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OPTIMAL CONTROL OF NONLINEAR EVOLUTION EQUATIONS

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In this paper we establish the existence of optimal solutions for a large class of strongly nonlinear evolution equations involving nonmonotone nonlinearities. An example of a nonlinear parabolic optimal control system illustrates the applicability of our work.

1. Introduction. The study of optimal control problems of infinite dimensional systems has attracted the interest of many mathematicians. This is exemplified by the books of Lions [9] and Ahmed-Teo [3] which summarize most of the work done in this area. So far, most of the attention was given to systems governed by linear or semilinear dynamical equations. Selectively we mention the works of Lions [9], [10], Zolezzi [14] and Ahmed-Teo [2]. More recently nonlinear optimal control problems were considered by Ahmed [1], Hou [8] and Cesari [6]. However, their continuity and growth hypotheses on the nonlinear term are restrictive and do not allow for the presence of nonmonotone nonlinearities. (See hypotheses $A(1)\to A(4)$ of Hou [8] and hypotheses $A_1\to A_5$, pp. 91—92 of Cesari [6].) Our work goes beyond the above papers and considers systems driven by a large class of strongly nonlinear evolutions.

2. Preliminaries. The mathematical setting of our problem is the following. The time horizon is $T=[0, b]$ and H is a separable Hilbert space. Let X be a subspace of H carrying the structure of a separable, reflexive Banach space, which embeds continuously and densely into H. Identifying H with its dual (pivot space), we have that $X\hookrightarrow H\hookrightarrow X^*$, with all embeddings being continuous and dense. We will also assume that they are compact. To have a concrete example in mind let Z be a bounded domain in $\mathbb{R}^n$ and $m\geq 1$ a positive integer. Set $X=H_0^m(Z)$, $H=L^2(Z)$ and $X^*=H_0^m(Z)^*=H^{-m}(Z)$. Then from the well-known Sobolev embedding theorem we have that $H_0^m(Z)\hookrightarrow L^2(Z)\hookrightarrow H^{-m}(Z)$ with all embeddings being continuous, dense and compact. Such a triple of spaces is usually known in the literature as a "Gelfand triple". Other names used are "evolution triple" or "spaces in normal position". By $\langle\cdot,\cdot\rangle$ we will denote the duality brackets for the pair (X, X^*) and by $(\cdot,\cdot)$ the inner product in H. The two are compatible in the sense that $\langle\cdot,\cdot\rangle|_{X\times H}=(\cdot,\cdot)$. Also by $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$) we will denote the norm of X (resp. of H, X^*).

Let $W(T)=\{x(\cdot)\in L^2(X): \dot{x}(\cdot)\in L^2(X^*)\}$, where the derivative in this definition should be understood in the sense of vector valued distributions. Furnished with the inner product $(x, y)_{W(T)}=(x,y)_{L^2(X)}+(\dot{x}, \dot{y})_{L^2(X^*)}$, $W(T)$ becomes a Hilbert space which is clearly separable (being a closed subspace of the separable Hilbert space $L^2(X)xL^2(X^*)$). Furthermore, it is well known that $W(T)\hookrightarrow C(T, H)=\{y: T\to H \text{ continuous}\}$ continuously; i. e. every element of $W(T)$ after possible modification on a Lebesgue null set is equal to a continuous function. Finally, since by hypothesis $X\hookrightarrow H$ compactly

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$W(T)\hookrightarrow L^2(H)$ compactly. For details we refer to Zeidler [13], Chapter 23. The control space will be modelled by a separable reflexive Banach space Y.

By $P_{f(c)}(Y)$ we will denote the family of nonempty, closed, (convex) subsets of Y. A multifunction $G: T\to P_f(Y)$ is said to be measurable if $GrG=\{(t, y)\in T\times Y: y\in G(t)\}\in B(T\times Y)=B(T)xB(Y)$ with $B(T\times Y)$ (resp. $B(T)$, $B(Y)$) being the Borel σ-field of $T\times Y$ (resp. of T, Y).

