Serdica
Bulgariacae mathematicae
publicationes

Сердика
Българско математическо
списание

CAUCHY-RIEMANN SUBMANIFOLDS OF LOCALLY CONFORMAL KAEHLER MANIFOLDS. III.

SORIN DRAGOMIR, RENATA GRIMALDI*

One classifies the totally-geodesic real surfaces of a generalized Hopf manifold with flat local Kaehler metrics. The only complex submanifolds $\psi: M^m \to CH^n$ of a complex Hopf manifold having harmonic components ψ^i are the Kaehlerian submanifolds. Let M^m be a generic Cauchy-Riemann submanifold of a locally conformal Kaehler (l.c.K.) manifold M^{2n}; if the holomorphic distribution of M^m is completely integrable and its leaves are totally-geodesic in M^m, then they are totally-umbilical in M^{2n}. The normal bundle of a complex submanifold of a l.c.K. manifold of positive holomorphic bisectional curvature has no parallel cross-sections.

1. Introduction and statement of results. Let $(M^{2n}, \bar{g}, J)$ be a Hermitian manifold of complex dimension n, where $\bar{g}$ denotes the Hermitian metric, while J is the complex structure. Cf. P. Libermann [20], M^{2n} is said to be a *locally conformal Kaehler* (l.c.K.) manifold if there exists an open covering $\{U_i\}_{i\infty \in I}$ of M^{2n} and a family $\{f_i\}_{i\in I}$ of smooth real-valued functions $f_i \in C(U_i)$, $i \in I$, such that the local metrics $g_i = \exp(-f_i)\bar{g}$ are Kaehler. Any such two local metrics are conformally related, i. e. $g_j = \exp(f_i - f_j) g_i$, and therefore are homothetic. Consequently, the local l-forms df_i glue up to a globally defined (closed) 1-form $\bar{\omega}$ on M^{2n}, i.e. the *Lee form.*

Let $\bar{\nabla}$ be the Levi-Civita connection of $(M^{2n}, \bar{g})$. A l.c.K. manifold whose Lee form is parallel with respect to $\bar{\nabla}$ is termed a *generalized Hopf manifold*, cf. I. Vaisman [27], (or a *PK-manifold*, according to the terminology in [28]).

The geometry of l.c.K. manifolds has been intensely studied in the last decade cf. [16], [18], [26] and [30]. Especially the local geometric structure of *PK*-manifolds is completely known today due to a deep result of I. Vaisman, i. e. theorem 3.7. in [28, p. 275]. In turn, the study of the geometry (of the second fundamental form) of submanifolds in l.c.K. manifolds is of recent interest, cf. K. Matsumoto [21], B. Y. Chen & P. Piccinni [8], S. Ianus & al. [17], L. Ornea [22]. With the present note we continue the investigations initiated in [10], [11], [12] and [13] and obtain the following results:

Theorem 1. *Let $\psi: M^m \to M^{2n}$ be an isometric immersion of an m-dimensional Riemannian manifold (M^m, g) in the generalized Hopf manifold M^{2n}. If the local Kaehler metrics of M^{2n} are flat (i. e. M^{2n} is a P_0K-manifold) and ψ is totally-geodesic, then M^m has a flat normal connection. Moreover, the induced form $\omega = \psi^*\bar{\omega}$ is parallel. Consequently, either $\omega = 0$, and then M^m is an elliptic real space-form, or $\omega \neq 0$ everywhere, and then M^m is tangent to the Lee field of M^{2n}.*

The Ricci curvature of an arbitrary totally-geodesic submanifold M^m of a P_0K-manifold is expressed by

* Work partially supported by G.N.S.A.G.A. of C.N.R. and M.P.I., Italy.

$$\mathrm{Ric} = \frac{m-2}{4}\{\|\omega\|^2 g - \omega\otimes\omega\}, \tag{1.1}$$

where $g=\psi^*\bar{g}$, i. e. M^m is *quasi-Einstein,* cf. S. Goldberg & I. Vaisman, [16, p. 118]. Combining (1.1) and Theorem 1 one obtains the complete classification of totally-geodesic surfaces:

Corollary. *Let M^2 be a totally-geodesic real surface ($m=2$) of the P_0K-manifold M^{2n}. Then M^2 is a real space-form $M^2(k)$ where either $k=c^2/2$ if $M^2(k)$ is normal to the Lee field of M^{2n}, or $k=0$ if $M^2(k)$ is tangent to the Lee field.*

Here $c\neq 0$ denotes the (constant) length of the Lee form of M^{2n}, while k stands for the (constant) sectional curvature of M^2.

Let M^m be a *Cauchy-Riemann* (C.R.) submanifold of the generalized Hopf manifold M^{2n}, cf. e.g. K. Yano & M. Kon [37, p. 79], i.e. M^m carries a pair of complementary (with respect to g) distributions D, $D^\perp$, such that D is holomorphic, (i.e. $J_x(D_x)=D_x$, $x\in M^m$) and $D^\perp$ is totally-real, (i.e. $J_x(D^\perp_x)\subseteq E(\psi)_x$, $x\in M^m$). Here $E(\psi)\to M^m$ denotes the normal bundle of the given immersion ψ. As to complex (i. e. $D^\perp=0$) submanifolds we obtain:

Theorem 2. *Any complex minimal submanifold M^m of a P_0K-manifold M^{2n} obeying to*

$$A_{h(X,Z)}Y = A_{h(Y,Z)}X \tag{1.2}$$

or any tangent vector fields X, Y, Z on M^m, is locally analytically homothetic to a complex Hopf manifold.

Here h denotes the second fundamental form of ψ, while A_ξ is the Weingarten operator (associated with the normal section ξ). It is well known that each complex submanifold of a Kaehlerian manifold is minimal. In turn, if the ambient space is only l.c.K., cf. Theorem 5.1. of [27, p. 252] one has $H=-\frac{1}{2}B^\perp$, i.e. the mean curvature vector of ψ and the normal component of the Lee field are colinear, such that generally $H\neq 0$, i.e. the minimality condition in Theorem 2 is not superfluous.

