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SOME DOUBLE INTEGRALS AND FOURIER SERIES FOR FOX'S *H*-FUNCTION OF TWO VARIABLES

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In this paper, we evaluate several double integrals and use them to find four Fourier series expansions for Fox's *H*-function of two variables.

1. Introduction. The subject of Fourier series for generalized hypergeometric functions of two and more variables is given a considerable place in the literature on special functions and two- or multi-dimensional boundary value problems.

The result presented in this paper are of quite general nature and their particular cases are scattered throughout the literature. Here some earlier results of Bajpai [3] are discussed only as particular cases.

During the past twenty years many mathematicians tried to present several Fourier series expansions for the *G*- and *H*-functions of two and several variables [8, 9, 14, 15, 16, 17, 19, 21, 22, 26, 29, 30, 31, 32]. The careful and serious study of these papers reveals that almost all of these Fourier series are not proper Fourier series for the *G*- or *H*-functions of two or several variables. They are presented in a form only to appear as Fourier series of these functions and may be viewed as manipulative forms of already known results on Meijer's *G*-function and Fox's *H*-function [21, 22, 31]. In order to support our contention, we wish to make the following comments on some well-known Fourier series for the generalized Fox's *H*-functions of two and more variables.

The Fourier series reproduced by Srivastava, Gupta and Goyal [31, pp. 168-169, (9.1.17), (9.1.18)] for Fox's *H*-function of two variables appear nothing but alternative forms of the Fourier series due to Bajpai [31, p. 76, (5.5.13), (5.5.14)]. It is interesting to note that the Fourier series [31, pp. 168-169, (9.1.17), (9.1.18)] involve only one variable Θ, i. e. they are Θ Fourier series for Fox's *H*-function of two variables $H[\varphi(\sin\Theta)^{2\lambda}, \eta(\sin\Theta)^{2\mu}]$. Therefore these results are special cases of our results concerning the function $H[\varphi(\sin\Theta)^{2\lambda}, (\sin\Phi)^{2\mu}]$ with two independent variables Θ, Φ. This explains why the Fourier series [31, 168-169, (9.1.17), (9.1.18)] have been presented in terms of single series, while we use presentation by means of double series. In general,only Fourier series for a function of two variables should involve two variables and should be presented in terms of a double series, as given by Carslaw and Jaeger [7, pp. 180-183].

The Fourier series, given by Srivastava and Panda [32, pp. 179-180, (3.27)—(3.30)] for Fox's *H*-function of several variables are in a sense, manipulative forms of the Fourier series due to Bajpai [3, pp. 705-706, (3.1), (3.2)]. Actually, the results of Srivastava and Panda involve only one variable Θ and have been presented in terms of single series. So, they concern some special cases only. In general, any Fourier series for a function of several variables should involve several variables and should be presented in terms of multiple series. Similarly, the Fourier series given by Gupta [17, pp. 47-48] is not a Fourier series for Fox's *H*-function of several variables, but

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it is a manipulative form of the Fourier series for Fox's H-function due to Bajpai [4, p. 33-34, (2.1)].

In an attempt to generalize Meijer's G-function and the functions of two variables [2], Agarwal [1] and Sharma [28] introduced Meijer's G-function of two variables. Since the H-function defined by Fox [12] is not a special case of the G-function of two variables, therefore several mathematicians tried to extend Fox's H-function to two variables. The works of Pathak [25], Goyal [13], Munot and Kalla [24], Varma [34] and Mittal and Gupta [23] are worth mentioning. The definitions of the generalized Fox's H-function of two variables given by the afore said authors are the same.

The significance of the H-function of two variables lies in the fact that it includes, as special cases, Fox's H-function, products of two H-functions and most of the known functions of one and two variables, e. g. Meijer's G-function, MacRobert's E-function, the G-function of two variables, Kampé de Fériet's function, Appell's functions F_1, F_2, F_3 and F_4, and their particular cases [10, 11, 21, 22, 31].

For sake of brevity in what follows, (a_p, α_p) stands for the set of parameters (a_1, α_1), ———, (a_p, α_p).