Finally recall that an operator $A: X\to X^*$ is said to be monotone if $<Ax-Ax'$, $x-x'\geq 0$ for all $x, x'\in X$ and is said to be hemicontinuous if $\lambda\to\langle A(x+\lambda y), z\rangle$ is continuous on $[0, 1]$ for all $x, y, z\in X$ (i. e. is weakly continuous along rays).

3. Existence theorem. *The Lagrange optimal control problem under consideration is the following:*

$$\left.\begin{cases} J(x, u)=\int\limits_0^b L(t, (Mx)(t), u(t))\,dt\to\inf=m \\ \text{s. t. } \dot{x}(t)+A(t, x(t))+g(t, x(t))=B(t)u(t) \text{ a. e.} \\ \qquad x(0)=x_0,\ u(t)\in U(t) \text{ a. e., } u(\cdot) \text{ measurable}\end{cases}\right\}(*).$$

We will need the following hypotheses on the data of our problem (*):

H(A): $A: TxX\to X^*$ is an operator s. t.

(1) for every $x\in X$, $t\to A(t, x)$ is measurable,

(2) for every $t\in T$, $x\to A(t, x)$ is monotone and hemicontinuous,

(3) $\langle A(t, x), x\rangle\geq c_1(t)\|x\|^2$ a. e. with $c_1(\cdot)\in L_+^\infty$,

(4) $\|A(t, x)\|_*\leq a(t)+b\|x\|$ a. e. with $a(\cdot)\in L_+^2$, $b\geq 0$.

H(g): $g: TxX\to H$ is a map s. t.

(1) for every $x\in X$, $t\to g(t, x)$ is measurable,

(2) for every $t\in T$, $x\to g(t, x)$ is continuous and sequentially weakly continuous,

(3) for all $x\in X$, $(g(t, x), x)\geq 0$ a, e.,

(4) $|g(t, x)|\leq a_1(t)+b_1\|x\|$ a. e. with $a_1(\cdot)\in L_+^2$, $b_1\geq 0$.

H(B): $B\in L^\infty(T, \mathscr{L}(Y, H))$($\mathscr{L}(Y, H)$ is the Banach space of bounded linear operators from Y into H).

H(U): $U: T\to P_{fc}(Y)$ is a measurable multifunction s.t. $U(t)\subseteq W\in P_{wkc}(Y)$ a. e.

H(L): Let E be a separable Banach space with norm $\|\cdot\|_E$ and let $L: T\times E\times Y\to \mathrm{R}=\mathrm{R}\cup\{+\infty\}$ be an integrand s.t.

(1) $(t, x, u)\to L(t, x, u)$ is measurable,

(2) $(x, u)\to L(t, x, u)$ is lower semicontinuous (l. s. c.).

(3) $u\to L(t, x, u)$ is convex,

(4) $\varnothing(t)-\lambda(\|x\|_E+\|u\|_Y)\leq L(t, x, u)$ a. e. with $\varnothing(\cdot)\in L^1$, $\lambda\geq 0$.

H(M): $M: L^2(X)\to L^2(E)$ is an operator s. t. for every sequence $\{x_n\}_{n\geq 1}$ weakly convergent in $W(T)$ to x, then $\{Mx_n\}_{n\geq 1}$ has a subsequence strongly convergent to Mx in $L^2(E)$.

Note that given an admissible control $u(\cdot)$ (i. e. a measurable map $u: T\to Y$ s. t. $u(t)\in U(t)$ a. e.), there exists a trajectory $x(\cdot)\in W(T)$ for system (*). This is a result of Hirano [7]. Hirano considered a time independent operator A, however it is easy to check that his work extends easily to the time dependent case $A(t, x)$ using hypothesis $H(A)$ above.

Now we are ready for our existence result. To avoid trivialities we will assume that $m<\infty$; i. e. there exists admissible "state-control" pair (x, u) s. t. $J(x, u)<\infty$.

