Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

ON BERNSTEIN POLYNOMIALS FOR RIEMANN INTEGRABLE FUNCTIONS

S. K. JASSIM

Direct and inverse theorems for Bernstein polynomials $B_n f$ are well known for continuous functions f in the sup-norm e. g. Ditzian and Totik (1987). Wickeren (1989) extended these results for Riemann integrable functions on [0, 1]. In terms of corresponding τ-moduli he proved the direct and inverse theorems in locally global norms. In this paper we prove the direct and inverse theorems in locally global norms in terms of τ-moduli $(\tau_2(f, \Delta(\delta))_p, 1 \leq p \leq \infty)$ which improve the above results.

1. Notations. Let $R = R[0, 1]$ be the space of Riemann integrable functions on [0, 1]. We denote by $\|f\|_p$ the L_p norm $(1 \leq p \leq \infty)$ and by $\|f\|_\infty$ the sup-norm L_∞ of the function f.

The Bernstein polynomial of $f \in R$ is defined by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{kn}(x); \quad p_{kn}(x) = \binom{n}{k} \chi^k (1-x)^{n-k}.$$

For $\chi \in [0, 1]$, $\delta > 0$, we set $\Delta(x, \delta) = \delta\Phi(x) + \delta^2$; $\Delta_n(x) = \Delta(x, n^{-1})$, where $\Phi(x) = \sqrt{x(1-x)}$.

Consider the following seminorms

$$\|f\|_{\delta,p} = \left(\int_0^1 \sup\{|f(y)| : y \in U(x, \delta)\}^p dx\right)^{\frac{1}{p}},$$

$$\|f|_{\delta,p}^{\Phi} = \left(\int_0^1 (\sup\{|f(y)| : y \in U(x, \Delta(x, \delta))\})^p dx\right)^{\frac{1}{p}},$$

where

$$U(x, \delta) = \{y \in [0, 1] : |x - y| \leq \delta\}.$$

Let us denote by $L_{\delta,p}$ or $L_{\delta,p}^{\Phi}$ the set of functions from L_∞ equipped with the norm $\|\circ\|_{\delta,p}$ or $\|\circ\|_{\delta,p}^{\Phi}$, respectively.

As a characteristic of the approximation of Bernstein polynomial we use the averaged moduli of smoothness

$$\tau_k(f, \Delta(\delta))_p = \|\omega_k(f, ., \Delta(., \delta))_\infty\|_p,$$

where

$$\omega_k(f, x, \Delta(x, \delta))_\infty = \sup\{|\Delta_h^k f(t)| : t, t + kh \in U(x, \Delta(x, \delta))\};$$

$$\tau_k(f, \Delta(\delta))_{p', \delta^2, p} = \|\omega_k(f, ., \Delta(., \delta))_{p'}\|_{p, \delta^2},$$

where the local $L_{p'}$ moduli ω_k are defined by

$$\omega_k(f, x, \Delta(x, \delta))_{p'} = \left((2\Delta(x, \delta))^{-1} \int_{-\Delta(x,\delta)}^{\Delta(x,\delta)} |\Delta_v^k f(x)|^{p'} dv\right)^{1/p'}.$$

Here $1 \leq p,\ p' \leq \infty$ and the finite difference $\Delta_v^k f(x)$ is defined as

$$\sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(x+mv), \text{ if } x,\ x+kv \in [0,\ 1]$$

and as 0 otherwise.