Fox's H-function of two variables is defined and represented as follows:

$$(1.1)\qquad H\left[\begin{matrix}\varphi\\ \\ \eta\end{matrix}\right]=H^{0,\, n_1;\, m_2,\, n_2;\, m_3,\, n_3}_{p_1,\, q_1;\, p_2,\, q_2;\, p_3,\, q_3}\left[\begin{matrix}\varphi\\ \eta\end{matrix}\middle|\begin{matrix}(a_{p_1};\, \alpha_{p_1},\, A_{p_1});(c_{p_2},\, \nu_{p_2});\,(e_{p_3},\, E_{p_3})\\ (b_{q_1},\, \beta_{q_1},\, B_{q_1});\,(d_{q_2},\, \delta_{q_2}),\,(f_{q_3},\, F_{q_3})\end{matrix}\right]$$

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Psi(s,\, t)\Theta(s)\Phi(t)\varphi^s\eta^t\,ds\,dt,$$

where L_1 and L_2 are suitable contours of Barnes type and

$$(1.2)\qquad \Psi(s,\, t)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)},$$

$$(1.3)\qquad \Theta(s)=\frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)\prod_{j=1}^{n_2}\Gamma(1-c_j+\nu_j s)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-Y_j s)},$$

$$(1.4)\qquad \Phi(t)=\frac{\prod_{j=1}^{m_3}\Gamma(f_j-F_j t)\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j t)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j t)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j t)}.$$

Following the result of Braaksma [5, p. 278], it can easily be shown that the function defined by (1.1), is an analytic function of φ and η if

$$(1.5)\qquad \sum_{j=1}^{p_1}\alpha_j+\sum_{j=1}^{p_2}\nu_j<\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{q_2}\delta_j,$$

$$(1.6)\qquad \sum_{j=1}^{p_1}A_j+\sum_{j=1}^{p_3}E_j<\sum_{j=1}^{q_1}B_j+\sum_{j=1}^{q_3}F_j.$$

According to the set of conditions given by Buschman [6, (5.8)], the integral (1.1) defining the H-function of two variables, is convergent under the following conditions:

$$(1.7)\quad W_1=\sum_{j=1}^{n_1}\alpha_j-\sum_{j=n_1+1}^{p_1}\alpha_j-\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{m_2}\delta_j-\sum_{j=m_2+1}^{q_2}\delta_j+\sum_{j=1}^{n_2}\nu_j-\sum_{j=n_2+1}^{p_2}\nu_j>0,$$

$$(1.8)\quad W_2=\sum_{j=1}^{n_1}A_j-\sum_{j=n_1+1}^{p_1}A_j-\sum_{j=1}^{q_1}B_j+\sum_{j=1}^{m_3}F_j-\sum_{j=m_3+1}^{q_3}F_j+\sum_{j=1}^{n_3}E_j-\sum_{j=n_3+1}^{p_3}E_j>0,$$

$|\arg\varphi|<\frac{1}{2}W_1\pi$ and $|\arg\eta|<\frac{1}{2}W_2\pi$.

For details, see [22, 31].

Fox's H-function of two variables has further been generalized to Fox's H-function of several variables by Saxena [27] and Srivastava and Panda [33], as a generalization of Meijer's G-function of several variables, studied by Khadia and Goyal [18].

The following basic formula is required for the proofs given below,

$$(1.9)\quad \int_0^\pi \sin(2n+1)\Theta(\sin\Theta)^{1-2\rho}d\Theta=\frac{\sqrt{(\pi)}\,\Gamma\left(\frac{3}{2}-\rho\right)\Gamma(\rho+n)}{\Gamma(\rho)\Gamma(2-\rho+n)},\quad \operatorname{Re}(3-2\rho)>0,\ n=0,1,2,\ldots;$$

which follows from [20, p. 80].

