy set $S_\alpha: X\to I$ by the equalities $S_\alpha(x)=\alpha x$ if $0\leq x\leq 1/\alpha$ and $S_\alpha(x)=1$ if $1/\alpha<x\leq 1$; at last, let $S_\infty=1$. Consider the family of fuzzy sets $\sigma=\{S_\alpha, S^c_\alpha: \alpha\in[0, +\infty]\}\subset I^X$ (where $S^c_\alpha=1-S_\alpha$) and let $\tau$ be the fuzzy topology on $X$ having $\sigma$ as a subbase. Obviously each $U\in\sigma$ is clopen and hence $(X, \tau)$ is zero-dimensional. (Evidently, $(x, \tau)$ is a $W_0$-space, too.) On the other hand, it is easy to notice that $\iota\tau$ is the usual (metric) topology on the closed unit interval $I$, and hence $(X, \iota\tau)$ is not zero-dimensional.

**3. Universal zero-dimensional spaces and embedding theorem.** It is well known that a topological $T_0$-space is zero-dimensional iff it is homeomorphic to a subspace of the generalized Cantor cube $D^k$ (where $D$ is the discrete two-point space $\{0, 1\}$ and $k$ is a cardinal which can be taken equal to the weight of the space). Our aim is to find a similar characterization of zero-dimensionality in the fuzzy setting.

**Construction $\mathcal{F}(T)$.** Patterned after the Hutton unit interval [4], for each linearly ordered space $X$ a fuzzy topological space $\mathcal{F}(X)$ was defined and studied in [12], [13]. $\mathcal{F}(X)$ contains the space $X$ as a crisp kernel and it can be considered as a fuzzification of $X$; in a certain sense $\mathcal{F}(X)$ can play in fuzzy topology a role similar to that of the space $X$ in general topology. We need here this construction in the case when $X$ is the three point space $T=\{0, 1, 2\}$ endowed with the natural ordering $\leq$. In this simple case the construction $\mathcal{F}(T)$ can be explicitly described as follows.

The elements of $\mathcal{F}(T)$ are the functions $z_\alpha: T\to I$, $\alpha\in I$, where $z_\alpha(0)=1$, $z_\alpha(1)=\alpha$, and $z_\alpha(2)=0$. In an obvious way $\mathcal{F}(T)$ can be endowed with the linear ordering "$<$" defined by $z_\alpha<z_\beta$ iff $\alpha\leq\beta$. The fuzzy topology $\tau_{\mathcal{F}(T)}$ on $\mathcal{F}(T)$ can be defined by the

subbase $\sigma=\{L_0, L_1, L_2, R_0, R_1, R_2\}\subset I^{\mathcal{F}(T)}$, where $L_0=R_2=0$, $L_2=R_0=1$, $R_1(z_\alpha)=\alpha$, and $L_1(z_\alpha)=\alpha^c(=1-\alpha)$ our notation here is compatible with [12], [13]). On the whole the fuzzy topology on $\mathcal{F}(T)$ looks like $\tau_{\mathcal{F}(T)}=\{0, 1, R_1, L_1, R_1\wedge L_1, R_1\vee L_1\}$. Noticing that $L_1^c=R_1$ we conclude that all fuzzy sets in $\tau_{\mathcal{F}(T)}$ are clopen and hence $\mathcal{F}(T)$ is zero-dimensional.

The fuzzy topological space $\mathcal{F}(T)$ will play a fundamental role in the sequel which is similar to the role of $D$ in general topology.

Remark 2. The fuzzification $\mathcal{F}(D)$ of the two-point set $D$ consists of the single element $z\colon D\to I$ such that $z(0)=1$ and $z(1)=0$ and therefore it is quite inadequate for the study of zero-dimensionality. On the other hand, the fuzzification $\mathcal{F}(P_n)$ of the $n$-point set $P_n\{0, 1, \ldots, n-1\}$, where $n\geqq 3$, could be used in the sequel just as well instead of $\mathcal{F}(T)$.

As it is shown below the cube $\mathcal{F}(T)^k$ where $k$ is an infinite cardinal is a universal space for zero-dimensional fuzzy topological $W_0$-spaces of weight $\leqq k$ (see Theorem 1). Recall that a fuzzy space $(X, \tau)$ is called a $W$-space if for every two distinct points $x, y\in X$ there exists $U\in\tau$ such that $U(x)\neq U(y)$ (see [14], [15], [17]; cf also [7]); this property can be considered as the weakest condition of $T_0$ type in fuzzy topology By the weight of a fuzzy space $X$ we call the least cardinality $w(X)$ of its bases (see e. g. [12]).