Let $W_p^k[0,\ 1]$ be the space of functions g with $g^{(k-1)}$ absolutely continuous and $g^{(k)} \in L_p[0,\ 1]$, endowed with seminorm

$$\| g^{(k)} \Delta(\delta)^k \|_{p[0,\ 1]} = \left(\int_0^1 | \Delta(x,\ \delta)^k g^{(k)}(x) |^p \, dx \right)^{1/p}.$$

2. Assertions. Let $0 < \delta \leq \delta'$. Then for $f \in L_\infty[0,\ 1]$, we have

$$(1) \qquad \|f\|_p \leq \|f\|^{\Phi}_{\delta,p} \leq \|f\|^{\Phi}_{\delta',p} \leq \|f\|^{\Phi}_{\delta',\infty},$$

$$(2) \qquad \|f\|_{\delta,p} \leq \|f\|^{\Phi}_{\sqrt{\delta},p}.$$

The first inequality follows immediately from the definitions of the norms; the second is trivial since $\delta \leq \Delta(x,\ \sqrt{\delta})$.

We also mention [4] that for every $n \geq 1$

$$(3) \qquad \left(\frac{1}{n} \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) \right|^p \right)^{1/p} \leq c \, \|f\|_{1/n,p}.$$

In the following lemma we give some of the properties of $\tau_k(f,\ \Delta(\delta))_p$.

Lemma 1. *For* $1 \leq p,\ p' \leq \infty$, $f \in L_\infty[0,\ 1]$ *and* $g \in W_p^k[0,\ 1]$ *we have*

$$(4) \qquad \tau_k(f,\ \Delta(\delta))_p \leq 2^k \|f\|^{\Phi}_{\delta,p},$$

$$(5) \qquad \tau_k(g,\ \Delta(\delta))_p \leq C_k \| \Delta(\delta)^k g^{(k)} \|_p,$$

$$(6) \qquad \tau_k(f, \Delta(\lambda\delta))_p \leq C \lambda^{2k+1} \tau_k(f,\ \Delta(\delta))_p, \quad \lambda > 0,$$

$$(7) \qquad \tau_k(f,\ \Delta(\delta))_{p',\delta^2,p} \leq C_k \, \tau_k(f,\ \Delta(\delta))_p.$$

Proof. To prove (4) note that

$$\omega_k(f,\ x,\ \Delta(x,\ \delta))_\infty \leq 2^k \sup\{ |f(t+mh)| : m=0,\ 1, \dots, k;$$
$$t,\ t+kh \in U(x,\ \Delta(x,\ \delta))\} = 2^k \| f(\cdot) \|_{\infty(U(\cdot,\ \Delta(\cdot,\ \delta)))}.$$

Hence

$$\tau_k(f,\ \Delta(\delta))_p \leq 2^k \|f\|^{\Phi}_{\delta,p}.$$

Now to prove (5) using Hölder inequality $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, we get

$$\omega_k(g,\ x,\ \Delta(x,\ \delta))_\infty \leq \sup\Big\{ \int_{-|h|}^{|h|} \cdots \int_{-|h|}^{|h|} | g^{(k)}(t+v_1+\dots+v_k) | \, dv_1 \cdots dv_k :$$
$$t,\ t+kh \in U(x,\ \Delta(x,\ \delta))\Big\}$$

$$\leq \sup\Big\{ \int_{-|h|}^{|h|} \cdots \int_{t+v_2+v_3+\dots+v_k-|h|}^{t+v_2+v_3+\dots+v_k+|h|} | g^{(k)}(u) | \, du \, dv_2 \cdots dv_k :$$
$$t,\ t+kh \in U(x,\ \Delta(x,\ \delta))\Big\}$$

$$\leqq \sup\{(2|h|)^{k-1}\int_{t-k|h|}^{t+k|h|}|g^{(k)}(u)|\,du:\ t,\ t+kh\in U(x,\ \Delta(x,\ \delta))\}$$

$$\leqq (4\Delta(x,\ \delta)/k)^{k-1}\int_{U(x,\ \Delta(x,\ \delta))}|g^{(k)}(u)|\,du$$

$$\leqq C_k\,\Delta(x,\ \delta)^{k-1}\,(\int_{U(x,\ \Delta(x,\ \delta))}|g^{(k)}(u)|^p\,du)^{1/p}(\int_{U(x,\ \Delta(x,\ \delta))}1^q\,du)^{1/q}$$

$$\leqq C_k\,\Delta(x,\ \delta)^{k-1/p}\,\|g^{(k)}\|_{p(U(x,\ \Delta(x,\ \delta)))}.$$