2. Some additional integrals. Here we evaluate some additional integrals of the H-function of two variables, used further on so as to obtain some Fourier series for these functions.

$$(2.1)\quad \int_0^\pi \sin(2m+1)\Theta(\sin\Theta)^{1-2\rho}H\begin{bmatrix}\varphi(\sin\Theta)^{2\lambda}\\ \eta\end{bmatrix}d\Theta=\sqrt{(\pi)}\,H^{0,\,n_1;\,m_2+1,\,n_2+1;\,m_3,\,n_3}_{p_1\,q_1;\,p_2+2;\,q_2+2;\,p_3,\,q_3}$$

$$\left[\begin{matrix}\varphi\\ \eta\end{matrix}\middle|\begin{matrix}(a_{p_1};\alpha_{p_1},A_{p_1});\ (\rho-\frac{1}{2},\lambda),(c_{p_2},\nu_{p_2}),(\rho,\lambda);\ (e_{p_3},E_{p_3})\\ (b_{q_1};\beta_{q_1},B_{q_1});\ (\rho+m,\lambda),(d_{q_2},\delta_{q_2}),(\rho-m-1,\lambda);\ (f_{q_3},F_{q_3})\end{matrix}\right],$$

$$\operatorname{Re}(3-2\rho)+2\lambda\min_{1\le j\le m_2}[\operatorname{Re}d_j/\delta_j]>0;$$

$$(2.2)\quad \int_0^\pi \sin(2n+1)\Phi(\sin\Phi)^{1-2\sigma}H\begin{bmatrix}\varphi\\ \eta(\sin\Phi)^{2\mu}\end{bmatrix}d\Phi$$

$$=\sqrt{(\pi)}\,H^{0,\,n_1;\,m_2,\,n_2;\,m_3+1,\,n_3+1}_{p_1,\,q_1;\,p_2,\,q_2;\,p_3+2,\,q_3+2}\left[\begin{matrix}\varphi\\ \eta\end{matrix}\middle|\begin{matrix}(a_{p_1};\alpha_{p_1},A_{p_1});\\ (b_{q_1};\beta_{q_1},B_{q_1});\end{matrix}\right.$$

$$\left.\begin{matrix}(c_{p_2},\nu_{p_2});\ (\sigma-\frac{1}{2},\mu),(e_{p_3},E_{p_3}),(\sigma,\mu)\\ (d_{q_2},\delta_{q_2});\ (\sigma+n,\mu);(f_{q_3},F_{q_3})\ (\sigma-n-1,\mu)\end{matrix}\right]$$

$$\operatorname{Re}(3-2\sigma)+2\mu\min_{1\le j\le m_3}[\operatorname{Re}f_j/F_j]>0;$$

$$(2.3)\quad \int_0^\pi \sin(2m+1)\Theta(\sin\Theta)^{1-2\rho}H\begin{bmatrix}\varphi(\sin\Theta)^{-2\lambda}\\ \eta\end{bmatrix}d\Theta$$

$$= \sqrt{(\pi)}\, H^{0,\, n_1;\, m_2+1,\, n_2+1;\, m_3,\, n_3}_{p_1,\, q_1;\, p_2+2,\, q_2+2;\, p_3,\, q_3}\left[\begin{matrix}\varphi\\ \eta\end{matrix}\,\middle|\,\begin{matrix}(a_{p_1};\, \alpha_{p_1},\, A_{p_1});\\ (b_{q_1};\, \beta_{q_1},\, B_{q_2});\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\rho-m,\, \lambda),\, (c_{p_2},\, \nu_{p_2}),\, (2-\rho+m,\, \lambda);\, (e_{p_3},\, E_{p_3})\\ \left(\frac{3}{2}-\rho,\, \lambda\right),\, (d_{q_2},\, \delta_{q_2}),\, (1-\rho,\lambda);\, (f_{q_3},\, F_{q_3})\end{matrix}\right],$$

$$\operatorname{Re}(1-2\rho)-2\lambda \max_{1\leq j\leq n_2} [\operatorname{Re}(c_j-1)\nu_j]>0;$$

