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COMMUTATIVITY CONDITIONS OF ONE-SIDED s-UNITAL RINGS WITH CERTAIN CONSTRAINTS

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In this paper we prove the following commutativity theorem: Let $n>0, m, t$, and s (resp. $m>0, n, t$ and s) be fixed non-negative integers such that $(n, t, m, s)\neq(1, 0, 1, 0)$, and let R be a left (resp. right) s-unital ring satisfying the polynomial identity $x^t[x^n, y]=[y^m, x]y^s$ for all $x, y\in R$. If further R possesses $Q(n)$ property (that is, for all $x, y\in R$, $n[x, y]=0$ implies $[x, y]=0$), then R is commutative. If $Q(n)$ property is replaced by m and n are relatively prime integers, then R must be commutative. Also, the commutativity of R has been proved under different sets of conditions.

1. Introduction. The advent of the twentieth century marks the beginning of the investigation of classes of rings which turn out to be commutative under certain constraints. An earlier example is the Wedderburn theorem, which states that a finite division ring is necessarily commutative. While studying the commutativity of rings, one encounters the difficulty that there is no clear cut way to allow cancellation among the elements of rings, which is permissible in case of groups. The objective of the present paper is to investigate the commutativity of left and right s-unital rings satisfying the polynomial identity

$$x^t[x^n, y]=[y^m, x]y^s \text{ for all } x, y\in R,$$

where n, m, s, and t are fixed integers, under different set of conditions.

To establish commutativity of a ring R with the above polynomial identity, we need some additional properties on R (commutators). They frequently concern the torsion freeness of the commutators in R, like the following property:

Let n be some positive integer. Then

$$Q(n)\text{: for all } x, y\in R,\ n[x, y]=0 \text{ implies } [x, y]=0.$$

The property $Q(n)$ is an H-property in the sense of [9]. Obviously, every n-torsion free ring R has the property $Q(n)$, and every ring has the property $Q(1)$.

2. Preliminary results. Throughout this paper let R denote an associative ring (with or without unity 1), $Z(R)$ the center of R, $C(R)$ the commutator ideal of R, $N(R)$ the set of all nilpotent elements in R, and $N'(R)$ the set of all zero-divisors in R. For any x and y in a ring R (resp. group G), we write as usual $[x, y]=xy-yx$ (resp. $[x, y]=xyx^{-1}y^{-1}$). By $GF(q)$ we mean the Galois field (finite field) with q elements, and by $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. In $(GF(q))_2$, we set $e_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $e_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $e_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, and $e_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$.

Definition 1. *A ring R is called a left (resp. right) s-unital if $x\in Rx$ (resp. $x\in xR$), for all $x\in R$. Further, R is called an s-unital ring if it is both left as well as right s-unital, that is $x\in Rx\cap xR$ for each $x\in R$.*

Definition 2. *If R is an s-unital (resp. left or right s-unital) ring, then for any finite subset F of R, there exists an element $e\in R$ such that $ex=xe=x$ (resp. $ex=x$*

or $xe=x$*) for all* $x\in F$*. Such an element* e *is called the pseudo (resp. pseudo left or pseudo right)-identity of* F *in* R*.*

We shall require the following well-known results:

Lemma 1 ([2, *Lemma* 2]). *Let* R *be a ring with unity* 1, *and let* x *and* y *be elements in* R*. If* $kx^m[x, y]=0$ *and* $k(x+1)^m[x, y]=0$ *for some integers* $m\geqq 1$ *and* $k\geqq 1$ *then necessarily* $k[x, y]=0$*.*

Lemma 2 ([14, *Lemma* 3]). *Let* x *and* y *be elements in a ring* R*. If* $[x, [x, y]]=0$*, then* $[x^k, y]=kx^{k-1}[x, y]$ *for all integers* $k\geqq 1$*.*