Hence using lemma 4 in [3], we obtain

$$\tau_k(g,\ \Delta(\delta))_p\leqq C_k\,(\int_0^1\Delta(x,\ \delta)^{kp-1}\int_{U(x,\ \Delta(x,\ \delta))}|g^{(k)}(u)|^p\,du\,dx)^{1/p}$$

$$\leqq C_k\,(\int_0^1\frac{1}{\Delta(x,\ \delta)}\int_{U(x,\ \Delta(x,\ \delta))}\Delta(u,\ \delta)^{kp}\,|g^{(k)}(u)|^p\,du\,dx)^{1/p}$$

$$\leqq C_k\,(\int_0^1\frac{1}{|U(x,\ \Delta(x,\ \delta))|}\int_{U(x,\ \Delta(x,\ \delta))}|g^{(k)}(u)\,\Delta(u,\ \delta)^k|^p\,du\,dx)^{1/p}\leqq C_k\,\|\Delta(\delta)^k\,g^{(k)}\|_p.$$

Property (6) was proved in [3].
Now to prove (7) let $y\in U(x,\ \Delta(x,\ \delta))$ and $|\nu|\leqq\Delta(y,\ \delta)$. In view of $\Delta(x,\ \delta)/a\leqq\Delta(y,\ \delta)\leqq a\,\Delta(x,\ \delta)$, $a\geqq 1$ (see [5]), we have $y,\ y+k\nu\in[x-\Delta(x,\ b\delta),\ x+\Delta(x,\ b\delta)]$, $b=ka+1$, $a\geqq 1$ and $\omega_k(f,\ y,\ \Delta(y,\ \delta))_{p'}\leqq\omega_k(f,\ x,\ \Delta(x,\ b\delta))_\infty$ which with (6) gives

$$\|\omega_k(f,\cdot,\ \Delta(\cdot,\ \delta))_{p'}\|_{\delta^2,p}\leqq\tau_k(f,\ \Delta(b\delta))_p\leqq C_k\,\tau_k(f,\ \Delta(\delta))_p.$$

Lemma 2. [6]. *For every $k,\ n\in N$; $n\geqq 6$ and for each $f\in R$, there is $g_{k,n}\in W_p^k[0,\ 1]$, such that*

$$|g_{k,n}(x)-f(x)|\leqq C_k\,\omega_k(f,\ x,\ \Delta(x,\ \frac{1}{\sqrt{n}}))_1 \text{ for } x\in[0,\ 1], \tag{8}$$

$$\|\Delta(1/\sqrt{n})^k\,g_{k,n}^{(k)}\|_p\leqq C_k\,\|\omega_k(f,\cdot,\ \Delta(\cdot,\ 1/\sqrt{n}))_1\|_p. \tag{9}$$

We want to mention that

$$\|\omega_k(f,\cdot,\ \Delta(\cdot,\ \delta))_1\|_p\leqq\|\omega_k(f,\cdot,\ \Delta(\cdot,\ \delta))_1\|_{\delta^2,p}=\tau_k(f,\ \Delta(\delta))_{1,\delta^2p}, \tag{10}$$

$$\|f-g_{k,n}\|^{\Phi}_{1/\sqrt{n},\ p}\leqq C_k\,\tau_k(f,\ \Delta(1/\sqrt{n}))_p. \tag{11}$$