Theorem 3.1: *If hypotheses $H(A)$, $H(g)$, $H(U)$, $H(L)$, $H(M)$ hold and $x_0\in H$, then problem (*) has a solution; i. e. there exists an admissible "state-control" pair $(x, u)\in W(T)xL^2(Y)$ s. t. $J(x, u)=m$.*

Proof: First we will establish some a priori bounds for the trajectories of (*). So let $x(\cdot)\in W(T)$ be such a trajectory. We have:

$$\langle \dot{x}(t),\ x(t)\rangle+\langle A(t,\ x(t)),\ x(t)\rangle+(g(t,\ x(t)),\ x(t))=(B(t)u(t),\ x(t)) \text{ a. e.,}$$

$$\Rightarrow \frac{d}{dt}|x(t)|^2+2\widehat{c}_1\|x(t)\|^2\leq(B(t)u(t),\ x(t)) \text{ a. e. with } \widehat{c}_1=\|c_1(\cdot)\|_\infty$$

(see hypotheses $H(A)$ (3) and $H(g)$ (3),

$$\Rightarrow \frac{d}{dt}|x(t)|^2+2\widehat{c}_1\|x(t)\|^2\leq\epsilon|B(t)u(t)|^2+\frac{1}{\epsilon}|x(t)|^2 \text{ a. e.}$$

(Cauchy's inequality with $\epsilon>0$).

Let $\epsilon=\beta^2/2\widehat{c}_1$, where $\beta>0$ is such that $|\cdot|\leq\beta\|\cdot\|$ (such a $\beta>0$ exists since $X\hookrightarrow H$ continuously). We have

$$\frac{d}{dt}|x(t)|^2\leq\frac{\beta^2}{2\widehat{c}_1}|B(t)u(t)|^2 \text{ a. e.,}$$

$$\Rightarrow |x(t)|^2\leq\frac{\beta^2}{2\widehat{c}_1}|W|^2\|B\|_\infty^2+|x_0|^2,$$

$\Rightarrow |x(t)|\leq M_1$ for all $t\in T$ and with $M_1>0$, independent of $x(\cdot)$.

Also if $\epsilon=1/2$, we have

$$|x(b)|^2+2\widehat{c}_1\int_0^b\|x(t)\|^2dt\leq\frac{1}{2}|W|^2\int_0^b\|B(t)\|^2_{\mathscr{L}(Y,H)}dt+\frac{1}{2}M_1^2b+|x_0|^2,$$

$\Rightarrow \|x\|_{L^2(X)}\leq M_2$ for some $M_2>0$, independent of the trajectory $x(\cdot)$.

Finally let $\eta(\cdot)\in L^2(X)$ and by $((\cdot,\ \cdot))_0$ denote the duality brackets for the dual pair $(L^2(X),\ L^2(X^*))$. We have:

$$\langle \dot{x}(t),\ \eta(t)\rangle+\langle A(t,\ x(t)),\ \eta(t)\rangle+(g(t,\ x(t)),\ \eta(t))=(B(t)u(t),\ \eta(t)) \text{ a. e.,}$$

$$\Rightarrow ((\dot{x},\ \eta))_0\leq\int_0^b\|A(t,\ x(t))\|_*\cdot\|\eta(t)\|dt+\int_0^b|g(t,\ x(t))|\,.\,|\eta(t)|dt+\int_0^b|B(t)u(t)|\,.\,|\eta(t)|dt$$

$$\leq(\|a\|_2+b\|x\|_{L^2(X)})\cdot\|\eta\|_{L^2(X)}+\widehat{\beta}(\|a_1\|_2+b_1\|x\|_{L^2(X)})\,.\,\|\eta\|_{L^2(X)}+\widehat{\beta}|W|\,.\,\|B\|_2\|\eta\|_{L^2(X)}$$

with $\widehat{\beta}>0$ depending only on $\beta>0$ for which $|\cdot|\leq\beta\|\cdot\|$. Since $\eta\in L^2(X)$ was arbitrary we deduce that $\|\dot{x}\|_{L^2(X^*)}\leq M_3$ with $M_3>0$ independent of the trajectory $x(\cdot)$. From all the above estimates and the definition of the space $W(T)$, we deduce that there exists $M_4>0$ independent of the trajectory $x(\cdot)$ s. t. $\|x\|_{W(T)}\leq M_4$.