Lemma 3 ([17, *Lemma*]). *Let* R *be a ring with unity* 1, *and let* x *and* y *be elements in* R*. If* $(1-y^k)x=0$*, then* $(1-y^{km})x=0$ *for any integers* $k>0$ *and* $m>0$*.*

Lemma 4 ([4, *Theorem* 4 (*C*)]). *Let* R *be a ring with unity* 1*. Suppose that for each* $x\in R$*, there exists a pair* n *and* m *of relatively prime positive integers for which* $x^n\in Z(R)$ *and* $x^m\in Z(R)$*. Then* R *is commutative.*

Lemma 5. *Let* x *and* y *be elements in a ring* R*. Suppose that there exists relatively prime positive integers* n *and* m *such that* $n[x, y]=m[x, y]=0$*. Then* $[x, y]=0$*.*

Proof. If $n>0$ and $m>0$ are relatively prime integers, then there exist integers t and s such that $1=tn+sm$. Thus $[x, y]=tn[x, y]+sm[x, y]=0$.

The following theorems play an important role in proving our results. The first and the second are due to I. N. Herstein [7, Theorem 18] and [8, Theorem] respectively, and the third due to T. P. Kezlan [10, Theorem] and H. E. Bell [3, Theorem 1] (also see [9, Proposition 2]).

Theorem *H*. *Let* R *be a ring, and let* $n>1$ *be a fixed integer. Suppose that*

$$(x^n-x)\in Z(R) \text{ for all } x\in R,$$

then R *is commutative.*

Theorem *H'*. *If for every* x *and* y *in a ring* R *we can find a polynomial* $p_{x,y}(t)$ *with integral coefficients which depend on* x *and* y *such that*

$$[x-x^2p_{x,y}(x), y]=0,$$

then R *is commutative.*

Theorem *KB*. *Let* f *be a polynomial in* n *non-commuting indeterminates* $x_1, x_2, \ldots, x_n$ *with relatively prime integral coefficients. Then the following are equivalent:*

(1) *For any ring* R *satisfying* $f=0$*,* $C(R)$ *is a nil ideal.*

(2) *Every semi-prime ring satisfying* $f=0$ *is commutative.*

(3) *For every prime* p*,* $(GF(p))_2$ *fails to satisfy* $f=0$*.*

3. **Results.** The objective of the present paper is to prove the following results:

Theorem 1. *Let* $n>0$*,* m*,* t*, and* s *(resp.* $m>0$*,* n*,* t*, and* s*) be fixed non-negative integers such that* $(n, t, m, s)\neq(1, 0, 1, 0)$*. Let* R *be a left* (resp. right) *s-unital ring satisfying the polynomial identity*

$$x^t[x^n, y]=[y^m, x]y^s \text{ for all } x, y\in R. \tag{1}$$

Further, if R *possesses* $Q(n)$ *property, then* R *is commutative.*

First we prove the following lemma:

Lemma 6. *Let* $n>0$*,* m*,* t*, and* s *(resp.,* $m>0$*,* n*,* t*, and* s*) be fixed non-negative integers such that* $(n, t, m, s)\neq(1, 0, 1, 0)$*. Suppose that* R *is a left (resp. right) s-unital ring satisfying the polynomial identity* (1)*. Then* R *is an s-unital ring.*

Proof. Let x and y be arbitrary elements of R, and choose an element $e\in R$ such that $ex=x$ and $ey=y$ (resp., $xe=x$ and $ye=y$). If either $m\neq 1$ or $s>0$ (resp., $n\neq 1$ or $t>0$), then (1) gives

$$y=ye^n+y^m ey^s-y^{m+s}=y(e^n+y^{m-1}ey^s-y^{m+s-1})\in yR$$

(resp.

$$x=e^m x-x^{n+t}+x^t ex^n=(e^m-x^{t+n-1}+x^t ex^{n-1})x\in Rx).$$

On the other hand, if $(m, s)=(1, 0)$ (resp. $(n, t)=(1, 0)$), then $(n, t)\neq(1, 0)$ (resp. $(m, s)\neq(1, 0)$). Thus

$$x=xe-x^{t+n}+x^{t+n}e=x(e-x^{t+n-1}+x^{t+n-1}e)\in xR$$

(resp.,

$$y=ey-y^{m+s}+ey^{m+s}=(e-y^{m+s-1}+ey^{m+s-1})\in R).$$

Therefore, R is a right (resp., left) s-unital ring, and hence R is an s-unital ring.