(2.4)
$$\int_0^{\pi} \sin(2n+1)\,\Phi\,(\sin\Phi)^{1-2\sigma} H\left[\begin{matrix}\varphi\\ \eta\,(\sin\Phi)^{-2\mu}\end{matrix}\right]d\Phi$$

$$= \sqrt{(n)}\, H^{0,\, n_1;\, m_2,\, n_2;\, m_3+1,\, n_3+1}_{p_1,\, q_1;\, p_2,\, q_2;\, p_3+2,\, q_3+2}\left[\begin{matrix}\varphi\\ \eta\end{matrix}\,\middle|\,\begin{matrix}(a_{p_1};\, \alpha_{p_1},\, A_{p_1});\\ (b_{q_1};\, \beta_{q_1},\, B_{q_1});\end{matrix}\right.$$

$$\left.\begin{matrix}(c_{p_2},\, \nu_{p_2});\, (1-\sigma-n,\, \mu),\, (e_{p_3},\, E_{p_3}),\, (2-\sigma+n,\, \mu)\\ (d_{q_2},\, \delta_{q_2});\, \left(\frac{3}{2}-\sigma,\, \mu\right),\, (f_{q_3},\, F_{q_3}),\, (1-\sigma,\, \mu)\end{matrix}\right]$$

$$\operatorname{Re}(1-2\sigma)-2\mu \max_{1\leq j\leq n_3} [\operatorname{Re}(e_j-1)/E_j]>0;$$

and where $\lambda>0$, $\mu>0$, $W_1>0$, $W_2>0$, $|\arg\varphi|<\frac{1}{2}W_1\pi$, $|\arg\eta|<\frac{1}{2}W_2\pi$.

Sketch of the proof. To establish (2.1), express the H-function in the integrand as (1.1), change the order of the Θ-integral and (s, t)-integral, evaluate the inner-integral with the help of (1.9) and use (1.1), so as to obtain the value of the integral (2.1).

The integrals (2.2), (2.3) and (2.4) can be evaluated by the same procedure.

The integrals (2.1)—(2.4) may be considered as analogues of the integral [3, p. 703, (2.1)].

3. Some double integrals involving the H-function of two variables. The following integrals have been evaluated:

(3.1)
$$\int_0^{\pi}\int_0^{\pi} \sin(2m+1)\Theta\,\sin(2n+1)\Theta(\sin\Phi)^{1-2\rho}(\sin\Phi)^{1-2\sigma}\, H\left[\begin{matrix}\varphi(\sin\Theta)^{2\lambda}\\ \eta(\sin\Phi)^{2\mu}\end{matrix}\right]d\Theta d\Phi$$

$$= \pi H^{0,\, n_1;\, m_2+1,\, n_2+1;\, m_3+1,\, n_3+1}_{p_1,\, q_1;\, p_2+2,\, q_2+2;\, p_3+2,\, q_3+2}\left[\begin{matrix}\varphi\\ \eta\end{matrix}\,\middle|\,\begin{matrix}(a_{p_1};\, \alpha_{p_1},\, A_{p_1});\\ (b_{q_1};\, \beta_{q_1},\, B_{q_1});\end{matrix}\right.$$

$$\left.\begin{matrix}\left(\rho-\frac{1}{2},\lambda\right)(c_{p_2},\, \nu_{p_2}),\, (\rho,\, \lambda);\, \left(\sigma-\frac{1}{2},\, \mu\right),\, (e_{p_3},\, E_{p_3}),\, (\sigma,\, \mu)\\ (\rho+m,\, \lambda),\, (d_{q_2},\, \delta_{q_2}),\, (\rho-m-1,\, \lambda);\, (\sigma+n,\, \mu),\, (f_{q_3},\, F_{q_3}),\, (\sigma-n-1,\, \mu)\end{matrix}\right]$$

$$\operatorname{Re}(3-2\rho)+2\lambda \min_{1\leq j\leq m_2} [\operatorname{Re} d_j/\delta_j]>0,\quad \operatorname{Re}(3-2\sigma)+2\mu \min_{1\leq j\leq m_3} [\operatorname{Re} f_j/F_j]>0;$$