Now, let $\{(x_n,\ u_n)\}_{n\geq1}\subseteq W(T)\times L^2(Y)$ be a minimizing sequence of admissible "state-control" pairs for (*); i. a. $J(x_n,\ u_n)\downarrow m$. From the previous a priori estimation, we know that $\{x_n\}_{n\geq1}$ is bounded in $W(T)$. Since the latter is a separable Hilbert space, by passing to a subsequence if necessary, we may assume that $x_n\xrightarrow{w}x$ in $W(T)$. But recall (see for example Zeidler [13], p. 450) that $W(T)\hookrightarrow L^2(H)$ compactly. So $x_n\xrightarrow{s}x$ in $L^2(H)$. Also we may assume that $u_n\xrightarrow{w}u$ in $L^2(Y)$ as $n\to\infty$.

Let $\widehat{A}(\cdot)$, $G(\cdot)$ and $\widehat{B}(\cdot)$ be the Nemitsky operators corresponding to $A(\cdot,\ \cdot)$, $g(\cdot,\ \cdot)$ and $B(\cdot)$ respectively. We have:

$$((\dot{x}_n,\ x_n-x))_0+((\widehat{A}(x_n),\ x_n-x))_0+((G(x_n),\ x_n-x))_0=((\widehat{B}u_n,\ x_n-x))_0.$$

From the integration by parts formula for space $W(T)$ (see Zeidler [13], proposition 23.23, p. 422), we have

$$((\dot{x}_n-\dot{x},\ x_n-x))_0=\int_0^b<\dot{x}_n(t)-\dot{x}(t),\ x_n(t)-x(t)>dt=\frac{1}{2}|x_n(b)-x(b)|^2,$$

$$\Rightarrow((\dot{x}_n,\ x_n-x))_0=((\dot{x},\ x_n-x))_0+\frac{1}{2}|x_n(b)-x(b)|^2.$$

Since $W(T)\hookrightarrow C(T,\ H)$ continuously, we have

$$((\dot{x},\ x_n-x))_0+\frac{1}{2}|x_n(b)-x(b)|^2\to 0 \text{ as } n\to\infty.$$

$$\Rightarrow((\dot{x}_n,\ x_n-x))_0\to 0 \text{ as } n\to\infty.$$

Also note that $((\widehat{B}u_n,\ x_n-x))_0=((\widehat{B}u_n,\ x_n-x))_{L^2(H)}\to 0$ (here $((\cdot,\ \cdot))_{L^2(H)}$ denotes the inner product in the Hilbert space $L^2(H)$). Hence we have $\lim((\widehat{A}(x_n)+G(x_n),\ x_n-x))_0=0$. Invoking proposition 2, p. 603 of Hirano [7], we get

$$\widehat{A}(x_n)+G(x_n)\xrightarrow{w}\widehat{A}(x)+G(x) \text{ as } n\to\infty \text{ in } L^2(X^*).$$

Note that since $x_n\xrightarrow{w}x$ in $W(T)\Rightarrow\dot{x}_n\xrightarrow{w}\dot{x}$ in $L^2(X^*)$. So if $\eta(\cdot)\in L^2(X)$, we have:

$$((\dot{x}_n,\ \eta))_0+((\widehat{A}(x_n),\ \eta))_0+((G(x_n),\ \eta))_0=((\widehat{B}u_n,\ \eta))_0,$$

$$\to((\dot{x},\ \eta))_0+((\widehat{A}(x),\ \eta))_0+((G(x),\ \eta))_0=((\widehat{B}(u),\ \eta))_0.$$