Proof of Theorem 1. *According to Lemma 6, R is an s-unital ring. Therefore, in view of Proposition 1 of [9], it suffices to prove the theorem for R with unity 1.*

We distinguish two cases:

(I) First, we consider the case $n>1$. Let $x=e_{12}$, and $y=e_{11}$, in $(GF(p))_2$, for any prime p. Then it is easy to check that x and y fail to satisfy the polynomial identity (1), for $m\geqq 1$. Next, if $m=0$, let $x=e_{11}$ and $y=e_{12}$. Again, we find x and y fail to satisfy (1). Therefore, by Theorem KB, we have $C(R)\subseteq N(R)$.

Now, let $a\in N(R)$. Then there exists a positive integer q such that

(2) $$a^{k'}\in Z(R) \text{ for all } k'\geqq q, \text{ where } q \text{ is minimal.}$$

Suppose $q>1$. Replace x by a^{q-1} in (1) to obtain

$$(a^{q-1})^t[(a^{q-1})^n, y]=[y^m, a^{q-1}]y^s \text{ for all } y\in R.$$

In view of (2) and the fact that $(q-1)n\geqq q$, the last identity implies

$$[y^m, a^{q-1}]y^s=0 \text{ for all } y\in R.$$

In (1), replace x by $(1+a^{q-1})$ to get

$$(1+a^{q-1})^t[(1+a^{q-1})^n, y]=[y^m, (1+a^{q-1})]y^s \text{ for all } y\in R.$$

Thus, for all $y\in R$, we have

$$n(1+a^{q-1})^t[a^{q-1}, y]=(1+a^{q-1})^t[(1+a^{q-1})^n, y]=[y^m, (1+a^{q-1})]y^s$$
$$=[y^m, a^{q-1}]y^s=0.$$

But $(1+a^{q-1})$ is an invertible element in R. Hence,

$$n[a^{q-1}, y]=0 \text{ for all } y\in R.$$

Since R possesses $Q(n)$ property, the last polynomial identity implies that

$$[a^{q-1}, y]=0 \text{ for all } y\in R.$$

Thus we have a cotradiction to the minimality of q, (see (2)). This contradiction shows that $q=1$. So $N(R)\subseteq Z(R)$. Therefore,

(3) $$C(R)\subseteq N(R)\subseteq Z(R).$$

Since, by (3), we have $[x, [x, y]]=0$ for all $x, y\in R$, we shall routinely apply Lemma 2 without explicit mention.

Now, if $m=0$, then (1) becomes

$$nx^{t+n-1}[x, y]=0 \text{ for all } x, y\in R.$$

Replacing x by $(x+1)$ and applying Lemma 1 to the last polynomial identity gives $n[x, y]=0$ for all $x, y\in R$. As R has $Q(n)$ property, we obtain $[x, y]=0$, for all $x, y\in R$. Therefore, R is commutative.

Suppose $m\geqq 1$. In view of (3), the polynomial identity (1) becomes

(4) $$nx^{t+n-1}[x, y]=m[y, x]y^{s+m-1}=my^{s+m-1}[y, x] \text{ for all } x, y\in R.$$

Let $l=(2^{t+n}-2)$. Then $l>1$, for $n>1$. By repeated use of (1), we have

$$lx^t[x^n, y]=(2^{t+n}-2)x^t[x^n, y]=(2x)^t[(2x)^n, y]-2x^t[x^n, y]$$
$$=(2x)^t[(2x)^n, y]-2[y^m, x]y^s=(2x)^t[(2x)^n, y]-[y^m, (2x)]y^s=0,$$

for all $x, y\in R$. Let $k=ln>1$. Then $kx^{t+n-1}[x, y]=0$ for all $x, y\in R$. Replacing x by $(x+1)$ and applying Lemma 1, we get $k[x, y]=0$ for all $x, y\in R$. So $[x^k, y]=kx^{k-1}[x, y]=0$ for all $x, y\in R$. Therefore,