Theorem 1. *The cube $\mathcal{F}(T)^k$ is a universal space for all zero-dimensional fuzzy topological $W_0$-spaces of weight $\leqq k$ ($k\geqq\aleph_0$). In order words $X$ is a zero-dimensional fuzzy topological $W_0$-space such that $w(X)\leqq k$ iff $X$ is homeomorphic to a subspace of $\mathcal{F}(T)^k$.*

Proof. Assume that $X$ is homeomorphic to the subspace $\mathcal{F}(T)^k$. Then $X$ being a subspace of a product of a zero-dimensional fuzzy spaces is zero-dimensional, too (see Propositions 2, 3). Besides, $\mathcal{F}(T)$ is obviously a $W_0$-space, and therefore $X$ is a $W_0$-space, too. At last, for an infinite cardinal $k$ the weight of $\mathcal{F}(T)^k$ is equal to $k$ and hence the weight of $X$ does not exceed $k$.

Conversely, assume that $(X, \tau)$ is a zero-dimensional fuzzy topological $W$-space of weight $\leqq k$ and let $C(X, \mathcal{F}(T))$ denote the set of all continuous mappings from $X$ into $\mathcal{F}(T)$. According to [14], [15] to show that $X\subset\mathcal{F}(T)^k$ (up to homeomorphism) it is sufficient to prove that there exists a family $\Phi\subset C(X, \mathcal{F}(T))$, $|\Phi|\leqq k$, which separates points of $X$ (i. e. for every pair of distinct points $x, y\in X$ there exists $\varphi\in\Phi$ such that $\varphi(x)\neq\varphi(y)$) and separates points and closed fuzzy sets of $X$ (i. e. for each $x\in X$, each closed fuzzy set $A$ and each $\varepsilon>0$ there exists $\varphi\in\Phi$ such that $A(x)\geqq\overline{\varphi(A)}(\varphi(x))-\varepsilon$ [14], [15]). (We emphasize that the assumption that all the spaces under consideration satisfy the $W_0$-separation axiom is essential for the validity of this fact.)

Let $\beta$ be a clopen base of the space $(X, \tau)$such that $|\beta|\leqq k$ and for each $U\in\beta$ let a mapping $f_U\colon X\to\mathcal{F}(T)$ be defined by the equality $f_U(x)=z_{U(x)}$. It is easy to see that $f_U$ is continuous. Indeed, $f_U^{-1}(L_0)=f_U^{-1}(R_2)=0$; $f_U^{-1}(L_2)=f_U^{-1}(R_0)=1$; $f_U^{-1}(L_1)(x)=L_1(f_U(x))=L_1(z_{U(x)})=U^c(x)$; $f_U^{-1}(R_1)(x)=R_1(f_U(x))=R_1(z_{U(x)})=U(x)$, and hence the preimages of all elements of the subbase $\sigma$ are open fuzzy sets in $X$. Therefore to finish the proof it is sufficient to show that the family $\Phi=\{f_U\colon U\in\beta\}$ separates the points of $X$ and separates points and closed fuzzy sets in $X$.

The first fact is obvious. Indeed, if $x, y\in X$ and $x\neq y$, then there exists a set $U\in\tau$ such that $U(x)\neq U(y)$. Besides, it is easy to verify that without loss of generality we can take $U\in\beta$. Then $f_U(x)\neq f_U(y)$.

For the second statement take some point $x\in X$, a closed fuzzy set $A$ and fix some $\varepsilon>0$. Since $\beta$ is a base it is clear that there exists $U\in\beta$ such that $U^c\leqq A^c$ and $U^c(x)>A^c(x)-\varepsilon$. Noticing that $f_U(U)(z_t)=\sup_{f_U(x)=z_t} U(x)=t=R_1(z_t)$, we conclude that $(f_U(U)=R_1$ is a clopen set. However, this means that $\overline{f_U(A)}(f_U(x))\leqq\overline{f_U(U)}(f_U(x))=f_U(U)$ $f_U(x))=U(x)<A(x)+\varepsilon$. Hence $\Phi$ separates points and closed fuzzy sets in $X$.

Corollary 1. *If $X$ is a zero-dimensional fuzzy topological $W_0$-space, then $X$ is completely regular.*

(If $X$ satisfies the assumptions of the Corollary, then $X$ can be embedded in $\mathcal{F}(T)^k$. On the other hand, $\mathcal{F}(T)$ can in an obvious way be considered as a subspace of the Hutton fuzzy unit interval $\mathcal{F}(I)$ and hence $X$ is embeddable into $\mathcal{F}(I)^k$. Applying [14] or [15] (cf also [5]), we conclude that $X$ is completely regular.)

Observe, however, that a zero-dimensional fuzzy topological $W$ space need not be Hausdorff [10], [11], nor even a $T_0$-space [10]: as one can easily see the space $\mathcal{F}(T)$ itself is not $T_0$ (if $z_\alpha, z_\beta \in \mathcal{F}(T)$, $U$ is open in $\mathcal{F}(T)$ and $U(z_\alpha)=0$, then obviously, $U(z_\beta)=0$, too.)