Since $\eta\in L^2(X)$ is arbitrary, we deduce that

$$\dot{x}+\widehat{A}(x)+G(x)=\widehat{B}(u) \text{ in } L^2(X^*),$$

$$\Rightarrow\dot{x}(t)+A(t,\ x(t))+g(t,\ x(t))=B(t)u(t) \text{ a. e.},$$

$$x(0)=x_0.$$

Furthermore since $u_n\xrightarrow{w}u$ in $L^2(Y)$ from Theorem 3.1 of [12] we have $u(t)\in U(t)$ a. e..So $(x,\ u)$ is an admissible "state-control" pair for (*).

Next from hypothesis $H(M)$ and by passing to a subsequence if necessary, we have

$$M(x_n)\xrightarrow{s}M(x) \text{ in } L^2(E),$$

$$\Rightarrow M(x_n)(t)\xrightarrow{s}M(x)(t) \text{ a. e. in } E.$$

Because of hypothesis H(L), we can apply Theorem 2.1 of Balder [5], to get

$$J(x,\ u)=\int_0^b L(t,\ (Mx)(t),\ u(t))\,dt\leq\lim J(x_n,\ u_n)=m.$$

But since $(x,\ u)\in W(T)\times L^2(Y)$ is an admissible "state-control" pair, we must have $J(x,\ u)=m\Rightarrow(x,\ u)$ is the desired solution of (*).

Q.E.D.

Remark. An interesting byproduct of the above proof is that the set of admissible "state-control" pairs is weakly compact in $W(T)\times L^2(Y)$.

4. **An example.** In this section we illustrate the applicability of Theorem 3.1, with an example of an optimal control problem of a nonlinear parabolic distributed parameter system.

So let Z be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma=\partial Z$. The optimal control system under consideration is the following:

$$\left\{\begin{array}{l} J(x,\ u)=\int_0^b\int_Z L(t,\ z,\ \eta(x(t,\ z)),\quad u(t,\ z))\,dzdt\to\inf=m \\ \text{s. t. } \dfrac{\partial x(t,\ z)}{\partial t}+\sum_{|\alpha|\leq m}(-1)^{|\alpha|}D\alpha A_\alpha(t,\ z,\ \theta(x(t,\ z))) \\ +g(t,\ z,\ \eta(x(t,\ z)))=c(t)u(t,\ z) \text{ on } T\times Z \\ D\gamma x(t,\ z)|_{T\times\Gamma}=0,\quad |\gamma|\leq m-1,\quad x(0,\ z)=x_0(z),\quad |u(t,\ z)|\leq r(t,\ z) \text{ a. e.} \end{array}\right\} (**).$$

Here $\alpha=(\alpha_1,\ \dots,\ \alpha_n)$ is an n-tuple of nonnegative integers (multi-index) $|\alpha|=\alpha_1+\dots+\alpha_n$ (the "length" of the multi-index), $D^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$, where $D_i=\dfrac{\partial}{\partial z_i}$, $\eta(x)=\{D^\beta x:\ |\beta|\leq m-1\}$ and $\theta(x)=\{D^\alpha x:\ |\alpha|\leq m\}$.

We will need the following hypotheses on the data of (**):

H(A)$_1$: $A_\alpha:\ T\times Z\times \mathrm{R}^{n_m}\to\mathrm{R}$ ($n_m=\dfrac{(n+m)!}{n!\,m!}$) is a function s. t.

(1) $(t,\ z)\to A_\alpha(t,\ z,\ \theta)$ is measurable,

(2) $\theta\to A_\alpha(t,\ z,\ \theta)$ is continuous,

(3) $|A_\alpha(t,\ z,\ \theta)|\leq a(t,\ z)+b(z)|\theta|$ a. e. with $a(\cdot,\ \cdot)\in L^2(T\times Z)$, $b(\cdot)\in L^\infty(Z)$,

(4) $\sum_{|\alpha|\leq m}(A_\alpha(t,\ z,\ \theta)-A_\alpha(t,\ z,\ \theta'))(\theta_\alpha-\theta'_\alpha)\geq 0$,

(5) $\sum_{|\alpha|\leq m}A_\alpha(t,\ z,\ \theta)\,\theta_2\geq c_1(z)\sum_{|\beta|=m}|\theta_\beta|$ with $c_1(\cdot)\in L^\infty(Z)$.