Any isometric immersion ψ of a Riemannian manifold in the Euclidean space is known to be minimal if and only if it is harmonic. If the ambient space is a complex Hopf manifold (endowed with the standard l.c.K. structure, see e.g. [28]), then we obtain:

Theorem 3. *Let ψ^i be the local components of an isometric immersion $\psi: M^m \to CH^n$. Let B, $B^\perp$ be respectively the tangential and normal components of the Lee field of CH^n. Then ψ^i are harmonic (with respect to any coordinate system) if and only if the mean curvature vector H of ψ is given by $H=-\frac{1}{2}B^\perp$ and $(m-2)B=0$. Consequently the only complex submanifolds of CH^n with ψ^i harmonic are the Kaehler submanifolds.*

Let M^m be a C.R. submanifold of the P_0K-manifold M^{2n}. Then M^m admits an f-structure, cf. K. Yano [34], P defined by $PX=\tan(JX)$, for any tangent vector field X on M^m. Suppose $\omega\neq 0$ everywhere. Let $U=\|\omega\|^{-1}B$, $B=\omega^+$, (where $^+$ denotes raising of indices by g), and $V=-PU$. We obtain the following:

Theorem 4. *For any totally-geodesic C.R. submanifold M^m of a P_0K-manifold the tangent vector fields U, V are Killing (with respect to g) provided that ω has no singular points.*

Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}. Let $p=\dim_C D_x$, $q=\dim_R D^\perp_x$, $x\in M^m$. If $p=0$, then M^m is termed *totally-real.* If $q=2n-m$, i.e. $J_x(D^\perp_x)=E(\psi)_x$, $x\in M^m$, then M^m is a *generic* C.R. submanifold. Moreover, M^m is said to be

D-geodesic, cf. A. Bejancu [3, p. 39], if $h(X, Y)=0$ for any $X, Y \in D$. Let $\overline{\theta}=\overline{\omega}\circ J$ be the *anti-Lee form* of M^{2n}. Let $\theta=\psi^*\overline{\theta}$. We obtain the following:

Theorem 5. *Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}.*
i) *The holomorphic distribution D of M^m is completely integrable and its leaves are totally-geodesic in M^m if and only if either M^m is totally-real or for any $X, Y \in D$, $Z \in D^\perp$, one has*

$$\overline{g}(h(X, Y), JZ)+\frac{1}{2}g(X, Y)\theta(Z)=0 \tag{1.3}$$

and any leaf of D is tangent to B. Moreover, if (1.3) *holds and M^m is generic, then all leaves of D are totally-umbilical in M^{2n}.*
ii) *If D is completely integrable and its leaves are totally-geodesic in M^{2n}, then M^m is D-geodesic. Conversely, if M^m is D-geodesic and tangent to the Lee field of M^{2n}, then the holomorphic distribution of M^m gives rise to a complex foliation on M^m whose leaves are totally-geodesic in M^{2n}.*

Let $\pi: G_2(M^m)\to M^m$ be the Grassmann bundle of all 2-planes tangent to M^m. Let Riem: $G_2(M^m)\to \mathbb{R}$ be the Riemannian sectional curvature of (M^m, g). Then a 2-plane $p_0 \in G_2(M^m)$ is termed *anti-holomorphic* if $J(p_0)$ and p_0 are orthogonal; if additionally $p_0\subseteq D^\perp_{\pi(p_0)}$, then p_0 is said to be *D-anti-holomorphic*. Next, cf. A. Bejancu [3, p. 96] the *D-anti-holomorphic sectional curvature* of the C.R. submanifold M^m is the restriction of Riem to the D-anti-holomorphic planes of M^m. Moreover $D^\perp$ is said to be *D-parallel* if $\nabla_X Y \in D^\perp$ for any $X \in D$, and $Y \in D^\perp$. Here ∇ denotes the Levi-Civita connection of (M^m, g). We obtain the following:

Theorem 6. *Let M^m be a C.R. submanifold of the complex Hopf manifold $CH^n(c)$. Let us assume that* i) $D^\perp$ *is D-parallel,* ii) *there is a constant $A>0$ such that*

$$\|h\|^2+\frac{1}{2}\|\omega\|^2\leq\frac{c^2}{2}-2A. \tag{1.4}$$

Then all D-anti-holomorphic sectional curvatures of M^m are $\geq A$.

Let $\nabla^\perp$ be the normal connection of the submanifold $\psi: M^m\to M^{2n}$ of the Riemannian manifold M^{2n}. A cross-section ξ is said to be *parallel* if $\nabla^\perp\xi=0$. We obtain the following:

Theorem 7. *Let M^m be a complex submanifold of a l.c.K. manifold of positive holomorphic bisectional curvature. Then M^m admits no parallel sections in the normal bundle.*

2. Basic formulae. Let M^{2n} be a l.c.K. manifold and $\{\overline{g}_i\}_{i\in I}$ its local metrics; since each $\overline{g}_i$ is Kaehler, one obtains $d\overline{\Omega}=\overline{\omega}\wedge\overline{\Omega}$, where $\overline{\Omega}$ is the *Kaehler* 2-form of M^{2n}, i.e. $\overline{\Omega}(X, Y)=\overline{g}(X, JY)$. Clearly, if $\overline{\omega}=0$, then $\overline{g}$ is a Kaehler metric. The l.c.K. manifold M^{2n} is said to be strongly non-Kaehler if its Lee form has no singular points, i.e. $\overline{\omega}_x\neq 0$, for any $x\in M^{2n}$. There exist various examples of complex manifolds which admit no Kaehler metrics and, in turn, possess natural l.c.K. metrics, cf. e.g. F. Tricerri [26]. For instance, let λ be a fixed complex number, $0<|\lambda|<1$. Let G_λ be the 0-dimensional Lie group of analytic transformations of $C^n-\{0\}$, $n>1$, generated by $z\to\lambda z$, $z\in C^n-\{0\}$. Cf. [19, p. 137], vol. II, G_λ is a properly discontinuous group acting freely on $C^n-\{0\}$, and thus the factor space $CH^n=(C^n-\{0\})/G_\lambda$ admits a naturally induced structure of complex manifold. This is the well-known *complex Hopf manifold*. Its first Betti number is $b_1(CH^n)=1$, and thus CH^n admits no Kaehler metrics. Yet the Hermitian metric $ds^2=|z|^{-2}\delta_{ij}dz^i\otimes d\overline{z}^j$, $|z|^2=\delta_{ij}Z^i\overline{z}^j$, $z=(z^1, \ldots, z^n)$, of $C^n-\{0\}$, is G_λ-invariant and thus gives rise to a well-defined Hermitian metric g on CH^n. This was observed to be a l.c.K. metric on CH^n, (see [28]); it is referred to as the *Boothby metric* of CH^n. The complex Hopf manifold endowed with the Boothby metric possesses several particular features, i.e. its Lee form $\bar{\omega}=d\log|z|^2$ is parallel (with respect to the Levi-Civita connection of $\bar{g}$), its local Kaehler metrics (i.e. $\delta_{ij}dz^i\otimes d\bar{z}^j$) are flat, i.e. CH^n is a P_0K-manifold. Moreover, $\|\bar{\omega}\|=2$. Of course, $(CH^n, \bar{g})$ is strongly non-Kaehler.