Indeed (10) follows from (1), then to prove (11) using (8) and (7), we get

$$\|f-g_{k,n}\|^{\Phi}_{\frac{1}{\sqrt{n}},\ p}\leqq C_k\,(\int_0^1\sup\{\omega_k(f,\ t,\ \Delta(t,\ \frac{1}{\sqrt{n}}))_1 : t\in U(x,\Delta(x,\frac{1}{\sqrt{n}}))\}^p\,dx)^{\frac{1}{p}}$$

$$=C_k\,(\int_0^1\sup\{\frac{1}{2\Delta(t,\ \frac{1}{\sqrt{n}})}\int_{-\Delta(t,\ \frac{1}{\sqrt{n}})}^{\Delta(t,\ \frac{1}{\sqrt{n}})}|\Delta_\nu^k f(t)|\,d\nu:\ t\in U(x,\ \Delta(x,\ \frac{1}{\sqrt{n}}))\}^p\,dx)^{\frac{1}{p}}$$

$$\leqq C_k(\int_0^1|\frac{1}{2\Delta(x,\ \frac{1}{\sqrt{n}})}\int_{-c\Delta(x,\ \frac{1}{\sqrt{n}})}^{c\Delta(x,\ \frac{1}{\sqrt{n}})}|\Delta_\nu^k f(t)|\,d\nu\,|^p\,dx)^{\frac{1}{p}}$$

$$\le C_k \|\omega_k(f,\ x,\ c\Delta(x, \frac{1}{\sqrt{n}}))_\infty\|_p \le C_k \tau_k(f,\ \Delta(\frac{1}{\sqrt{n}}))_p.$$

Lemma 3. *For $f \in R$, we have*

$$\|B_n f\|^{\Phi}_{\frac{1}{\sqrt{n}},p} \le C\|f\|_{\frac{1}{n},p}. \tag{12}$$

Proof. Using Jensen inequality ($\sum_{k=0}^{n} P_{k,n}(t) = 1,\ t \in [0,1]$) and (3), we obtain ($\|P_{kn}\|^{\Phi}_{\frac{1}{\sqrt{n}},1} \le \frac{c}{n}$, [q])

$$\|B_n f\|^{\Phi}_{\frac{1}{\sqrt{n}},p} = \int_0^1 \sup\{|\sum_{k=0}^{n} f(\frac{k}{n}) P_{kn}(t):\ t \in U(x,\ \Delta(x, \frac{1}{\sqrt{n}}))\}^p\, dx)^{\frac{1}{p}}$$

$$\le (\int_0^1 \sup\{\ \sum_{k=0}^{n} |f(\frac{k}{n})|^p P_{kn}(t):\ t \in U(x,\ \Delta(x, \frac{1}{\sqrt{n}}))\}\, dx)^{\frac{1}{p}}$$

$$\le (\sum_{k=0}^{n} |f(\frac{k}{n})|^p \int_0^1 \sup\{P_{kn}(t):\ t \in U(x,\ \Delta(x, \frac{1}{\sqrt{n}}))\}\, dx)^{\frac{1}{p}}$$

$$\le (\frac{c}{n} \sum_{k=0}^{n} |f(\frac{k}{n})|^p)^{1/p} \le C\|f\|_{\frac{1}{n},p}.$$

Lemma 4. *For $g \in W_p^2[0, 1]$, we have*

$$\|B_n g - g\|^{\Phi}_{\frac{1}{\sqrt{n}},p} \le C\|\Delta(\frac{1}{\sqrt{n}})^2 g''\|_p. \tag{13}$$

Proof. (13) is true for $v = \infty$ (see [7]),

$$\|B_n g - g\|^{\Phi}_{\frac{1}{\sqrt{n}},\infty} = \|B_n g - g\|_\infty \le \frac{7}{n}\|\Phi(\cdot)^2 g''(\cdot)\|_\infty \le 7\|\Delta(\frac{1}{\sqrt{n}})^2 g''\|_\infty$$

and for $p = 1$ (see [9]).
Then using the interpolation property of L_p and $L_{\delta,p}$ (see [4]) spaces ($1 \le p \le \infty$, δ is fixed), we obtain the preposition of lemma 4.