(3.2)
$$\int_0^{\pi}\int_0^{\pi} \sin(2m+1)\Theta\,\sin(2n+1)\Phi(\sin\Theta)^{1-2\rho}(\sin\Phi)^{1-2\sigma}\, H\left[\begin{matrix}\varphi(\sin\Theta)^{2\lambda}\\ \eta(\sin\Phi^{-2\mu}\end{matrix}\right]d\Theta d\Phi$$

$$= \pi H^{0,\, n_1;\, m_2+1,\, n_2+1;\, m_3+1,\, n_3+1}_{p_2,\, q_2;\, p_2+2,\, q_2+2;\, p_3+2,\, q_3+2} \left[\begin{matrix} \varphi \\ \eta \end{matrix} \middle| \begin{matrix} (a_{p_1},\ \alpha_{p_1},\ A_{p_1}); \\ (b_{q_1},\ \beta_{q_1},\ B_{q_2}); \end{matrix} \right.$$

$$\left. \begin{matrix} \left(\rho-\frac{1}{2},\ \lambda\right), (c_{p_2},\ \nu_{p_2}), (\rho,\ \lambda);\ (1-\sigma-n,\ \mu)\ (e_{p_3},\ E_{p_3}), (2-\sigma+n,\ \mu) \\ (\rho+m,\ \lambda), (d_{q_2},\ \delta_{q_2}), (\rho-m-1,\ \lambda);\ \left(\frac{3}{2}-\sigma,\ \mu\right), (f_{q_3},\ F_{q_3}), (1-\sigma,\ \mu) \end{matrix} \right]$$

$$\operatorname{Re}(3-2\rho)+2\lambda \min_{1\le j\le m_2} [\operatorname{Re} d_j/\delta_j] > 0,\ \operatorname{Re}(1-2\sigma) - 2\mu \max_{1\le j\le n_3},\ [\operatorname{Re}(e_j-1)/E_j] > 0;$$

$$(3.3)\quad \int_0^{\pi}\int_0^{\pi} \sin(2m+1)\Theta \sin(2n+1)\Phi (\sin\Theta)^{1-2\rho} (\sin\Phi)^{1-2\sigma} H\left[\begin{matrix} \varphi(\sin\Theta)^{-2\lambda} \\ \eta(\sin\Phi)^{2\mu} \end{matrix} \right] d\Theta d\Phi$$

$$= \pi H^{0,\, n_1;\, m_2+1,\, n_2+1;\, m_3+1,\, n_3+1}_{p_1\, q_1;\, p_2+2,\, q_2+2,\, p_3+2,\, q_3+2} \left[\begin{matrix} \varphi \\ \eta \end{matrix} \middle| \begin{matrix} (a_{p_1};\ \alpha_{p_1},\ A_{p_1}); \\ (b_{q_1};\ \beta_{q_1},\ B_{q_1}); \end{matrix} \right.$$

$$\left. \begin{matrix} (1-\rho-m,\ \lambda), (c_{p_2},\ \nu_{p_2}), (2-\rho+m,\ \lambda);\ (\sigma-\frac{1}{2},\ \mu), (e_{p_3},\ F_{p_3}), (\sigma,\ \mu) \\ (\frac{3}{2}-\rho,\ \lambda), (d_{q_2},\ \delta_{q_2}), (1-\rho,\ \lambda);\ (\sigma+n,\ \mu), (f_{q_3},\ F_{q_3}), (\sigma-n-1,\ \mu) \end{matrix} \right],$$

$$\operatorname{Re}(1-2\rho)-2\lambda \max_{1\le j\le n_2} [\operatorname{Re} c_j-1)/\nu_j] > 0;\ \operatorname{Re}(3-2\sigma)+2\mu \min_{1\le j\le m_3},\ [\operatorname{Re} f_j/F_j] > 0;$$

$$(3.4)\quad \int_0^{\pi}\int_0^{\pi} \sin(2m+1)\Theta \sin(2n+1)\Phi(\sin\Theta)^{1-2\rho} (\sin\Phi)^{1-2\sigma} H\left[\begin{matrix} \varphi(\sin\Theta)^{-2\lambda} \\ \eta(\sin\Phi)^{-2\mu} \end{matrix} \right] d\Theta d\Phi$$