Let M^{2n} be a l.c.K. manifold. The Levi-Civita connections ∇^i of the local Kaehler metrics $\bar{g}_i$ are known to glue up to a globally defined torsion-free linear connection $\bar{D}$ on M^{2n}, i.e. the *Weyl connection*. It is expressed by

$$\bar{D}_X Y=\bar{\nabla}_X Y-\frac{1}{2}\{\bar{\omega}(X)Y+\bar{\omega}(Y)X-\bar{g}(X, Y)\bar{B}\}. \tag{2.1}$$

Here $\bar{B}=\bar{\omega}^+$ (raising of indices is understood with respect to $\bar{g}$). The tangent vector fields $\bar{B}$ and $\bar{A}=-J\bar{B}$ are referred to as the *Lee* and *anti-Lee fields* of the l.c.K. manifold M^{2n}. Each ∇^i is almost-complex, such that J is parallel with respect to the Weyl connection, i.e. $\bar{D}J=0$. Thus (2.1) leads to:

$$\bar{\nabla}_X JY=J\bar{\nabla}_X Y+\frac{1}{2}\{\bar{\theta}(Y)X-\bar{\omega}(Y)JX-\bar{g}(X, Y)\bar{A}-\bar{\Omega}(X, Y)\bar{B}\}. \tag{2.2}$$

As a consequence of (2.1) the curvature tensor fields $\bar{K}$, $\bar{R}$ of $\bar{D}$, $\bar{\nabla}$ respectively are related by

$$\bar{K}(X, Y)Z=\bar{R}(X, Y)Z-\frac{1}{2}\{\bar{L}(X, Z)Y-\bar{L}(Y, Z)X \tag{2.3}$$
$$+\bar{g}(X, Z)\bar{L}(Y, \cdot)^+-\bar{g}(Y, Z)\bar{L}(X, \cdot)^+\}-\frac{1}{4}\|\bar{\omega}\|^2(X\wedge Y)Z,$$

where $\bar{L}=\bar{\nabla}\bar{\omega}+\frac{1}{2}\bar{\omega}\otimes\bar{\omega}$. See also S. I. Goldberg [14, p. 115].

Let M^m be a submanifold of the l.c.K. manifold M^{2n}; we shall need the Gauss and Weingarten formulae

$$\bar{\nabla}XY=\nabla XY+h(X, Y),\ \bar{\nabla}_X\xi=-A_\xi X+\nabla_X^\perp\xi \tag{2.4}$$

for any tangent vector fields X, Y on M^m, respectively any cross-section ξ in $E(\psi)\to M^m$. Let $\tan_x$, nor_x be the natural projections assoctiated with the direct sum decomposition $T_x(M^{2n})=T_x(M^m) * E(\psi)_x, x\in M^m$. We set, as usual, $FX=\mathrm{nor}(JX)$, $t\xi=\tan(J\xi)$, $f\xi=\mathrm{nor}(J\xi)$, where X is tangential, while ξ is normal. We define covariant derivatives of P, F, t and f in terms of ∇, $\nabla^\perp$ in the usual manner, i.e. cf. [37, p. 77]. Set $A=\tan(\bar{A})$, $B=\tan(\bar{B})$, $A^\perp=\mathrm{nor}(\bar{A})$ and $B^\perp=\mathrm{nor}(\bar{B})$. By (2.2) and (2.4) one obtains the following identities:

$$(\nabla_X P)Y=A_{FY}X+th(X, Y)+\frac{1}{2}\{\theta(Y)X-\omega(Y)PX-g(X, Y)A-\Omega(X, Y)B\}, \tag{2.5}$$

$$(\nabla_X F)Y=-h(X, PY)+fh(X, Y)-\frac{1}{2}\{\omega(Y)FX+g(X, Y)A^\perp+\Omega(X, Y)B^\perp\}, \tag{2.6}$$

$$(\nabla_X t)\xi=A_{f\xi}X-PA_\xi X+\frac{1}{2}\{\bar{\theta}(\xi)X-\bar{\omega}(\xi)PX-\bar{\Omega}(X, \xi)B\}, \tag{2.7}$$

$$(\nabla_X f)\xi=-h(X, t\xi)-FA_\xi X-\frac{1}{2}\{\bar{\omega}(\xi)FX+\bar{\Omega}(X, \xi)\bar{B}\}, \tag{2.8}$$

where $\Omega=\psi^*\bar{\Omega}$. Suppose now that $\bar{M}^{2n}$ is a P_0K-manifold. Then by (2.3) the curvature of $\bar{M}^{2n}$ has the following expression:

$$(2.9)\quad \bar{R}(X,Y)Z=\frac{1}{4}\{[\bar{\omega}(X)Y-\bar{\omega}(Y)X]\bar{\omega}(Z)+[\bar{g}(X,Z)\bar{\omega}(Y)-\bar{g}(Y,Z)\bar{\omega}(X)]\bar{B}\}+\frac{1}{4}\|\bar{\omega}\|^2(X\wedge Y)Z.$$

Consequently, the Gauss-Codazzi-Ricci equations (i.e. eq. (2.6)—(2.7) and (2.11) in ref- [4, p. 45 – 47]) of M^m in the P_0K-manifold $\bar{M}^{2n}$ are

$$(2.10)\quad R(X,Y)Z=A_{h(Y,Z)}X-A_{h(X,Z)}Y+\frac{1}{4}\{[\omega(X)Y-\omega(Y)X]\omega(Z)+[g(X,Z)\omega(Y)-g(Y,Z)\omega(X)]B\}+\frac{1}{4}\|\bar{\omega}\|^2\{g(Y,Z)X-g(X,Z)Y\},$$

$$(2.11)\quad (\nabla_X h)(Y,Z)-(\nabla_Y h)(X,Z)=\frac{1}{4}\{g(X,Z)\omega(Y)-g(Y,Z)\omega(X)\}B^{\perp},$$

$$(2.12)\quad \bar{g}(R^{\perp}(X,Y)\xi,\eta)=g([A_\xi,A_\eta]X,Y).$$

Here R, $R^{\perp}$ denote respectively the curvature tensor fields of ∇, $\nabla^{\perp}$.

