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NON-EXISTENCE OF PROPER SEMI-INVARIANT SUBMANIFOLDS OF AN ALMOST r-PARACONTACT RIEMANNIAN MANIFOLD OF P-SASAKIAN TYPE

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The main objective of the paper is to prove that an almost r-paracontact Riemannian manifold of P-Sasakian type does not admit any proper semi-invariant submanifold.

Introduction. Recently, in [1], semi-invariant submanifolds of a Sasakian manifold have been defined and studied. It has been shown that a Sasakian manifold always admits a proper semi-invariant submanifold [1]. Subsequently, in [5], it was proved that a Para-Sasakian manifold [4] does not admit any proper semi-invariant submanifold. In [2], a generalization of a Para-Sasakian manifold [4] called an almost r-paracontact Riemannian manifold of P-Sasakian type has been defined and studied. In this paper, we mainly investigate the problem analogous to that in [1] and [5] and prove that there does not exist any proper semi-invariant submanifold of an almost r-paracontact Riemannian manifold of P-Sasakian type. Some interesting results concerning integrability of the distributions which arise naturally on the semi-invariant submanifold have also been established.

Note. Throughout this paper (i) the term almost r-paracontact Riemannian shall be abbreviated to al. r-p. c. R., and (ii) the indices α and β will run over $1, \dots r$.

1. Preliminaries. Let $\bar{M}$ be an al. r-p. c. R. manifold [2] with structure $(\varphi, \xi_\alpha, \eta^\alpha, g)$, where φ is a (1, 1) tensor field, $\{\xi_\alpha\}$ are r vector fields, $\{\eta^\alpha\}$ are r 1-forms and g is an associated Riemannian metric on $\bar{M}$. These tensor fields are related by [2]

$$\varphi^2 = I - \eta^\alpha \otimes \xi_\alpha,\ \eta^\alpha(\xi)_\beta = \delta^\alpha_\beta,\ \varphi(\xi_\alpha) = 0,\ \eta^\alpha \circ \varphi = 0, \tag{1.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y),\ g(\xi_\alpha, X) = \eta^\alpha(X), \tag{1.2}$$

$$\Phi(X, Y) \equiv g(\varphi X, Y) = g(X, \varphi Y) = \Phi(Y, X), \tag{1.3}$$

$$(\bar{\nabla}_X \Phi)(Y, Z) = g((\bar{\nabla}_X \varphi)YZ), \tag{1.4}$$

for arbitrary vector fields X and Y tangent to $\bar{M}$, where $\bar{\nabla}$ is the Riemannian connection of $\bar{M}$.

Let M be a submanifold of $\bar{M}$. Let the induced metric on M also be denoted by g. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{1.6}$$

respectively, for all vector fields X and Y tangent to M and for each vector field N normal to M, where ∇ is the induced Riemannian connection on M, h is the second fundamental form of the immersion, and $A_N X$ and $\nabla^\perp_X N$ are tangential and normal parts of $\bar{\nabla}_X N$. From (1.8) and (1.9) it follows that

(1.7) $$g(h(X, Y), N) = g(A_N X, Y).$$

A submanifold M of an al. r-p. c. R. manifold $\bar{M}$ is called a *semi-invariant submanifold* of $\bar{M}$ if the following conditions are satisfied [6]:

(a) $TM = D \oplus D^\perp \oplus A$, where A is the r-dimensional distribution spanned by the structure vector fields $\xi_1, \ldots, \xi_r$ and D, $D^\perp$ are differentiable distributions on M;

(b) the distribution D is invariant by φ, that is $\varphi(D_x) = D_x$ for each $x \in M$;

(c) the distribution $D^\perp$ is anti-invariant by φ, that is $\varphi(D_x^\perp) \subset T_x M^\perp$ for each $x \in M$, where $T_x M^\perp$ is the normal space to M at the point x.

From the conditions (a), (b) and (c) it follows that the distributions D, $D^\perp$ and A are mutually orthogonal. If both the distributions D and $D^\perp$ are non-zero, then the semi-s invariant submanifold is called a proper semi-invariant submanifold. For any vector bundle H on M [resp. $\bar{M}$] we denote by $\Gamma(H)$ the module of all differentiable section of H on a neighbourhood coordinates on M [resp., $\bar{M}$].

Next, we recall some special classes of an al. r-p. c. R. manifold. An al. r-p. c. R manifold $\bar{M}$ with structure $(\varphi, \xi_\alpha, \eta^\alpha, g)$ is said to be [2]

(i) of *s-paracontact type* if

(1.8) $$\varphi X = \bar{\nabla}_X \xi_\alpha;\ \alpha = 1, \ldots, r;$$

(ii) of *P*-Sasakian type if it is of *s*-paracontact type and

(1.9) $$(\bar{\Delta}_X \Phi)(Y, Z) = -\sum_\beta \eta^\beta(Y)[g(X, Z) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Z)] - \sum_\beta \eta^\beta(Z)[g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)],$$

(iii) of *SP*-Sasakian type if it is of *s*-paracontact type and

(1.10) $$\Phi(X, Y) = e[g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)];\ e^2 = 1,$$

for any vector fields X, Y and Z on $\bar{M}$.