$$= \pi H^{0,\, n;\, m_2+1,\, n_2+1;\, m_3+1,\, n_3+1}_{p_1,\, q_1,\, p_2+2,\, q_2+2;\, p_3+2,\, q_3+2} \left[\begin{matrix} \varphi \\ \eta \end{matrix} \middle| \begin{matrix} (a_{p_1},\ \alpha_{p_1},\ A_{p_1}); \\ (b_{q_1},\ \beta_{q_1},\ B_{q_1}); \end{matrix} \right.$$

$$\left. \begin{matrix} (1-\rho-m,\ \lambda), (c_{p_2},\ \nu_{p_2}), (2-\rho+m,\ \lambda);\ (1-\sigma-n,\ \mu), (e_{p_3},\ E_{p_3}), (2-\sigma+m,\ \mu) \\ (\frac{3}{2},\ \rho,\ \lambda), (d_{q_2},\ \delta_{q_2}), (1-\rho,\ \lambda);\ (\frac{3}{2}-\sigma,\ \mu), (f_{q_3},\ F_{q_3}), (1-\sigma,\ \mu) \end{matrix} \right],$$

$$\operatorname{Re}(1-2\rho)-2\lambda \max_{1\le j\le n_2} [\operatorname{Re}(c_j-1)/\nu_j] > 0;\ \operatorname{Re}(1-2\sigma)\ 2\mu \max_{1\le j\le n_3},\ [\operatorname{Re}(e_j-1)/E_j] > 0;$$

and where $\lambda>0$, $\mu>0$, $W_1>0$, $W_2>0$, $|\arg\varphi|<\frac{1}{2} W_1\pi$, $|\arg\eta|<\frac{1}{2} W_2\pi$.

Sketch of the proof. To establish (3.1), evaluate the Φ-integral of (3.1) with the help of (2.2) and then evaluate the resulting Θ-integral with the help of (2.1). Thus, the value of (3.1) is obtained.

By using the same procedure as above, the integrals (3.2), (3.3) and (3.4) can be found with the help of (2.1) and (2.4), (2.2) and (2.3), and (2.4) respectively.

4. The Fourier series. The Fourier series to be established are

$$(4.1)\quad (\sin\Theta)^{1-2\rho} (\sin\Phi)^{1-2\sigma} H\left[\begin{matrix} \varphi(\sin\Theta)^{2\lambda} \\ \eta(\sin\Phi)^{2\mu} \end{matrix} \right]$$

$$= \frac{4}{\pi} \sum_{u,\, v=0}^{\infty} H^{0,\, n_1;\, m_2+1,\, n_2+1;\, m_3+1,\, n_3+1}_{p_1,\, q_1;\, p_2+2,\, q_2+2;\, p_3+2,\, q_3+2} \left[\begin{matrix} \varphi \\ \eta \end{matrix} \middle| \begin{matrix} (a_{p_1},\ \alpha_{p_1},\ A_{p_1}); \\ (b_{q_1};\ \beta_{q_1},\ B_{q_1}); \end{matrix} \right.$$

$$\left.\begin{matrix}(\rho-\tfrac{1}{2},\lambda),(c_{p_2},\nu_{p_2}),(\rho,\lambda);(\sigma-\tfrac{1}{2},\mu),(e_{p_3},E_{p_3}),(\sigma,\mu)\\(\rho+u,\lambda),(d_{q_2},\delta_{q_2}),(\rho-u-1,\lambda);(\sigma+v,\mu),(f_{q_3},F_{q_3})(\sigma-v-1,\mu)\end{matrix}\right]$$

$$\times\sin(2u+1)\Theta\sin(2v+1)\Phi,$$

valid under the conditions of (3.1);