Lemma 5. *If $f \in R$, $g \in W_p^2[0, 1]$, then*

$$\|\Phi^2 B_n'' f\|_p \le cn\|f\|_{\frac{1}{n},p}, \tag{14}$$

$$\|B_n'' f\|_p \le cn^2\|f\|_{\frac{1}{n},p}, \tag{15}$$

$$\|\Phi^2 B_n'' g\|_p \le \|(\Phi^2 + \frac{1}{n}) g''\|_p, \tag{16}$$

$$\|B_n'' g\|_p \le \|g''\|_p. \tag{17}$$

Proof. The inequality (14) is true for $p = \infty$ (see [1]) and for $p = 1$ (see [9]). From the interpolation property of the spaces L_p and $L_{\delta,p}$ (δ-fixed) we have (14) for every $p \in [1, \infty]$. Analogously (15) is true because it is valid for $p = \infty$ (see [10]) and for $p = 1$ (see [9]).

To establish (16), we need the identity (see [1])

$$\Phi^2(x)B''_n g(x)=n^2 \sum_{k=1}^{n-1} \Delta^2_{\frac{1}{n}}\, g\left(\frac{k-1}{n}\right) \Phi^2\left(\frac{k}{n}\right) P_{kn}(x).$$

Note that $\Phi^2(y)\leq\Phi^2(z)+\frac{1}{n}$ if $|y-z|\leq\frac{1}{n}$. So that for $g\in W^2_p[0, 1]$ (see [10])

$$|\Delta^2_{\frac{1}{n}}\, g\left(\frac{k-1}{n}\right)|\Phi^2\left(\frac{k}{n}\right)\leq \Phi^2\left(\frac{k}{n}\right) \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} |g''\left(\frac{k}{n}+s+t\right)|\, ds\, dt$$

(18)
$$\leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \int_{\frac{k}{n}+t-\frac{1}{2n}}^{\frac{k}{n}+t+\frac{1}{2n}} \left(\Phi^2(v)+\frac{1}{n}\right) |g''(v)|\, dv\, dt.$$

Then from (18) and by using Jensen inequality and Hölder inequality two times, we get

$$\|\Phi^2 B''_n g\|_p^p=n^{2p} \int_0^1 |\sum_{k=1}^{n-1} \Delta^2_{\frac{1}{n}}\, g\left(\frac{k-1}{n}\right)\Phi^2\left(\frac{k}{n}\right) p_{kn}(x)|^p\, dx$$

$$\leq n^{2p} \sum_{k=1}^{n-1} |\Delta^2_{\frac{1}{n}}\, g\left(\frac{k-1}{n}\right) \Phi^2\left(\frac{k}{n}\right)|^p \int_0^1 p_{kn}(x)\, dx$$

$$\leq n^{2p-1} \sum_{k=1}^{n-1} \left(\int_{-\frac{1}{2n}}^{\frac{1}{2n}} \int_{\frac{k}{n}+t-\frac{1}{2n}}^{\frac{k}{n}+t+\frac{1}{2n}} \left(\Phi^2(v)+\frac{1}{n}\right) |g''(v)|\, dv\, dt\right)^p$$

$$\leq n^{2p-1} \sum_{k=1}^{n-1} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \left(\int_{\frac{k}{n}+t-\frac{1}{2n}}^{\frac{k}{n}+t+\frac{1}{2n}} \left(\Phi^2(v)+\frac{1}{n}\right) |g''(v)|\, dv\right)^p dt \left(\frac{1}{n}\right)^{p-1}$$

$$\leq n^{p} \sum_{k=1}^{n-1} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \int_{\frac{k}{n}+t-\frac{1}{2n}}^{\frac{k}{n}+t+\frac{1}{2n}} \left(\Phi^2(v)+\frac{1}{n}\right)^p |g''(v)|^p\, dv \,.\left(\frac{1}{n}\right)^{p-1} dt$$

$$\leq n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \left(\sum_{k=1}^{n-1} \int_{\frac{k}{n}+t-\frac{1}{2k}}^{\frac{k}{n}+t+\frac{1}{2n}} \left(\Phi^2(v)+\frac{1}{n}\right)^p |g''(v)|^p\, dv\right) dt$$

$$=n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \int_{t+\frac{1}{2n}}^{1+t-\frac{1}{2n}} \left(\Phi^2(v)+\frac{1}{n}\right)^p |g''(v)|^p\, dv\, dt$$

$$\leq n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \int_0^1 \left(\Phi^2(v)+\frac{1}{n}\right)^p |g''(v)|^p\, dv\, dt=\|\left(\Phi^2(.)+\frac{1}{n}\right) g''(.)\|_p^p.$$