3. Totally-geodesic submanifolds of generalized Hopf manifolds. Suppose ψ is totally-geodesic, i. e. $h=0$. By (1.12) in [4, p. 41] and (2.4), it follows that M^m has a flat normal connection, i.e. $R^{\perp}=0$. As $\bar{\nabla}\bar{\omega}=0$, the Gauss formula in (2.4) leads to

$$(3.1)\quad \nabla_X\omega=A_{B^{\perp}}X.$$

Therefore, if $h=0$, then ω is parallel, too. Thus $\|\omega\|=\text{const}$. Consequently, either $\omega=0$ i.e. M^m is normal to the Lee field of $\bar{M}^{2n}$, or $\omega_x\neq 0$, at any $x\in M^m$. If this is the case then $B^{\perp}=0$, i.e. M^m is tangent to the Lee field of $\bar{M}^{2n}$, as a consequence of the Codazz equation (2.11). The proof is by contradiction. Indeed, if $B^{\perp}$ were non-vanishing at some $x\in M^m$, then $\langle u,w\rangle\omega_x(v)-\langle v,w\rangle\omega_x(u)=0$, for any $u,v,w\in T_x(M^m)$. Here $\langle,\rangle=g_x$. For u arbitrary, we may choose $v=w$, $\|v\|=1$, $\langle u,v\rangle=0$. Thus $\omega_x=0$, is a contradiction. Our Theorem 1 is completely proved. Of course, we are concerned with the case of a non-Kaehler (i.e. $\|\bar{\omega}\|\neq 0$) ambient P_0K-manifold. Thus, if M^m is normal to $\bar{B}$, then $B^{\perp}$ is nowhere vanishing.

To prove the corollary, let us put $c=\|\bar{\omega}\|$, $c>0$. Thus, if $\omega=0$, then M^m is an elliptic space-form (the constant sectional curvature equals $c^2/2$) by (2.10), provided that $h=0$. For the remaining situation (i.e. when $B^{\perp}=0$) the Ricci curvature of M^m, obtained by suitable contraction of indices in (2.10)) is given by (1.1). This yields our corollary. Indeed, if $B^{\perp}=0$, then (1.1) holds and thus M^2 is Ricci flat. Consequently $R=0$, i.e. M^2 is flat (for surfaces the two notions are known to coincide).

4. Complex submanifolds of generalized Hopf manifolds. Let us examine now the case of invariant (i.e. $J_x(T_x(M^m))=T_x(M^m)$, $x\in M^m$) submanifolds M^m in the PK-manifold $\bar{M}^{2n}$. Using (2.2) and the Gauss formula in (2.4), one obtains

$$(4.1)\quad h(X,JY)=Jh(X,Y)-\frac{1}{2}\{g(X,Y)A^{\perp}+\Omega(X,Y)B^{\perp}\}.$$

Note that (4.1) also furnishes $h(JX,JY)=-h(X,Y)-g(X,Y)B^{\perp}$. Therefore, the mean curvature vector $H=\frac{1}{m}$ Trace(h) of ψ is expressed by

$$(4.2)\quad H=-\frac{1}{2}B^{\perp}.$$

Hence, if ψ is minimal $(H=0)$, then by (3.1) one has $\|\omega\|=\text{const}$. Therefore, as $c\neq 0$ a minimal invariant submanifold of a PK-manifold is a strongly non-Kaehler PK-manifold itself. Let us prove now our Theorem 2. To this end, let $g_i=\psi^*\bar{g}_i$, $i\in I$. Set $D_XY=\tan(\bar{D}_XY)$, for any tangent vector fields X, Y on M^m. Then D is a torsion-free linear connection on M^m. By (2.1), (2.4), it is related to the Levi-Civita connection of (M^m, g) by

$$D_XY=\nabla_XY-\frac{1}{2}\{\omega(X)Y-\omega(Y)X-g(X,Y)B\}. \tag{4.3}$$

Clearly D is the Levi-Civita connection of the induced local metrics g_i. Let K be its curvature tensor field. Let $S(X,Y)=\text{nor}(\bar{D}_XY)$, where X, Y are tangential. Again (2.1), (2.4) furnish

$$S(X,Y)=h(X,Y)+\frac{1}{2}g(X,Y)B^{\perp}. \tag{4.4}$$

By Equation (2.6) of [4, p. 45] one has

$$g_i(K(X,Y)Z,W)=\bar{g}_i(S(X,W),S(Y,Z))-\bar{g}_i(S(X,Z),S(Y,W)) \tag{4.5}$$

provided that M^{2n} is a P_0K-manifold (i.e. $\bar{K}=0$). Actually (4.5) is the Gauss equation of $M_i=M^m\cap U_i$ in the flat Kaehler manifold $(U_i, \bar{g}_i, J)$. It is supposed tacitly that the imbedding $\psi: M^m\to M^{2n}$ is regular, such that M_i is open in M^m. Assume now that M^m is invariant and minimal. Let $m=2s$. Then, since $\omega\neq 0$, by our (4.4)—(4.5), M^{2s} is a P_0K-manifold if and only if (1.2) holds. As $s\geq 2$ (indeed, if $s=1$, then $d\Omega=0$ and M^2 is Kaehler; thus $\omega=0$, a contradiction) one may apply Theorem 3.8. in [28, p. 277] to obtain our Theorem 2.

Let $\psi: M^m\to CH^n$ be an isometric immersion of the Riemannian manifold (M^m, g) in the complex Hopf manifold endowed with the Boothby metric. Due to the Vaisman theorem, i.e. theorem 3.8. of [28, p. 277], a P_0K-manifold with $\|\omega\|=c$ will be denoted by $CH^n(c)$. To unify notation $CH^n=CH^n(2)$. If $\{E_a\}_{1\leq a\leq m}$ is a tangential orthonormal frame, the Laplacian (on functions) Δ of M^m is given by

$$\Delta f=\delta^{ab}\{E_a(E_b(f))-(\nabla_{E_a}E_b)(f)\} \tag{4.6}$$

for any $f\in C^\infty(M^m)$. Let (U, x^i) be a local system of real analytic coordinates on CH^n Let $\psi^i=x^i\circ\psi$, $1\leq i\leq 2n$, be the equations of M^m in CH^n. The Weyl connection $\bar{D}$ of CH^n satisfies

$$\bar{D}_{E_a}E_b=E_a(E_b\psi^i)\frac{\partial}{\partial x^i}. \tag{4.7}$$

On the other hand, (2.1) leads to

$$\bar{D}_{E_a}E_a=\bar{\nabla}_{E_a}E_a+\frac{1}{2}\bar{B}-\omega(E_a)E_a. \tag{4.8}$$

At this point (4.6)—(4.8) and (2.4) lead to

$$\Delta\psi^i=mH^i-B(\psi^i)+\frac{m}{2}\bar{B}(\psi^i). \tag{4.9}$$

Thus ψ^i are harmonic if and only if (4.2) holds and $(m-2)$ $B=0$. Let M^m be invariant. Then (4.2) holds. Thus $\Delta\psi^i=0$ yields either $m=2$ (and thus $d\Omega=0$) or $m\neq 2$ and then $B=0$, by (4.9).