(4.2)
$$(\sin\Theta)^{1-2\rho}(\sin\Phi)^{1-2\sigma}H\left[\begin{matrix}\varphi(\sin\Theta)^{2\lambda}\\\eta(\sin\Phi)^{-2\mu}\end{matrix}\right]$$

$$=\frac{4}{\pi}\sum_{u,v=0}^{\infty}H^{0,n_1;m_2+1,n_2+1;m_3+1,n_3+1}_{p_1,q_1;p_2+2,q_2+2;p_3+2,q_3+2}\left[\begin{matrix}\varphi\\\eta\end{matrix}\middle|\begin{matrix}(a_{p_1};\alpha_{p_1},A_{p_1});\\(b_{q_1},\beta_{q_1},B_{q_1});\end{matrix}\right.$$

$$\left.\begin{matrix}(\rho-\tfrac{1}{2},\lambda),(c_{p_2},\nu_{p_2}),(\rho,\lambda);(1-\sigma-v,\mu),(e_{p_3},E_{p_3}),(2-\sigma+v,\mu)\\(\rho+u,\lambda),(d_{q_2},\delta_{q_2}),(\rho-u-1,\lambda);(\tfrac{3}{2}-\sigma,\mu),(f_{q_3},F_{q_3}),(1-\sigma,\mu)\end{matrix}\right]$$

$$\times\sin(2u+1)\Theta\sin(2v+1)\Phi,$$

valid under the conditions of (3.2);

(4.3)
$$(\sin\Theta)^{1-2\rho}(\sin\Phi)^{1-2\sigma}H\left[\begin{matrix}\varphi(\sin\Theta)^{-2\lambda}\\\eta(\sin\Phi)^{2\mu}\end{matrix}\right]$$

$$=\frac{4}{\pi}\sum_{u,v=0}^{\infty}H^{0,n_1;m_2+1,n_2+1;m_3+1,n_3+1}_{p_1q_1;p_2+2,q_2+2;p_3+2,q_3+2}\left[\begin{matrix}\varphi\\\eta\end{matrix}\middle|\begin{matrix}(a_{p_1},\alpha_{p_1},A_{p_1});\\(b_{q_1},\beta_{q_1};B_{q_1});\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\rho-u,\lambda),(c_{p_2},\nu_{p_2}),(2-\rho+u,\lambda);(\sigma-\tfrac{1}{2},\mu),(e_{p_3},E_{p_3}),(\sigma,\mu)\\(\tfrac{3}{2}-\rho,\lambda),(d_{q_2},\delta_{q_2}),(1-\rho,\lambda);(\sigma+v,\mu),(f_{q_3},F_{q_3}),(\sigma-v-1,\mu)\end{matrix}\right]$$

$$\times\sin(2u+1)\Theta\sin(2v+1)\Phi,$$

valid under the conditions of (3.3);

(4.4)
$$(\sin\Theta)^{1-2\rho}(\sin\Phi)^{1-2\sigma}H\left[\begin{matrix}\varphi(\sin\Theta)^{-2\lambda}\\\eta(\sin\Phi)^{-2\mu}\end{matrix}\right]$$

$$=\frac{4}{\pi}\sum_{u,v=0}^{\infty}H^{0,n_1;m_2+1,n_2+1;m_3+1,n_3+1}_{p_1,q_1;p_2+2,q_2+2;p_3+2,q_3+2}\left[\begin{matrix}\varphi\\\eta\end{matrix}\middle|\begin{matrix}(a_{p_1};\alpha_{p_1},A_{p_1});\\(b_{q_1};\beta_{q_1},B_{q_1});\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\rho-u,\lambda),(c_{p_2},\nu_{p_2}),(2-\rho+u,\lambda);(1-\sigma-v,\mu),(e_{p_3},E_{p_3}),(2-\sigma+v,\mu)\\(\tfrac{3}{2}-\rho,\lambda),(d_{q_2},\delta_{q_2}),(1-\rho,\lambda);(\tfrac{3}{2}-\sigma,\mu),(f_{q_3},F_{q_3}),(1-\sigma,\mu)\end{matrix}\right]$$

$$\times\sin(2u+1)\Theta\sin(2v+1)\Phi,$$

valid under the conditions of (3.4).