Now to establish (17) using that $\sum_{k=1}^{n-1} \Phi^{-2}(x)\Phi^2\left(\frac{k}{n}\right) p_{kn}(x)=\frac{n-1}{n}$, $x\in[0, 1]$

and $\int_0^1 \Phi^{-2}(x)\Phi^2(\frac{k}{n})p_{kn}(x)\,dx \leqq \frac{1}{n}$, analogously we get

$$\| B''_n g\|^p_{p[0,1]} = n^{2p} \| \sum_{k=1}^{n-1} \Phi^{-2}(x)\Phi^2(\frac{k}{n})p_{kn}(x)\Delta^2_{\frac{1}{n}} g(\frac{k-1}{n})\|^p_p .$$

$$\leqq n^{2p}(\frac{n-1}{n})^{p-1}\sum_{k=1}^{n-1} |\Delta^2_{\frac{1}{n}} g(\frac{k-1}{n})|^p \int_0^1 \Phi^{-2}(x)\Phi^2(\frac{k}{n})p_{kn}(x)\,dx$$

$$\leqq n^{2p-1}\sum_{k=1}^{n-1} |\Delta^2_{\frac{1}{n}} g(\frac{k-1}{n})|^p$$

$$\leqq n^{2p-1}\sum_{k=1}^{n-1} (\int_{-\frac{1}{2n}}^{\frac{1}{2n}} \int_{\frac{k}{n}+t-\frac{1}{2n}}^{\frac{k}{n}+t+\frac{1}{2n}} |g''(v)|\,dv\,dt)^p$$

$$\leqq n^{2p-1}\sum_{k=1}^{n-1} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} (\int_{\frac{k}{n}+t-\frac{1}{2n}}^{\frac{k}{n}+t+\frac{1}{2n}} |g''(v)|\,dv)^p\,dt.\,(\frac{1}{n})^{p-1}$$

$$\leqq n^{p}\sum_{k=1}^{n-1} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \int_{\frac{k}{n}+t-\frac{1}{2n}}^{\frac{k}{n}+t+\frac{1}{2n}} |g''(v)|^p\,dv\,.(\frac{1}{n})^{p-1}\,dt$$

$$= n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \int_{t+\frac{1}{2n}}^{1+t-\frac{1}{2n}} |g''(v)|^p\,dv\,dt \leqq \| g''\|^p_p .$$

3. Direct and inverse theorems. The method of proving, the following preposition is analogous to that in [9].

Theorem 1. *If $f \in R$, then*

(19) $$\| B_n f - f\|^{\Phi}_{\frac{1}{\sqrt{n}},\,p} \leqq C\tau_2(f,\,\Delta(\frac{1}{\sqrt{n}}))_p .$$

Proof. Let $g = g_{2,n}$ be the function from lemma 2. In view of (12), (13), (2), (9), (11) and (7), we get

$$\| B_n f - f\|^{\Phi}_{\frac{1}{\sqrt{n}},\,p} \leqq \| B_n(f-g)\|^{\Phi}_{\frac{1}{\sqrt{n}},\,p} + \|f-g\|^{\Phi}_{\frac{1}{\sqrt{n}},\,p} + \| B_n g - g\|^{\Phi}_{\frac{1}{\sqrt{n}},\,p}$$

$$\leqq C\|f-g\|_{\frac{1}{n},\,p} + \|f-g\|^{\Phi}_{\frac{1}{\sqrt{n}}} + C\|\Delta(\frac{1}{\sqrt{n}})^2 g''\|_p$$

$$\leqq C\|f-g\|^{\Phi}_{\frac{1}{\sqrt{n}},\,p} + C\tau_2(f,\,\Delta(\frac{1}{\sqrt{n}}))_{1,\,\frac{1}{n},\,p} \leqq C\tau_2(f,\Delta(\frac{1}{\sqrt{n}}))_p .$$