H(g)$_1$: $gTxZ\times\mathrm{R}^{n_{m-1}}\to\mathrm{R}$ ($n_{m-1}=\dfrac{(n!+m-1)!}{n!(m-1)!}$) is a function s. t.

(1) $(t,\ z)\to g(t,\ z,\ \eta)$ is measurable,

(2) $\eta\to g(t,\ z,\ \eta)$ is continuous,

(3) $\sum_{|\beta|\leq m-1}g(t,\ z,\ \eta)\,\eta_\beta\geq 0$,

(4) $|g(t,\ z,\ \eta)|\leq a_1(t,\ z)+g_1(z)|\eta|$ a. e. with $a_1(\cdot,\ \cdot)\in L^2(T\times Z)$, $b_1(\cdot)\in L^\infty(Z)$.

H(c): $c(\cdot)\in L^\infty(T)$.

H(r): $r(\cdot,\ \cdot)\in L^\infty(T\times Z)$.

H(L)$_1$: $L:T\times Z\times\mathrm{R}^{n_{m-1}}\times\mathrm{R}\to\bar{\mathrm{R}}=\mathrm{R}\cup\{+\infty\}$ is an integrand s.t.

(1) $(t,\ z,\ \eta,\ u)\to L(t,\ u,\ \eta,\ u)$ is measurable,

(2) $(\eta,\ u)\to L(t,\ z,\ \eta,\ u)$ is lower semicontinuous (l. s. c.),

(3) $u\to L(t,\ z,\ \eta,\ u)$ is convex,

(4) $\varnothing(t,\ z)-\lambda(z)(\|\eta\|+|u|)\leq L(t,\ z,\ \eta,\ u)$ a. e. with $\varnothing(\cdot,\ \cdot)\in L^1(T\times Z)$, $\lambda(\cdot)\in L_+^\infty(Z)$.

Here $X=H_0^m(Z)$, $H=L^2(Z)$ and $X^*=H^{-m}(Z)$. From the Sobolev embedding theorem, we know that $(X,\ H,\ X^*)$ is a Gelfand triple with all embeddings compact. Also let $Y=L^2(Z)$.

Consider the time varying Dirichlet form $a(t,\ x,\ y)$ corresponding to the elliptic partial differential operator of our system; i. e.

$$a(t.\ x,\ y)=\int_Z\sum_{|\alpha|\leq m}A_\alpha(t,\ z,\ \eta(x(z)))\,D^\alpha y(z)\,dz,\ x,\ y\in H_0^m(Z).$$

Clearly from Fubini's theorem, we see that $t\to a(t,\ x,\ y)$ is measurable. Also using Cauchy's inequality, we get

$$|a(t,\ x,\ y)|\leq\sum_{|\alpha|\leq m}\|\widehat{A}_\alpha(t)x\|_2\cdot\|D^\alpha y\|_2\leq(\widehat{a}(t)+\widehat{b}\,|x\|_{H_0^m(Z)})\cdot\|y\|_{H_0^m(Z)}$$

with $\widehat{A}_\alpha(t)(\cdot)$ being the Nemitsky (superposition) operator corresponding to the function $A_\alpha(t,\ \cdot,\ \cdot)$ and $\widehat{a}(t)=\|a(t,\ \cdot)\|_2$, $\widehat{b}=\|b(\cdot)\|_\infty$.

So there exists a generally nonlinear operator $\widehat{A}: T\times H_0^m(Z)\rightarrow H^{-m}(Z)$ s. t. $\langle \widehat{A}(t, x), y\rangle = a(t, x, y)$.