(5) $$x^k\in Z(R) \text{ for all } x\in R.$$

Now, we consider the following cases:

(i) Let $m>1$. Then (1) gives

$$x^t[x^n, y]=m[y, x]y^{s+m-1} \text{ for all } x, y\in R.$$

Replace y by y^m in the above identity to obtain

$$x^t[x^n, y^m]=m[y^m, x]y^{m(s+m-1)} \text{ for all } x, y\in R.$$

So

$$mx^t[x^n, y]y^{m-1}=m[y^m, x]y^{m(s+m-1)},$$

and hence

$$m[y^m, x]y^{s+m-1}=m[y^m, x]y^{m(s+m-1)}.$$

Thus

$$m[y^m, x]y^{s+m-1}(1-y^{(m-1)(s+m-1)})=0.$$

In view of Lemma 3, we get

(6) $$m[y^m, x]y^{s+m-1}(1-y^{k(m-1)(s+m-1)})=0 \text{ for all } x, y\in R.$$

It is well known that R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i ($i\in I$, the index set). Each R_i satisfies (1), (3), (4), (5), and (6), but R_i is not necessarily having the property $Q(n)$. Thus, we let S be the intersection of all non-zero ideals of R_i. Hence, $S\neq 0$. Let v be a central zero-divisors in R. Then, it can be easily verified $Sv=0$.

Let $u\in N'(R_i)$. By (6), we have

$$m[u^m, x]u^{s+m-1}(1-u^{k(m-1)(s+m-1)})=0 \text{ for all } x\in R_i.$$

Suppose that

$$m[u^m, x]u^{s+m-1}\neq 0 \text{ for all } x\in R_i.$$

Then $u^{k(m-1)(s+m-1)}$, and $(1-u^{k(m-1)(s+m-1)})$ are central zero-divisors. So

$$(0)=S(1-u^{k(m-1)(s+m-1)})=S\neq(0).$$

Hence we have a contradiction. This contradiction shows that $m[u^m, x]u^{s+m-1}=0$ for all $x\in R_i$. Therefore,

(7) $$m^2[u, x]u^{s+2(m-1)}=0 \text{ for all } x\in R_i.$$

Now, for all $x, y\in R_i$, (4), and (7) gives

$$nx^{t+n-1}[x, u^m]=mnx^{t+n-1}[x, u]u^{m-1}=m^2[u, x]u^{s+2(m-1)}=0.$$

Replacing x by $(x+1)$, in the last identity and applying Lemma 1, yields

$$n[x, u^m]=0(n[u^m, x]=0) \text{ for all } x\in R_i.$$

Thus

$$nm[u, x]u^{m-1}=n[u^m, x]=0 \text{ for all } x\in R_i,$$

and hence

$$n^2x^{t+n-1}[x, u]=n(m[u, x]u^{s+m-1})=0 \text{ for all } x\in R_i.$$

As an application to Lemma 1, we obtain

(8) $$n^2[x, u]=0 \text{ for all } x\in R_i, \text{ and } u\in N'(R_i).$$

Next, let $c\in Z(R_i)$. Then for all $x, y\in R_i$, we have

$$(c^{t+n}-c)x^t[x^n, y]=(cx)^t[(cx)^n, y]-cx^t[x^n, y]=(cx)^t[(cx)^n, y]-c[y^m, x]y^s$$
$$=(cx)^t[(cx)^n, y]-[y^m, (cx)]y^s=0,$$

and hence

$$n(c^{t+n}-c)x^{t+n-1}[x, y]=0 \text{ for all } x, y\in R_i.$$

If we replace x by $(x+1)$, and using Lemma 1, we finally obtain

$$n(c^{t+n}-c)[x, y]=0 \text{ for all } x, y\in R_i,$$

which implies

$$(c^{t+n}-c)[x^n, y]=n(c^{t+n}-c)[x, y]x^{n-1}=0 \text{ for all } x, y\in R_i.$$