Remark 1. In view of (2), (12), (13), (9) and (8), we can get analogously

(20) $$\| B_n f - f \|_{\frac{1}{n}, p} \leqq C \tau_2 (f, \Delta(\frac{1}{\sqrt{n}}))_{1, \frac{1}{n}, p}.$$

Theorem 2. *If $f \in R$, then*

(21) $$\tau_2 (f, \Delta(\frac{1}{\sqrt{n}}))_p \leqq \frac{c}{n} \sum_{k=1}^{n} \| B_k f - f \|^{\Phi}_{\frac{1}{\sqrt{k}}, p}.$$

Proof. Let us apply the lemma given in [10] namely if $\mu_n, \nu_n, \psi_n \geqq 0$ with $\mu_1 = \nu_1 = 0$ satisfying $(1 \leqq k \leqq n)$

(22) $$\mu_n \leqq \frac{k}{n} \mu_k + \nu_k + \psi_k,$$

(23) $$\nu_n \leqq (\frac{k}{n})^2 \nu_k + \psi_k,$$

then it follows that

(24) $$\mu_n \leqq \frac{c}{n} \sum_{k=1}^{n} \psi_k.$$

Setting

$$\mu_n = n^{-1} \| \Phi^2 B''_n f \|_p + n^{-2} \| B''_n f \|_p,$$

$$\nu_n = n^{-2} \| B''_n f \|_p,$$

$$\psi_n = c \| B_n f - f \|^{\Phi}_{\frac{1}{\sqrt{n}}, p}.$$

We obtain in view of (14)—(17) and (1)

$$\mu_n \leqq n^{-1} \| \Phi^2 B''_n (B_k f) \|_p + n^{-2} \| B''_n (B_k f) \|_p + n^{-1} \| \Phi^2 B''_n (B_k f - f) \|_p + n^{-2} \| B''_n (B_k f - f) \|_p$$

$$\leqq n^{-1} \| \Phi^2 B''_k f \|_p + n^{-2} \| B''_k f \|_p + n^{-2} \| B''_k f \|_p + C \| B_k f - f \|^{\Phi}_{\frac{1}{\sqrt{n}}, p} + C \| B_k f - f \|^{\Phi}_{\frac{1}{\sqrt{n}}, p}$$

$$\leqq \{ n^{-1} \| \Phi^2 B''_k f \|_p + n^{-2} \| B''_k f \|_p \} + n^{-2} \| B''_k f \|_p + C \| B_k f - f \|^{\Phi}_{\frac{1}{\sqrt{n}}, p}$$

$$\leqq (\frac{k}{n}) \mu_k + (\frac{k}{n})^2 \nu_k + \psi_k,$$

which proves (22). Similarly, we establish (23). From (22) and (23), we obtain (24) which yields

(25) $$\| \Delta(\frac{1}{\sqrt{n}})^2 B''_n \|_p \leqq \frac{c}{n} \sum_{k=1}^{n} \| B_k f - f \|^{\Phi}_{\frac{1}{\sqrt{k}}, p}.$$