5. Proof of theorem 4. Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}. Note that P is D-valued, while F vanishes on D; thus $P^2+tF=-I$ and P^3+P

$=0$, i.e.P is an f-structure on M^m. This is stated in [37, p. 86] under the assumption that M^{2n} is Kaehler, but clearly holds for the general case of an almost Hermitian ambient space. We consider only C.R. submanifolds (of PK-manifolds) obeying $\omega_x \neq 0$, at any $x \in M^m$. Thus we may set $u = \|\omega\|^{-1}\omega$, $U = u^+$. The following identities are obvious:

$$(5.1) \qquad u = -v \circ P - u \circ t \circ F, \quad u(V) = v(U) = 0, \quad V = -PU,$$

where $v = u \circ P$, $V = v^+$. Indices are rised with respect to g. Next one has

$$(5.2) \qquad A = \|\omega\| V - tB, \quad A^{\perp} = -\|\omega\| FU - fB.$$

Clearly $\Omega(X, Y) = g(X, FY)$. Set $\alpha = \frac{1}{2}\|\omega\|$. Substitute $\omega = 2\alpha u$ in (3.1). Since ∇ is torsion-free and h symmetric, $d\alpha \otimes u$ must be symmetric, too. Thus $d\alpha = U(\alpha)\, u$. This yields

$$(5.3) \qquad \alpha\nabla u + U(\alpha)\, u \otimes u = \frac{1}{2}\,\bar{\omega} \circ h.$$

Applying the isomorphism $^+$ to (5.3), we also obtain

$$(5.4) \qquad \nabla U = \frac{1}{2\alpha} A_{B\perp} - U(\log \alpha)\, u \otimes U.$$

As ∇ is the Levi-Civita connection of (M^m, g) we may write

$$(5.5) \qquad 2g(\nabla_X U, Y) = 2(du)(X, Y) + (L_U g)(X, Y).$$

Here L denotes the Lie derivative. Note that $d\alpha \wedge u = 0$; therefore $d\omega = 0$ leads to $du = 0$. At this point (5.4)—(5.5) give

$$(5.6) \qquad (L_U g)(X, Y) = \frac{1}{\alpha}\, g(A_{B\perp} X, Y) - 2U(\log \alpha)\, u(X)\, u(Y).$$

Therefore, if M^m is tangent to the Lee field of M^{2n}, then U is a Killing vector field, for the induced metric g. To prove the second part of Theorem 4., note that (5.1), (5.4) lead to

$$(5.7) \qquad \nabla_X V = -(\nabla_X P)\, U.$$

At this point we may use (2.5) and the identities $\theta(U) = 0$, $\omega(U) = 2\alpha$, $\Omega(U, X) = v(X)$, such as to obtain $(\nabla_X F)\, U = -\alpha PX - \frac{1}{2} u(X)\, A + \frac{1}{2} v(X)\, B$. Let us substitute in (5.7). We obtain

$$(5.8) \qquad \nabla V = \alpha P + \frac{1}{2}\{u \otimes A - v \otimes B\}.$$

Finally, (5.8) and $(L_V^{\bullet} g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V)$ lead to $L_V g = 0$, provided that $h = 0$.

6. Holomorphic distributions with totally-geodesic leaves. In this paragraph we shall prove our Theorem 5. To this end, suppose D is integrable. Let L be a leaf of D and $i: L \to M^m$ the canonical inclusion. We denote by ∇^L, h^L respectively the Levi-Civita connection of i^*g and the second fundamental form of i. Let us assume $h^L = 0$. By the Gauss formula $\nabla_X Y = \nabla^L_X Y$, i.e. $\nabla_X Y \in D$, for all $X, Y \in D$. Using this fact, (4.4) and $\overline{D}J = 0$, one obtains

$$(6.1) \qquad \bar{g}(h(X, Y), JZ) + \frac{1}{2}\, g(X, Y)\,\theta(Z) = -\frac{1}{2}\,\Omega(X, Y)\,\omega(Z)$$

for any $X, Y \in D$, $Z \in D^{\perp}$. Now the left hand (respectively the right hand) member of (6.1) is symmetric (respectively skew-symmetric) in X, Y. Consequently, both sides of (6.1) vanish, one leading to (1.3), the other giving $\Omega(X, Y)\omega(Z)=0$. We distinguish two possibilities. Either $p=0$, i.e. M^m is totally-real, or $p\neq 0$, and then $\omega=0$ on $D^{\perp}$, i.e. $B \in D$. Conversely, let us see that (1.3) yields the involutivity of D. Let $X, Y \in D$, $Z \in D^{\perp}$. As $\overline{D}$ is torsion-free and almost-complex, we obtain $g([X, Y], Z)=\overline{g}(S(X, JY), JZ)-\overline{g}(S(Y, JX), JZ)$. Substitution from (4.4) leads to $g([X, Y], Z)=\overline{g}(h(X, JY), JZ)-\overline{g}(h(Y, JX), JZ)+\Omega(X, Y)\theta(Z)$, i.e. $g([X, Y], Z)=0$ as a consequence of (1.3).