Therefore,

(9) $$(c^{t+n}-c)[x^n, y]=0 \text{ for all } x, y\in R_i.$$

In particular, by (5), we get

(10) $$(y^{k(t+n)}-y^k)[x^n, y]=0 \text{ for all } x, y\in R_i.$$

If $n[x, y]\neq 0$, then (10) shows that $(y^{k(t+n)}-y^k)$ in R_i is a zero-divisor. Hence $(y^{k(t+n-1)+1}-y)$ is also a zero-divisor in R_i. Therefore, (8) implies

$$n^2[x, y^{k(t+n-1)+1}-y]=0 \text{ for all } x, y\in R_i.$$

If $n[x, y]=0$, then the same holds trivially. Thus

(11) $$n^2[x, y^{k(t+n-1)+1}-y]=0 \text{ for all } x, y\in R_i.$$

Since each R_i $(i\in I)$ satisfies (11), the original ring R also satisfies (11). But R has $Q(n)$ property. Hence combining (11) and Lemma 2, we finally get

$$(y^j-y)\in Z(R) \text{ for all } y\in R, \text{ and } j=(k(t+n-1)+1)>1.$$

Therefore, R is commutative by Theorem H.

(ii) Let $m=1$, and $s>0$. Then by (4) we have

$$nx^{t+n-1}[x, y]=[y, x]y^s \text{ for all } x, y\in R.$$

Replacing x by x^n in the above identity, gives

$$nx^{n(t+n-1)}[x^n, y]=[y, x^n]y^s \text{ for all } x, y\in R.$$

Following the proof of case (i), we can see that

$$n(1-x^{k(n-1)(t+n-1)})x^{t+n-1}[x^n, y]=0 \text{ for all } x, y\in R.$$

If $u\in N'(R_i)$, then we can easily prove $n^2u^{t+2(n-1)}[u, y]=0$ for all $y\in R_i$. Hence, $[y, u^n]=0$ $(nu^{n-1}[y, u]=0)$. Thus

$$[y, u]y^s = nu^{t+n-1}[u, y] = 0 \text{ for all } y \in R_i.$$

By applying Lemma 1, we have $N'(R_i) \subseteq Z(R_i)$ (see (8)). Next, if we proceed as in case (i), we can easily get $(c^{s+1} - c)[y, x] = 0$ for all $x, y \in R$, and $c \in Z(R_i)$. Following the argument in the previous case, we see that

$$(y^{ks+1} - y) \in Z(R) \text{ for all } y \in R.$$

Therefore, by Theorem H, R is commutative provided $s > 0$.

(iii) If $m = 1$ and $s = 0$, then (1) becomes

$$x^t[x^n, y] + [x, y] = 0 \text{ for all } x, y \in R.$$

Therefore, R is commutative by [11, Theorem].

(II) Next, we study the case $n = 1$. Then (1) becomes

$$x^t[x, y] = [y^m, x]y^s \text{ for all } x, y \in R. \tag{12}$$

If $m = 0$, then $x^t[x, y] = 0$, and hence $[x, y] = 0$ for all $x, y \in R$. Therefore, R is commutative. If $m > 1$, then by the symmetry of the proof of case (I), we can easily establish the commutativity of R

Next, let $m = 1$ and $s = 0$. Then $t > 0$, and

$$x^t[x, y] = [y, x] \text{ for all } x, y \in R.$$

Hence, R is commutative by T. P. Kezlan' Theorem [11].