Proof. To establish (4.1), let

$$f(\Theta,\ \Phi)=(\sin\Theta)^{1-2\rho}(\sin\Phi)^{1-2\sigma}H\left[\begin{matrix}\varphi(\sin\Theta)^{2\lambda}\\ \eta(\sin\Phi)^{2\mu}\end{matrix}\right] \tag{4.5}$$

$$=\sum_{u,\ v=0}^{\infty}C_{u,\ v}\sin(2u+1)\Theta\sin(2v+1)\Phi.$$

Equation (4.5) is valid, since $f(\Theta,\ \Phi)$ is continuous and of bounded variation in the open interval $(0,\ \pi)$.

Multiplying both sides of (4.5) by $\sin(2n+1)\Phi$ and integrating with respect to Φ from 0 to π, then using (2.2) and the orthogonality property of the sine functions, we get

$$\frac{2(\sin\Phi)^{1-2\rho}}{\sqrt{(\pi)}}H^{0,\ n_1;\ m_2,\ n_2;\ m_3+1,\ n_3+1}_{p_1,\ q_1;\ p_2,\ q_2;\ p_3+2,\ q_3+2}\left[\begin{matrix}\varphi(\sin\Theta)^{2\lambda}\\ \eta\end{matrix}\left|\begin{matrix}(a_{p_1};\ \alpha_{p_1},\ A_{p_1});\\ (b_{q_1};\ \beta_{q_1},\ B_{q_1});\end{matrix}\right.\right. \tag{4.6}$$

$$\left.\begin{matrix}(c_{p_2},\ \nu_{p_2});(\sigma-\frac{1}{2},\ \mu),\ (e_{p_3},\ E_{p_3}),\ (\sigma,\ \mu)\\ (d_{q_2},\ \delta_{q_2});\ (\sigma+n,\ \mu),\ (f_{q_3},\ F_{q_3}),\ (\sigma-n-1,\ \mu)\end{matrix}\right]$$

$$=\sum_{u=0}^{\infty}C_{u,\ n}\sin(2u+1)\Theta.$$

Multiplying both sides of (4.6) by $\sin\ (2m+1)\Theta$ and integrating with respect to Θ from 0 to π, and using (2.1) and the orthogonality property of the sine functions, we obtain

$$C_{m,n}=\frac{4}{\pi}H^{0,\ n_1;\ m_2+1,\ n_2+1;\ m_3+1,\ n_3+1}_{p_1,\ q_1;\ p_2+2,\ q_2+2;\ p_3+2,\ q_3+2}\left[\begin{matrix}\varphi\\ \eta\end{matrix}\left|\begin{matrix}(a_{p_1};\ \alpha_{p_1},\ A_{p_1});\\ (b_{q_1},\ \beta_{q_1},\ B_{q_1});\end{matrix}\right.\right. \tag{4.7}$$

$$\left.\begin{matrix}(\rho-\frac{1}{2},\ \lambda),\ (c_{p_2},\ \nu_{p_2}),\ (f,\ \lambda);\ (\sigma-\frac{1}{2},\ \mu),\ (e_{p_3},\ E_{p_3}),\ (\sigma,\ \mu)\\ (\rho+m,\ \lambda),\ (d_{q_2},\ \delta_{q_2}),\ (\rho-m-1,\ \lambda);\ (\sigma+n,\ \mu),\ (f_{q_3},\ F_{q_3}),\ (\sigma-n-1,\ \mu)\end{matrix}\right].$$

From (4.5) and (4.7), the Fourier series (4.1) is obtained.

On applying the same procedure, the Fourier series (4.2), (4.3) and (4.4) are established with the help of integrals (2.1) and (2.4), (2.2) and (2.3), and (2.4) respectively.

The Fourier series (4.1), (4.2), (4.3) and (4.4) can also be obtained with the help of the double integrals (3.1), (3.2), (3.3) and (3.4) respectively on the lines of Carslaw and Jaeger [7, pp. 180-183].

5. Particular cases. On specializing the parameters of Fox's H-function of two variables in (3.4) and (4.4) and simplifying, we obtain two known results earlier given by Bajpai [3, pp. 703-705, (2.1) and (3.1)].

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