Let ι_x, $\iota^{\perp}_x$ be the natural projections of the direct sum decomposition $T_x(M^m)=D_x \oplus D^{\perp}_x$. The next step is to show that under the assumptions (3.1) and $\iota^{\perp}B=0$, each leaf L of D is totally-geodesic in M^m. Using (2.1) and $\overline{D}J=0$, for any $X, Y \in D$, $Z \in D^{\perp}$, one has

$$g(h^L(X, Y), Z)=g(\nabla_X Y, Z)=\overline{g}(\overline{\nabla}_X Y, Z)=\overline{g}(\overline{D}_X Y, Z)-\frac{1}{2}g(X, Y)\overline{g}(\overline{B}, Z)$$
$$=\overline{g}(\overline{D}_X JY, JZ)-\frac{1}{2}g(X, Y)\omega(Z)=\overline{g}(S(X, JY), JZ)-\frac{1}{2}g(X, Y)\omega(Z).$$

Now we use (4.4), (1.3) and the fact that $D^{\perp}\to L$ is precisely the normal bundle of $i: L\to M^m$ such as to obtain

$$h^L=-\frac{1}{2}g \oplus \iota^{\perp}B. \tag{6.2}$$

Thus (1.3) by itself yields totally-umbilicity of $i: L\to M^m$. Finally, by $B \in D$, our i is also minimal. To prove the second part of the statement i), let M^m be a generic C.R. submanifold. Then the normal bundle of M^m in M^{2n} is precisely $J(D^{\perp})\to M^m$ and thus (1.3) gives

$$h(X, Y)=-\frac{1}{2}g(X, Y)B^{\perp} \tag{6.3}$$

for any $X, Y \in D$. Since $h^L=0$, the second fundamental form of L in M^{2n} is precisely (6.3).

Suppose now that D is integrable and its leaves are totally-geodesic in M^{2n}. Consequently, $\overline{\nabla}_X Y \in D$, for all $X, Y \in D$. Let ξ be a cross-section in $E(\psi)$: then $\overline{g}(h(X, Y), \xi)=\overline{g}(\overline{\nabla}_X Y, \xi)=0$, i.e. M^m is D-geodesic. Conversely, suppose M^m is D-geodesic. Then computation (similar to the proof of (6.2)) leads to

$$\iota^{\perp}[X, Y]=-\Omega(X, Y)tB^{\perp} \tag{6.4}$$

for any $X, Y \in D$. Our Theorem 5 is completely proved.

7. Pinching on C.R. submanifolds. Let M^m be a submanifold of the P_0K-manifold $CH^n(c)$, $c=\|\omega\|$. Let Riem: $G_2(M^m)\to \mathbb{R}$ be its Riemannian sectional curvature. Let X, Y be two orthonormal tangent vector fields on M^m; the Gauss equation (2.10) leads to

$$\mathrm{Riem}(\sigma_{XY})=\frac{c^2}{4}-\frac{1}{4}[\omega(X)^2+\omega(Y)^2]+\overline{g}(h(X, X), h(Y, Y))-\|h(X, Y)\|^2. \tag{7.1}$$

Here $\sigma_{XY}\in G_2(M^m)$ is the 2-plane spanned by X, Y. Let $\{\xi_a\}_{1\leq a\leq 2n-m}$ be a (locally defined) orthonormal frame of $E(\psi)\to M^m$. Set $A_a=A_{\xi_a}$, $1\leq a\leq 2n-m$. For any tangent vector fields X, Y on M^m, suitable contraction of indices in (2.10) furnishes the following expression of the Ricci curvature:

$$\text{Ric}(X, Y)=m\bar{g}(h(X, Y), H)-\sum_{a=1}^{2n-m} g(A_a^2X, Y) + \frac{1}{4}\{(m-1)c^2-\|\omega\|^2\}g(X, Y)-\frac{m-2}{4}\omega(X)\omega(Y). \tag{7.2}$$

Let $\{E_i\}_{1\leq i\leq m}$ be a (locally defined) tangential orthonormal frame. Set $h(E_i, E_j) = h_{ij}^a\xi_a$. Then

$$\|h\|^2=\sum_{a,i,j}(h_{ij}^a)^2. \tag{7.3}$$

Moreover, by (7.1), the sectional curvature of the 2-plane spanned by E_i, E_j, $i\neq j$, is expressed by

$$\text{Riem}(\sigma_{E_iE_j})=\frac{c^2}{4}-\frac{1}{4}\{\omega_i^2+\omega_j^2\}+\sum_a\{h_{ii}^a h_{jj}^a-(h_{ij}^a)^2\}, \tag{7.4}$$

where $\omega(E_i)=\omega_i$. We need to establish the following:

Lemma 7.1. *Let M^m be a proper* (i.e. $p\neq 0$, $q\neq 0$) *C.R. submanifold of the l.c.K. manifold M^{2n}. Then its totally-real distribution is D-parallel if and only if* (1.3) *holds and $B\in D$.*

Proof. The proof follows from $\bar{g}(h(X, Y), JZ)+\frac{1}{2}g(X, Y)\theta(Z)=g(JY, \nabla_X Z) -\frac{1}{2}\Omega(X, Y)\omega(Z)$, for any $X, Y\in D$, $Z\in D^\perp$, by computations similar to those carried on during the proof of Theorem 5.

Remark. Cf. our Theorem 5, the totally-real distribution of M^m is D-parallel if and only if D is integrable and its leaves are totally-geodesic in M^m.

Let us prove our Theorem 6. Let ν_x be the orthogonal complement of $J_x(D_x^\perp)$ in $E(\psi)_x$, $x\in M^m$. We choose an orthonormal frame on $CH^n(c)$ in the following manner. Let $p_0\in G_2(M^m)$ be a D-anti-holomorphic 2-plane on M^m and let $\{E_1, E_2\}$ be an orthonormal pair of tangent vector fields such that $E_i\in D$, $i=1, 2$, and $\{E_{1,x}, E_{2,x}\}$ span p_0, where $x=\pi(p_0)$. Let $\{E_A\}_{1\leq A\leq 2p}$ be an orthonormal frame of D, otherwise written $\{E_i, E_{i^*}\}$, $E_{i^*}=JE_i$, $i^*=i+p$, $1\leq i\leq p$. Next we consider $F_a\in D^\perp$, $1\leq a\leq q$, an orthonormal frame. Set $F_{a^*}=JF_a$, $a^*=a+q$. Finally, let $\{V_\alpha, V_{\alpha^*}\}$, $V_{\alpha^*}=JV_\alpha$, $1\leq\alpha\leq r$, $\alpha^*=\alpha+r$, be an orthonormal frame of ν. Here $2r=2n-m-q$. Then $\{E_A, F_a, F_{a^*}, V_\alpha, V_{\alpha^*}\}$ is an orthonormal frame on $CH^n(c)$, such that $\{E_i, E_{i^*}, F_a\}$ are tangential, while $\{F_{a^*}, V_\alpha, V_{\alpha^*}\}$ are normal. As $D^\perp$ is supposed to be D-parallel, by Lemma 7.1. we obtain