Hence from the measurability of $a(\cdot, x, y)$, we deduce that $t\rightarrow\widehat{A}(t, x)$ is weakly measurable and since $H^{-m}(Z)$ is a separable Hilbert space from the Pettis measurability theorem, we deduce that $t\rightarrow\widehat{A}(t, x)$ is a measurable map.

Also using hypothesis $H(A)_1$ (4), we can easily check that for every $t\in T$, $\widehat{A}(t, \cdot)$ is monotone. Furthermore from hypothesis $H(A)_1$ (5), we get that

$$\langle\widehat{A}(t, x), x\rangle\geqq\widehat{c}\,\|x\|^2_{H_0^m(Z)}$$

with $\widehat{c}=\|c(\cdot)\|_\infty$. Finally let $x_n\xrightarrow{s}x$ in $H_0^m(Z)$. Then since by Krasnoselski's theorem $\widehat{A}_\alpha(t)(\cdot)$ is continuous, we have

$$\widehat{A}_\alpha(t)(x_n)\xrightarrow{s}\widehat{A}_\alpha(t)(x) \text{ in } L^2(Z) \text{ as } n\rightarrow\infty.$$

But from Cauchy's inequality, we have

$$\|\widehat{A}(t, x_n)-\widehat{A}(t, x)\|_*\leqq\sum_{|\alpha|\leqq m}\|\widehat{A}_\alpha(t)(x_n)-\widehat{A}_\alpha(t)(x)\|_2\rightarrow 0 \text{ as } n\rightarrow\infty$$

$$\Rightarrow x\rightarrow\widehat{A}(t, x) \text{ is continuous, in particular then hemicontinuous.}$$

So we see that $\widehat{A}(t, x)$ mapping $T\times X$ into X^* satisfies hypothesis $H(A)$.

Next let $\widehat{g}: T\times H_0^m(Z)\rightarrow L^2(Z)$ be the Nemitsky operator corresponding to the function $g(t, z, \eta)$. From Krasnoselski's theorem we have that $\widehat{g}(t, \cdot)$ is continuous while since $H_0^m(Z)\hookrightarrow H_0^{m-1}(Z)$ compactly (Sobolev's embedding theorem; see for example Zeidler [13]), we get that $\widehat{g}(t, \cdot)$ is also sequentially weakly continuous. Furthermore for fixed $x\in H_0^m(Z)$, we have for every $h\in L^2(Z)$:

$$(\widehat{g}(t, x), h)_{L^2(Z)}=\int_Z g(t, z, \eta(x(z)))\,h(z)\,dz.$$

Invoking Fubini's theorem we have that $t\rightarrow(\widehat{g}(t, x), h)_{L^2(Z)}$ is measurable. Since $h\in L^2(Z)$ was arbitrary, we deduce that $t\rightarrow\widehat{g}(t, x)$ is weakly measurable and because $L^2(Z)$ is separable, once again from Pettis' theorem we conclude that $t\rightarrow\widehat{g}(t, x)$ is measurable. Also from hypothesis $H(g)_1$ (3) we get that $(\widehat{g}(t, x), x)_{L^2(Z)}\geqq 0$, while from $H(g)_1$ (4) we have that $\|\widehat{g}(t, x)\|_{L^2(Z)}\leq\widehat{a}_1(t)+\widehat{b}_1\|x\|_{H_0^m}$ with $\widehat{a}_1(t)=\|a_1(t, \cdot)\|_2\,\widehat{b}_1=\|b_1(\cdot)\|_\infty$. Hence we have checked that $\widehat{g}(t, x)$ going from $T\times X$ into H satisfies hypothesis $H(g)$.

Let $U(t)=\{v\in Y=L^2(Z): \|v\|_2\leq\|r(t, \cdot)\|_\infty\}\subseteq W=\{v\in Y=L^2(Z): \|v\|_2\leq\|r\|_\infty\}$. Clearly $U(\cdot)$ is measurable (since $t\rightarrow\|r(t, \cdot)\|_\infty$ is a measurable function) and $W\in P_{wkc}(Y)$. So we have satisfied hypothesis $H(U)$.