2. Non-existence of proper semi-invariant submanifolds. We first prove a lemma.

Lemma 2.1. *On an al. r.-p. c. R manifold $\bar{M}$ of s-paracontact type, the distribution T determined by η^α, s is involutive.*

Proof. Let $X, Y \in \Gamma(T)$. Here $\eta^\alpha(X) = 0$, $\eta^\alpha(Y) = 0$ and consequently in view of $(1.2)_2$, (1.8) and (1.3), it follows that $\eta^\alpha([X, Y]) = 0$, which completes the proof.

The above lemma provides the proof of the following

Theorem 2.1. *The distribution $D \oplus D^\perp$, of a semi-invariant submanifold of an al. r-p. c. R. manifold of s-paracontact type, is always integrable.*

Now we prove the main theorem of this paper.

Theorem 2.2. *For a semi-invariant submanifold M of an al. r-p. c. R. manifold $\bar{M}$ of P-Sasakian type, $\dim D^\perp = 0$. Consequently, an al. r-p. c. R manifold $\bar{M}$ of P-Sasakian type does not admit any proper semi-invariant submanifold.*

Proof. Let $X, Y \in \Gamma D^\perp$). Hence $\varphi X, \varphi Y \in \Gamma(TM^\perp)$ and in view of (1.7), (1.5), (1.3), (1.4), (1.9) and (1.6), it follows that

$$g(A_{\varphi X} Y, Z) = g(h(Y, Z)\varphi X) = g(h(Z, Y)\varphi X) = g(\bar{\nabla}_Z Y, \varphi X)$$

$$= g(\varphi(\bar{\Delta}_Z Y), X) = g(\bar{\nabla}_Z(\varphi Y) - (\bar{\nabla}_Z \varphi)Y, X) = g(\bar{\nabla}_Z(\varphi Y), X)$$

$$= -g(A_{\varphi Y} Z, X) = g(-A_{\varphi Y} X, Z);\ Z \in \Gamma(TM),$$

which implies that

(2.1) $$A_{\varphi X}Y + A_{\varphi Y}X = 0;\quad X, Y \in \Gamma(D^{\perp}).$$

Now taking account of $(1.2)_2$, (1.6), and (1.8), we get

$$\eta^{\alpha}(A_{\varphi Y}X) = -g(\bar{\nabla}_X(\varphi Y), \xi_{\alpha}) = g(\varphi Y, \bar{\nabla}_X \xi_{\alpha}) = g(\varphi Y, \varphi X) = g(X, Y);\ X, Y \in \Gamma(D^{\perp}).$$

Interchanging X and Y in this equation and then adding both the equations, in view of (2.1) we have

$$2g(X, Y) = \eta^{\alpha}(A_{\varphi Y}X + A_{\varphi X}Y) = 0;\ X, Y \in \Gamma(D^{\perp}).$$

Hence $\dim D^{\perp} = 0$. This completes the proof of the Theorem.

Since on al. r-p. c. R. manifold of SP-Sasakian type is always of P-Sasakian type [2], we immediately have the following

Corollary 2.1. *An al. r-p. c. R. manifold of SP-Sasakian type does not admit any proper semi-invariant submanifold.*

Theorem 2.1 and Theorem 2.2 lead to

Theorem 2.3. *If M is a semi-invariant submanifold of an al. r-p. c. R manifold of P-Sasakian type, then*

(i) the distribution D is integrable;

(ii) $\nabla_X(\varphi Y) - \nabla_Y(\varphi X) = \varphi[X, Y]$; $\quad X, Y \in \Gamma(D)$.

(iii) $h(X, \varphi Y) = h(\varphi X, Y)$;

Proof. Since in view of Theorem 2.2, $\dim D^{\perp} = 0$; taking into account Theorem 2.1, the distribution D becomes integrable. Next, in view of (1.4) and (1.9) it follows that

(2.2) $$g((\bar{\nabla}_X \varphi)Y - (\bar{\nabla}_Y \varphi)X Z = 0;\ X, Y \in \Gamma(D), Z \in \Gamma(T\bar{M}).$$

From (2.2), we get

(2.3) $$(\bar{\nabla}_X \varphi)Y - (\bar{\nabla}_Y \varphi)X = 0;\ X, Y \in \Gamma(D).$$

Using (1.5) in (2.3), we have

$$0 = (\bar{\nabla}_X \varphi)Y - (\bar{\nabla}_Y \varphi)X = \nabla_X(\varphi Y) - \nabla_Y(\varphi X) - \varphi[X, Y] + h(X, \varphi Y) - h(\varphi X, Y);\ X, Y \in \Gamma(D),$$

which on equating tangential and normal parts, yields (ii) and (iii) respectively.

Remark 2.1. In [3], various lemmas and theorems dealing with different properties of a semi-invariant submanifold of a P-Sasakian manifold have been proved. Since this manifold is a special case of the manifold considered in this paper when $r=1$, it does not admit any proper semi-invariant submanifold. In fact, in this case the distribution $D^{\perp}$ is always zero and the distribution D is always integrable. Finally let us note that the results in [3] are redundant.

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