Corollary 2. *If $X$ is a zero-dimensional fuzzy topological $W_0$-space, then there exists a strongly compact* [3] *(and hence also compact* [6]*) zero-dimensional fuzzy topological space $bX$ containing $X$ as a dense subspace.*

(Embed $X$ into $\mathcal{F}(T)^k$ and let $bX$ be the closure of $X$ in $\mathcal{F}(T)^k$. According to [12] $\mathcal{F}(T)$ is strongly compact and hence (by [3]) $bX$ is strongly compact, too.)

Let $\mathcal{F}^\lambda(T)$ be the laminated modification of $\mathcal{F}(T)$, i. e. $\mathcal{F}^\lambda(T)=(\mathcal{F}(T), \lambda\tau_{\mathcal{F}(T)})$. In a way quite similar to the proof of Theorem 1 it is quite easy to establish the following laminated version of the previous results:

Theorem $1^\lambda$. The cube $\mathcal{F}^\lambda(T)^k$ is a universal space for zero-dimensional laminated fuzzy topological $W_0$-spaces of weight $\leq k$ $(k \geq \aleph_0)$.

Corollary $2^\lambda$. *If $X$ is a zero-dimensional laminated fuzzy topological $W_0$-space, then there exists a strongly compact zero-dimensional laminated fuzzy topological $W_0$-space $bX$ containing $X$ as a dense subspace.*

## REFERENCES


1. C. L. Chang. Fuzzy topological spaces. *J. Math. Anal. Appl.*, **24**, 1968, 182-190.
2. R. Engelking. General Topology. Warszawa, 1977.
3. T. E. Gantner, R. C. Steinlage, R. C. Warren. Compactness in fuzzy topological spaces. *J. Math. Anal. Appl.*, **62**, 1978, 547-562.
4. B. Hutton. Normality in fuzzy topological spaces. *J. Math. Anal. Appl.*, **50**, 1975, 74-79.
5. A. K. Katsaras. Fuzzy proximities and fuzzy completely regular spaces. *J. Anal. St. Univ. Jaşi.*, **26**, 1980, 31-41.
6. R. Lowen. A comparison of different compactness notions in fuzzy topological spaces. *J. Math, Anal. Appl.*, **64**, 1978, 446-454.
7. R. Lowen, A. K. Srivastava. $FTS_0$: the reflective hull of the Sierpinski object in FTS. *Fuzzy Sets and Syst.*, **29**, 1989, 171-176.
8. Pu Pao-ming, Liu Ying-ming. Fuzzy topology. I. Neighborhood structure of fuzzy point and Moore-Smith convergence. *J. Math. Anal. Appl.*, **76**, 1980, 571-599.
9. Pu Pao-ming, Liu Ying-ming. Fuzzy topology II. Product and quotient spaces. *J. Math. Anal. Appl.*, **77**, 1980, 20-37.
10. S. E. Rodabaugh. Separation axioms and fuzzy real lines. *Fuzzy Sets and Syst.*, **11**, 1983, 163-183.
11. R. Srivastava, S. N. Lal, A. K. Srivastava. Fuzzy Hausdorff topological spaces. *J. Math. Anal. Appl.*, **81**, 1981, 497-506.
12. A. P. Šostak. A fuzzy modification of a linearly ordered space. *Coll. Math. Soc. Janos. Bolyai* (Topology and Applications), **41**, 1983, 581-604.
13. A. P. Šostak. A fuzzy modification of the category of linearly ordered spaces. *Comm. Math. Univ. Carol.*, **26**, 1985, No 3, 421-442.
14. A. P. Šostak. On complete regularity and $E$-regularity in fuzzy topology. *Mat. Vesn.*, **41**, 1989 No 3, 81-93.
15. A. P. Šostak. Two decades of fuzzy topology: basic ideas, concepts and results. *Russ. Math. Surveys*, **44**, 1989, No 6, 99-147.
16. A. P. Šostak. On some modifications of fuzzy topologies. *Mat. Vesn.*, **41**, 1989, No 1, 51-64.
17. A. P. Šostak. Relative closedness, $E$-compactness and compactness spectra of fuzzy sets in fuzzy topological spaces. *Latv. Math. Ann.*, **34**, 1990 (to appear).
18. R. H. Warren. Neighborhoods, bases and continuity in fuzzy topological spaces. *Rocky Mount. J. Math.*, **8**, 1978, 459-470.


*Department of Mathematics*
*Latvian University*
*Rainis boul. 19*
*Riga*
*226098, Latvia, USSR*

*Received 19.03.90*
*Revised 10.07.1990*