Finally, let $m = 1$, and $s > 0$ in (12). Then for any integer r, we can see that

$$x^{rt}[x, y] = [y, x]y^{rs} \text{ for all } x, y \in R. \tag{13}$$

Let $x = e_{11}$ and $y = e_{12}$. Then x and y fail to satisfy the identity

$$x^t[x, y] = [y, x]y^s \text{ for all } x, y \in R. \tag{14}$$

Therefore, by Theorem KB, we get $C(R) \subseteq N(R)$. If $u \in N(R)$, then (13) gives

$$x^{rt}[x, u] = [u, x]u^{rs} \text{ for all } x \in R, r > 0 \text{ and } s > 0.$$

But since $u \in N(R)$, $u^{rs} = 0$, for sufficiently large r. Thus $x^{rt}[x, u] = 0$ for all $x \in R$. Replace x by $(x+1)$, and apply Lemma 1, to get $[x, u] = 0$ for all $x \in R$, that is $u \in Z(R)$. Therefore,

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{15}$$

Now, let $d = (2^{s+1} - 2) > 1$, for $s > 0$. Then for all $x, y \in R$, (14) gives

$$\begin{aligned} d[y, x]y^s = (2^{s+1} - 2)[y, x]y^s &= [(2y), x](2y)^s - 2[y, x]y^s \\ &= [(2y), x](2y)^s - x^t[x, (2y)] = 0. \end{aligned}$$

Therefore, $d[y, x] = 0$ for all $x, y \in R$, and hence $[y, x^d] = dx^{d-1}[y, x] = 0$ for all $x, y \in R$. Thus

$$x^d \in Z(R) \text{ for all } x \in R. \tag{16}$$

By (14) and (15), we have $x^t[x, y] + y^s[x, y] = 0$, and hence

$$(x^t + y^s)[x, y] = 0 \text{ for all } x, y \in R. \tag{17}$$

Represent R as a subdirect sum of a family $\{R_\lambda \mid \lambda \in \Lambda\}$ of subdirectly irreducible rings which are homomorphic images of R. Clearly, each R_λ, has 1, and satisfies (14), (16) and (17). It is our aim to show that each zero-divisor in R_λ is central. For arbitrary $x \in R_\lambda$ and $b \in N'(R_\lambda)$, we get from (13) and (17) the result that

$$(x^{dt}+b^{ds})[x, b]=0=[x, b](x^{dt}+b^{ds}).$$

If $[x, b]\neq 0$, then b^{ds} and $(x^{dt}+b^{ds})$ are central zero-divisors of R_λ (see (16)). Let $S_1 \neq (0)$ be the heart of R_λ, that is, the intersection of all non-zero ideals. Thus

$$(0)=S_1(x^{dt}+b^{ds})=S_1x^{dt}+S_1b^{ds}=S_1x^{dt}\neq(0).$$

Therefore, we have a contradiction. Hence $[x, b]=0$ for all $x \in R_\lambda$, and hence

(18) $$N'(R_\lambda)\subseteq Z(R_\lambda).$$

Let $c_1 \in Z(R_\lambda)$. Then, we can prove that

$$(c_1^{s+1}-c_1)[x, y]=0 \text{ for all } x, y \in R_\lambda.$$

By following the argument given in case (I), the last identity and (16) forces that $(x^{ds+1}-x)$ is a zero-divisor. Hence, by (18), we get

(19) $$(x^{ds+1}-x)\in Z(R_\lambda) \text{ for all } x \in R_\lambda.$$

Thus the original ring R satisfies (19). Therefore,

$$(x^{ds+1}-x)\in Z(R) \text{ for all } x \in R,$$

which implies commutativity of R by Theorem H. This completes the proof.