Next let $E=L^2(Z)^{n_{m-1}}$ and let $\widehat{L}: T\times E\times Y\rightarrow\overline{\mathrm{R}}=\mathrm{R}\cup\{+\infty\}$ be defined by $\widehat{L}(t, y, u)=\int_Z L(t, z, \widehat{\eta}(y(z)), u(z))\,dz$, where $\widehat{\eta}(y(z))=(y_k(z))_{k=1}^{n_{m-1}}$. Let $L_k: T\times Z\times\mathrm{R}^{n_{m-1}}\times\mathrm{R}\rightarrow\mathrm{R}$ be Caratheodory integrands (i. e. measurable in (t, z), continuous in (η, u)) s. t. $L_k\uparrow L$ as $k\rightarrow\infty$ and $\varnothing(t, z)-\lambda(z)(\|\eta\|+|u|)\leq L_k(t, z, \eta, u)\leq k$. Such a sequence exists by Lemma 2, p. 535 of Balder [4]. Set $\widehat{L}_k(t, y, u)=\int_Z L_k(t, z, \widehat{\eta}(y(z)), u(z))dz$. Clearly for each $k\geqq 1$, $\widehat{L}_k(\cdot, \cdot, \cdot)$ is a Caratheodory map (i. e. measurable in t, continuous in (y, u)), thus it is jointly measurable. Furthermore from the monotone convergence theorem we have that $\hat{L}_k \uparrow \hat{L}$ as $k \to \infty$, hence $\hat{L}$ is measurable too. In addition using Fatou's lemma we can check that $\hat{L}(t, \cdot, \cdot)$ is l. s. c. and is also convex in u. So we have satisfied hypothesis $H(L)$. Let $x_0 = x_0(\cdot) \in L^2(Z)$.

Now let $M: L^2(T, H_0^m(Z)) \to L^2(T, L^2(Z)^{n_m - 1})$ be defined by $(Mx)(t, \cdot) = \eta(x(t, \cdot))$.

Since $H_0^k(Z) \hookrightarrow L^2(Z)$ compactly for $1 \leq k \leq m$, we have that $L^2(T, H_0^k(Z)) \hookrightarrow L^2(T, L^2(Z))$ compactly for $1 \leq k \leq m$ (see Zeidler [13], p. 450) and so we deduce that $M(\cdot)$ satisfies hypothesis $H(M)$.

Therefore system (**) admits the following equivalent abstract form:

$$\left.\begin{cases} \hat{J}(x, u) = \int_0^b \hat{L}(t, (Mx)(t), u(t))\,dt \to \inf = \hat{m} \\ \text{s. t. } \dot{x}(t) + \hat{A}(t, x(t)) + \hat{g}(t, x(t)) = c(t)u(t) \text{ a. e.} \\ \qquad x(0) = x_0 \\ \qquad u(t) \in U(t) \text{ a. e. and } u(\cdot) \text{ is measurable} \end{cases}\right\} (**)'.$$

This has the form of problem (**) and we have checked above that satisfies all the hypotheses of theorem 3.1. So applying that theorem we get:

Theorem 4.1. *If hypotheses $H(A)_1$, $H(g)_1$, $H(c)$, $H(r)$ and $H(L)_1$ hold and $x_0(\cdot) \in L^2(Z)$, then there exists admissible "state-control" pair $(x, u) \in L^2(T, H_0^m(Z)) \cap C(T, L^2(Z)) \times L^2(T \times Z)$ and $\frac{\partial x}{\partial t} \in L^2(T, H^{-m}(Z))$* s. t. $J(x, u) = m$.

Remark. Also using the compactness theorem of Nagy [11], we can say that the set of trajectories of (**) is compact in $C(T, L^2(Z))$.

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