$$h_{ij}^a+\frac{1}{2}\theta(F_a)\delta_{ij}=0. \tag{7.5}$$

Consider $Y\in D$. By (2.2), (2.4), one obtains

$$h(X, JY)=F\nabla_X Y+fh(X, Y)-\frac{1}{2}\{\omega(Y)FX+g(X, Y)A^\perp+\Omega(X, Y)B^\perp\} \tag{7.6}$$

for any X tangent to M^m. Suppose now $X\in D$. Since, as observed above, one may combine Lemma 7.1. and Theorem 5, it follows that the leaves of D are totally-geodesic in M^m, i.e. $\nabla_X Y\in D$. But $F=0$ on D such that (7.6) becomes

$$h(X, JY)=fh(X, Y)-\frac{1}{2}\{g(X, Y)A^\perp+\Omega(X, Y)B^\perp\}. \tag{7.7}$$

Consequently

$$h(JX, JY)=f^2h(X, Y)-\frac{1}{2}\{g(X, Y)[fA^\perp+B^\perp]-\Omega(X, Y)[fB^\perp+A^\perp]\} \tag{7.8}$$

for any $X, Y \in D$. The following identities hold:

$$Pt+tf=0, \quad Ft+f^2=-i \tag{7.9}$$

as direct consequences of definitions. But t is $D^\perp$-valued, while $P=0$ on $D^\perp$. Thus $Pt=0$. By (7.9) one also has $tf=0$. Now, if $Z \in D^\perp$, then $g(fFZ, \xi)=g(Z, tf\xi)=0$, such that $fF=0$. Consequently f vanishes on $J(D^\perp)$. Note also that the following identities hold:

$$\begin{aligned} &\theta(F_a)=(B^\perp)^a, \\ &(B^\perp)^a=-(A^\perp)^{a^*}=\bar{\omega}(V_a), \\ &(B^\perp)^{a^*}=(A^\perp)^a=\bar{\theta}(V_a). \end{aligned} \tag{7.10}$$

Now (7.8) leads to

$$\begin{aligned} &h^a_{ij}=h^a_{i^*j^*}=-\frac{1}{2}\theta(F_a)\delta_{ij}, \\ &h^\alpha_{i^*j^*}=-h^\alpha_{ij}-\bar{\omega}(V_a)\delta_{ij}, \\ &h^{\alpha^*}_{i^*j^*}=-h^{\alpha^*}_{ij}-\frac{1}{2}\bar{\theta}(V_a)\delta_{ij}. \end{aligned} \tag{7.11}$$

On the other hand, (7.4) shows that

$$\begin{aligned} \mathrm{Riem}(p_0)=\frac{c^2}{4}-\frac{1}{4}[\omega_1^2+\omega_2^2]+\sum_a\{h^a_{11}h^a_{22}-(h^a_{12})^2\} \\ +\sum_a\{h^\alpha_{11}h^\alpha_{22}+h^{\alpha^*}_{11}h^{\alpha^*}_{22}-(h^\alpha_{12})^2-(h^{\alpha^*}_{12})^2\}. \end{aligned} \tag{7.12}$$

Following the line in [4, p. 98], one obtains

$$\|h\|^2 \geq 2\sum_a\{(h^a_{12})^2-h^a_{11}h^a_{22}\}+2\sum_\alpha\{(h^\alpha_{12})^2-h^\alpha_{11}h^\alpha_{22}+(h^{\alpha^*}_{12})^2-h^{\alpha^*}_{11}h^{\alpha^*}_{22}\}. \tag{7.13}$$

Finally, (1.4), (7.12)–(7.13) lead to $\mathrm{Riem}(p_0)\geq A$, Q.E.D.

8. Proof of Theorem 7. Let M^{2n} be a l.c.K. manifold. Let σ, σ' be two holomorphic 2-planes on M^{2n}; we recall, cf. [23], [24], the concept of *holomorphic bisectional curvature* of M^{2n}, i.e. if $\pi(\sigma)=\pi(\sigma')=x$, $x\in M^{2n}$, and if $u\in\sigma$, $v\in\sigma'$ are two unit tangent vectors, then we define

$$\mathrm{Riz}(\sigma, \sigma')=\langle \bar{R}_x(v, J_xv)J_xu, u\rangle,$$

where $\langle\,,\rangle=\bar{g}_x$. The definition of $\mathrm{Riz}(\sigma, \sigma')$ does not depend upon the choice of unit vectors u, v in σ, σ', respectively.

The proof of our Theorem 7 is by contradiction. Let M^m be a complex submanifold of a l.c.K. manifold obeying $\mathrm{Riz}>0$. Suppose ξ is a parallel section in $E(\psi)$. Then $R^\perp(X, Y)\xi=0$ and the Ricci equation, i.e. eq. (2.11) of [4, p. 47] leads to

$$\bar{g}(\bar{R}(X, Y)\xi, \eta)=-g([A_\xi, A_\eta]X, Y). \tag{8.1}$$

Let X be a unit tangent vector field. Let $Y=JX$, $\eta=J\xi$, in (8.1). If $\xi\neq0$, then there is $x\in M^m$ such that $\xi_x\neq0$; let $N=\|\xi_x\|^{-1}\xi_x$. Let σ, σ' be the holomorphic 2-planes spanned by $\{u, J_xu\}$, $u=X_x$, and respectively by $\{N, J_xN\}$. Then we may combine (8.1) and the following:

Lemma 8.1. *For any cross-section ξ in the normal bundle of a complex submanifold M^m of a l.c.K. manifold the following identity holds:*

$$[A_\xi, A_{J\xi}]=-2J\{A_\xi+\frac{1}{2}\bar{\omega}(\xi)I\}^2, \tag{8.2}$$

where I denotes the identical transformation.

Indeed, as $A_\xi + \frac{1}{2}\bar{\omega}(\xi) I$ is self-adjoint, (8.1)-(8.2) lead to

$$\| \xi_x \|^2 \mathrm{Riz}\,(\sigma, \sigma') = -2 \| A_\xi X + \frac{1}{2} \bar{\omega}(\xi) X \|_x^2 \leqq 0 \tag{8.3}$$

a contradiction. All we need is to prove Lemma 8.1. This follows by computation from the identities

$$A_{J\xi} = JA_\xi + \frac{1}{2}\{\bar{\omega}(\xi) J - \bar{\theta}(\xi) I\},$$
$$JA_\xi = -A_\xi J - \bar{\omega}(\xi) J. \tag{8.4}$$

In turn, (8.4) is a consequence of (2.2), (2.4).