In Theorem 1 if the property $Q(n)$ is replaced by n and m are relatively prime positive integers, then R must be commutative. Indeed, we have the following result:

Theorem 2. *Let $n>1$, $m>1$, s, and t be fixed non-negative integers. Suppose that R is a left* (resp. right) s-unital *ring satisfying the polynomial identity* (1). *Then R is commutative provided that n and m are relatively prime.*

Proof. According to Lemma 6 and [9, Proposition 1], we prove the theorem for R with unity 1. Let $n>1$ and $m>1$. By following the same argument given in the proof of Theorem 1 (see case (I)), we can prove that

$$n[a^{q-1}, y]=0=m[a^{q-1}, y] \text{ for all } y \in R, \text{ and } a \in N(R).$$

But n and m are relatively prime. Hence Lemma 5, yields $[a^{q-1}, y]=0$, for all $y \in R$. Thus we have a contradiction ($q>1$, see (2)). Therefore, $N(R)\subseteq Z(R)$.

Next, let $x=e_{11}$ and $y=e_{12}$. Then x and y fail to satisfy (1). In view of Theorem KB, and the above argument, we conclude that

(20) $$C(R)\subseteq N(R)\subseteq Z(R).$$

By following the proof of case (I) in Theorem 1 (see (5)) along with (20), we prove

(21) $$x^k \in Z(R) \text{ for all } x \in R.$$

Furthermore, we obtain

$$[x^{n^2}, u]=0=[x^{m^2}, u] \text{ for } x \in R \text{ and } u \in N'(R).$$

Since n and m are relatively prime, Lemma 4, gives $[x, u]=0$ for all $x \in R$ and hence

$$N'(R)\subseteq Z(R).$$

As we have in the proof of Theorem 1, we get

$$n(c^{t+n}-c)[x, y]=0 \text{ for all } x, y \in R \text{ and } c \in Z(R).$$

A variation of the argument yields

$$m(c^{t+n}-c)[x, y]=0 \text{ for all } x, y \in R \text{ and } c \in Z(R).$$

Now, since n and m are relatively prime, Lemma 5 shows that

$$(c^{t+n}-c)[x, y]=0 \text{ for all } x, y \in R \text{ and } c \in Z(R).$$

By (21), if $y^k \in Z(R)$, then

$$(y^{k(t+n)}-y^k)[x, y]=0 \text{ for all } x, y \in R.$$

Next, one can proceed exactly as in the proof of Theorem 1 to show that

$$(y^{k(t+n-1)}-y) \in Z(R) \text{ for all } x, y \in R.$$

Therefore, R is commutative by Theorem H. This completes the proof.

Now, we establish the commutativity of a left (resp. right) s-unital ring under different set of conditions:

Theorem 3. *Let $n>1$ and $m>1$ be fixed relatively prime integers, and let s and t be fixed non-negative integers. If R is a left* (resp. right) *s-unital ring satisfying both identities*

(22) $$x^t[x^n, y]=[y^n, x]y^s \text{ and } x^t[x^m, y]=[y^m, x]y^s \text{ for all } x, y \in R,$$

then R is commutative.

Proof. According to Lemma 6 and Proposition 1 of [9], we prove the theorem for R with unity 1. Following the proof of Theorem 1 (case (I)) and Theorem 2, we can easily modify the proof to show that

$$C(R) \subseteq N(R) \subseteq Z(R),$$

under the present hypotheses. The argument of subdirectly irreducible rings can then be carried out for both n and m, yielding integers $j>1$ and $k>1$ such that R satisfies the identities

$$[x^j-x, y^{m^2}]=0 \text{ and } [x^k-x, y^{n^2}]=0 \text{ for all } x, y \in R.$$

Let $p(x)=(x^j-x)^k-(x^j-x)$. Then

$$0=[p(x), y^{m^2}]=m^2y^{m^2-1}[p(x), y] \text{ for all } x, y \in R,$$

and

$$0=[p(x), y^{n^2}]=n^2y^{n^2-1}[p(x), y] \text{ for all } x, y \in R.$$

The relative primeness of m and n yields

(23) $$y^r[p(x), y]=0 \text{ for all } x, y \in R \text{ and } r=\max\{m^2-1, n^2-1\}.$$

Therefore, by applying Lemma 1, we get $p(x) \in Z(R)$. Since $p(x)=x-x^2q(x)$ with $q(x)$ having integral coefficients, then Theorem H' shows that R is commutative.