Now let $m \in N$ with $\frac{n}{2} \leqq m \leqq n$ such that $\| B_m f - f \|^{\Phi}_{\frac{1}{\sqrt{m}}, p} \leqq \| B_k f - f \|^{\Phi}_{\frac{1}{\sqrt{k}}, p}$ for any $\frac{n}{2} \leqq k \leqq n$. In view of (4), (5) and (25), we get

$$\tau_2 (f, \Delta(\frac{1}{\sqrt{n}}))_p \leqq \tau_2 (f - B_m f, \Delta(\frac{1}{\sqrt{m}}))_p + \tau_2 (B_m f, \Delta(\frac{1}{\sqrt{m}}))_p$$

$$\leqq c\,\|f-B_mf\|^{\Phi}_{\frac{1}{\sqrt{m}},p}+c\,\|\Delta(\frac{1}{\sqrt{m}})^2B''_mf\|_p$$

$$\leqq\frac{c}{n}\sum_{\frac{n}{2}\leqq k\leqq n}\|B_kf-f\|^{\Phi}_{\frac{1}{\sqrt{k}},p}+\frac{c}{m}\sum_{k=1}^{m}\|B_kf-f\|^{\Phi}_{\frac{1}{\sqrt{k}},p}$$

$$\leqq\frac{c}{n}\sum_{k=1}^{n}\|B_kf-f\|^{\Phi}_{\frac{1}{\sqrt{k}},p},$$

which completes the proof of the theorem.

Remark 2. In view of $\tau_2(f,\Delta(\delta))_{p',\delta^2,p}\leqq c\,\|f\|_{\delta^2,p}$, $1\leqq p'\leqq p$, using (2), (7), (5) and the inequalities (14)-(17), we can get that

(26) $$\tau_2(f,\Delta(\frac{1}{\sqrt{n}}))_{p',\frac{1}{n},p}\leqq\frac{c}{n}\sum_{k=1}^{n}\|B_kf-f\|_{\frac{1}{k},p}.$$

The inequality (26) is the inverse theorem for the direct result (20).

From theorems 1 and 2 we have

Corollary. *Let* $f\in R[0,1]$, $1\leqq p\leqq\infty$ and $0<\alpha<1$. *Then*

$$\|B_nf-f\|^{\Phi}_{\frac{1}{\sqrt{n}},p}=0(\frac{1}{n^{\alpha}})$$

iff

$$\tau_2(f,\Delta(\frac{1}{\sqrt{n}}))_p=0(\frac{1}{n^{\alpha}}).$$

REFERENCES

1. H. Berens, G. G. Lorentz. Inverse theorems for Bernstein polynomials. *Indiana Univ. Math. J.*, 21, 1972, 693-708.
2. Z. Ditzian, V. Totik. Moduli of smoothness. New York, 1987.
3. V. H. Hristov, K. G. Ivanov. Operators for onesided approximation by algebraic polynomials in $L_p[-1,1]^{\alpha}$. *Mathematica Balkanica* (New series), 2, 1988, 374-390.
4. V. H. Hristov. Best onesided approximation and mean approximation by interpolation polynomials of periodic functions. *Mathematica Balkanica* (New series), 3, 1989, 418-429.
5. K. G. Ivanov. On a new characteristic of functions. I. *Serdica*, 8, 1982, 262-279.
6. K. G. Ivanov. A constructive characteristic of the best algebraic approximation in $L_p[-1,1]$ $(1\leqq p\leqq\infty)$. Constructive Function Theory '81. *Bulg. Acad. Sci.*, 1983, 357-367.
7. K. G. Ivanov. Approximation by Bernstein polynomials in L_p metric. Constructive Function Theory' 84. *Bulg. Acad. Sci.*, 1984, 421-429.
8. S. K. Jassim. Best onesided approximation with algebraic polynomials. *Serdica* (to appear).
9. E. V. Wickeren. Direct and inverse theorems for Bernstein polynomials in the space of Riemann integrable functions. *Constructive approximation*, 5, 1989, 189-198.
10. E. V. Wickeren. Steckin — Marchaud-type inequalities in connection with Bernstein polynomials. *Constructive approximation*, 2, 1986, 331-337.

Institute of Mathematics
Bulgarian Academy of Sciences
1113 Sofia, Bulgaria

Received 27. 06. 1990
Revised 25. 10. 1990