REFERENCES

1. M. Barros, F. Urbano. Submanifolds of complex Euclidean space which admit a holomorphic distribution. *Quart. J. Math. Oxford,* 2 (34), 1983, 141-143.
2. A. Beancu. On the geometry of leaves on a C.R. submanifold. *An. St. Univ. "Al I. Cuza"*, Jasi, 2 (25), 1979, 393-398.
3. A. Bejancu. Geometry of C.R. submanifolds. Tokyo, 1986.
4. B. Y. Chen. Geometry of submanifolds. New York, 1973.
5. B. Y. Chen, K. Ogiue. On the scalar curvature and sectional curvatures of a Kaehler submanifolds. *Proc. Amer. Math. Soc.,* **1** (41), 1973, 247-250.
6. B. Y. Chen, K. Ogiue. A characterization of the complex sphere. *Michigan Math. J.,* **21**, 1974, 231-232.
7. B. Y. Chen, M. Okumura. Scalar curvature, inequality and submanifold. *Proc. Amer. Math. Soc.,* **3** (38), 1973, 605-608.
8. B. Y. Chen, P. Piccinni. The canonical foliations of a locally conformal Kaehler manifold. *Ann. Matem. pura appl.,* **4** (CXLI), 1985, 289-305.
9. B. Y. Chen, H. Lue. On normal connection of Kaehler submanifolds. *J. Math. Soc. Japan,* **4** (27), 1975, 550-556.
10. S. Dragomir. Cauchy-Riemann submanifolds of locally conformal Kaehler manifolds. I-II. *Geometriae Dedicata,* **28**, 1988. 181-197; *Atti Sem. Fis. Univ. Modena,* **37**, 1989, 1-11.
11. S. Dragomir. On submanifolds of Hopf manifolds. *Israel J. Math.,* **2** (61), 1988, 199-210.
12. S. Dragomir. Totally real submanifolds of generalized Hopf manifolds. *Le Matematiche,* XLII, 1987, 3-10.
13. S. Dragomir, R. Grimaldi. Isometric immersions of Riemann spaces in a real Hopf manifold. *J. Math. pures et appl.,* **68**, 1989, 335-364.
14. S. Goldberg. Curvature and Homology. New. York, 1962.
15. S. Goldberg, S. Kobayashi. Holomorphic bisectional curvature. *J. Diff. Geometry,* **1**, 1967, 225-233.
16. S. Goldberg, I. Vaisman. On compact locally conformal Kaehler manifolds with non-negative sectional curvature. *Ann. Fac. Sci. Toulouse,* **2**, 1980, 117-123.
17. S. Ianus, K. Matsumoto, L. Ornea. Complex hypersurfaces of a generalized Hopf manifold. *Publ. de l' Inst. Math.,* N. S., **42**, 1987, 123-129.
18. T. Kashiwada. Some properties of locally conformal Kaehler manifolds. *Hokkaido Math. J.,* **8**, 1979, 191-198.
19. S. Kobayashi, K. Nomizu. Foundations of differential geometry. Intersci. Publ., Vol. I-II. New York, 1963, 1969.
20. P. Libermann. Sur les structures presque complexes et autres structures infinitésimales réguliers. *Bull. Soc. Math. France,* **83**, 1955, 195-224.
21. K. Matsumoto. On submanifolds of locally conformal Kaehler manifolds. *Bull. Yamagata Univ.,* N. S., **11**, 1984, 33-38.
22. L. Ornea. On C.R. submanifolds of locally conformal Kaehler manifolds. *Demonstratio Math.,* **4** (19), 1986, 863-869.
23. G. B. Rizza. Sulla curvatura delle faccette di una varietà Kaehleriana. *Ann. Matem. pura appl.,* **4** (XLVII), 1959, 81-90.

24. G. B. Rizza. Problemi di curvatura su di una varietà quasi hermitiana. *Ann. Matem. pura appl.*, 4 (XCII), 1972, 37-63.
25. G. B. Rizza. Varietà parakaehleriane. *Ann. Matem. pura appl.*, 4 (XCVIII), 1974, 47-61.
26. F. Tricerri. Some examples of locally conformal Kaehler manifolds. *Rend. Sem. Mat. Univers. Politecn. Torino*, **1** (40), 1982, 81-92.
27. I. Vaisman. Generalized Hopf manifolds. *Geometriae Dedicata*, **13**, 1982, 231-255.
28. I. Vaisman. Locally conformal Kaehler manifolds with parallel Lee form. *Rendiconti di Matem.*, **12**, 1779, 263—284.
29. I. Vaisman. On locally conformal almost Kaehler manifolds. *Israel J. Math.*, **24**, 1976, 338-351.
30. I. Vaisman. Remarkable operators and commutation formulas on locally conformal Kaehler manifolds. *Comp. Math.*, **40**, 1980, 287-300.
31. I. Vaisman. A theorem on compact locally conformal Kaehler manifolds. *Proc. Amer. Math. Soc.*, 2 (75), 1979, 279-283.
32. I. Vaisman. Some curvature properties of locally conformal Kaehler manifolds. *Trans. Amer. Math. Soc.*, **259**, 1980, 439-447.
33. I. Vaisman. On locally and globally conformal Kaehler manifolds. *Trans. Amer. Math. Soc.*, **262**, 1980, 533-542.
34. K. Yano. On a structure defined by a tensor field of type (1, 1) satisfying $f^3+f=0$. *Tensor*, N.S., **14**, 1963, 99-109.
35. K. Yano, M. Kon. C.R. submanifolds of Kaehlerian and Sasak an manifolds. *Progress in Math.*, **30**, Boston-Basel-Stuttgart, 1983.
36. K. Yano, M. Kon. Generic submanifolds. *Ann. Matem. pura appl.*, **123**, 1980, 59-92.
37. K. Yano, M. Kon. Differential Geometry of C.R. submanifolds. *Geometriae Dedicata*, **10**, 1981, 369-391.

Received 7. 02. 1990

S. Dragomir
State University of New York at Stony Brook
Mathematics Department
Stony Brook, N. Y., 11794, USA

R. Grimaldi
Università de Palermo
Dipartimento di Matematica ed Applicazioni
Via Archirafi 34
90123 Palermo, Italia