4. Some corollaries and examples. As a corollary of Theorem 1, we have the following:

Corollary 1. *Let $n>0$ and m (resp. $m>0$ and n) be fixed non-negative integers. Suppose a left (resp. right) s-unital ring R satisfies the polynomial identity*

(24) $$[xy, x^n-y^m]=0 \text{ for all } x, y \in R.$$

If further, R has the property $Q(n)$, then R is commutative.

Proof. Actually, R satisfies the polynomial identity

$$x[x^n, y]=[y^m, x]y \text{ for all } x, y\in R.$$

Hence, the assertion is clear, by Theorem 1.

Now, replacing the property $Q(n)$ by n and m are relatively prime positive integers yields the following corollary of Theorem 2.

Corollary 2. *Let $m>1$ and $n>1$ be fixed integers. Suppose a left (resp., right) s-unital ring R satisfies the polynomial identity (24). If n and m are relatively prime, then R is commutative.*

Corollary 3. *Let $k>1$ be a fixed integer. If R is a left (resp., right) s-unital ring satisfying the polynomial identities*

$$x^k y-yx^k=y^k x-xy^k \text{ for all } x, y\in R \text{ and } k=n,\ k=n+1,$$

then R is commutative.

Proof. We notice that R satisfies the polynomial identities

$$[x^n, y]=[y^n, x] \text{ and } [x^{n+1}, y]=[y^{n+1}, x] \text{ for all } x, y\in R.$$

Therefore, R is commutative, by Theorem 3.

Corollary 4. *Let R be a left (resp., right) s-unital ring and let $n>1$ be a fixed integer. If R^+, the additive group of R, has the property $Q(n)$, and R satisfies the identity*

$$x^n y-yx^n=y^n x-xy^n \text{ for all } x, y\in R,$$

then R is commutative.

Proof. Actually, R satisfies

$$[x^n, y]=[y^n, x] \text{ for all } x, y\in R.$$

Hence, R is commutative, by Theorem 1.

Example 1 ([12, Remark]). Let K be a field. Then, the non-commutative ring

$$R=\begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix},$$

has a right identity element and satisfies the polynomial identity $x[x, y]=0$ for all x, $y\in R$. Hence, in case $m=0$ and $n>0$, Theorem 1 need not be true for right s-unital rings.

The hypothesis of R to be a left (right) s-unital ring or the existence of a unity element 1, in R is not superfluous in Theorem 1. This is shown by the following Harmanci's examples [6].

Example 2. Let

$$A_1=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad B_1=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } C_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

be elements of the ring of all 3×3 matrices over Z_2, the ring of integers *mod* 2. If R is the subring generated by the matrices A_1, B_1, and C_1, then for each integer $n\geqq 1$ and x, $y\in R$, $[x^n, y]=[y^n, x]$ holds. However, R is not commutative.

Example 3. Let

$$A_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\quad B_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } C_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

be elements of the ring of all 3×3 matrices over Z_2, the ring of integers *mod 2*. If R is the ring generated by the matrices A_2, B_2, and C_2, then for each integer $n\geq 1$, and all $x, y\in R$, $[x^n, y]=[y^n, x]$ is satisfied, but R is not commutative.

Remark. In Theorem 1, the restriction on $Q(n)$ property is essential. To see this, we consider Example 2 and use the Dorroh construction (with the ring of integers mod 2) to get a ring R with unity 1. This ring R satisfies $[x^2, y]=[y^2, x]$, for all x, $y\in R$, and is not commutative (see [4, Remark]).

5. Groups. Finally, a close look to the symmetric group S_3, we find that S_3 satisfies the identity

$$x^6[x^6, y]=[y^6, x]y^6 \text{ for all } x, y\in S_3,$$

but S_3 is not an abelian group. If G is any group satisfying the polynomial identity (1), then it is interesting to study commutativity of G under certain constraints.

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