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ON THE RANK OF A CURVATURE TENSOR OF A FINSLER MANIFOLD

MASAKI FUKUI, CONSTANTIN UDRIŞTE

1. The rank of a curvature tensor. Let M be a Riemannian manifold of dimension n and g_{ij} be its metric tensor. The tangent space V of any fixed point has g_{ij} as an inner product. Then the exterior product $\Lambda^2 V$ is a vector space of dimension $n(n-1)/2$ with an inner product

$$G_{ijkl}=(g_{ik}g_{jl}-g_{il}g_{jk})/2. \tag{1.1}$$

The Riemannian curvature tensor R_{ijkl} of M satisfies

$$R_{ijkl}=-R_{jikl}=-R_{ijlk}, \tag{1.2}$$

and so R_{ij}^{kl} can be regarded as a linear endomorphism of $\Lambda^2 V$. The rank of this endomorphism is called the rank of the curvature tensor, and it coincides with the rank of the matrix (R_{ij}^{kl}).

On the other hand, any Riemannian space of constant curvature is obviously a locally symmetric space which is characterized by

$$R_{hijk,l}=0 \tag{1.3}$$

where we denote by "," the covariant differentiation. Relating to this topic one of the authors proved:

Theorem (Udrişte). If a Riemannian manifold M of dimension n is locally symmetric and the curvature tensor has the maximal rank $n(n-1)/2$, then M is a space of constant curvature.

Later similar theorems for Kaehlerian and Sasakian manifolds have been obtained by K. Sato [2] and T. Takahashi [3]. In the present paper, we consider the case of a Finsler manifold. Notations and terminologies are referred to Matsumoto's monograph [1].

2. Finslerian analogy. Let M be a Finsler manifold and $L(x, y)$ the fundamental function, where x is a point of M and y is an element of support, that is, a non-zero tangent vector at x. The h-curvature and (v)-h torsion of the Cartan connection are denoted by R_{jk}^{hi} and R_{jk}^{i}, respectively. Then one of Ricci formulas for X_i is written as

$$X^i_{|j|k}-X^i_{|k|j}=X^r R^i_{r\,jk}-X^i|_r R^r_{jk}, \tag{2.1}$$

where we denote by "|" and "|" h- and v-covariant differentiations, respectively. These tensors satisfy the Bianchi identity

$$\mathfrak{S}_{(hjk)}\{R_{h\,jk}^{\,i}-C_{hr}^{i}R_{jk}^{r}\}=0. \tag{2.2}$$

Since the h-curvature satisfies the skew-symmetric properties (1.2) just like as the Riemannian case, we can define the rank of Cartan's h-curvature tensor R_{hijk} of a Finsler manifold.

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In general, a Finsler manifold is called an isotropic manifold when it holds that

$$R_{hijk}=R(g_{hj}g_{ik}-g_{hk}g_{ij}) \tag{2.3}$$

We shall show:

Theorem. *Let M be a Finsler manifold of dimension n. If Cartan's h-curvature tensor has the maximal rank n (n−1)/2 at any (x, y) and $R_{hijk|l}=0$, $R_{hijk|l}=0$ hold identically, then M is an isotropic Finsler manifold.*

Proof. In consideration of Ricci's formula for R_{hijk}, we get

$$R_{rijk}R^r_{hlm}+R_{hrjk}R^r_{ilm}+R_{hirk}R^r_{jlm}+R_{hijr}R^r_{klm}=0. \tag{2.4}$$

Since the linear endomorphism R is regular, there exists Q_{lmst} such that $R_{rhlm}Q^{lm}_{st}=g_{rs}g_{ht}-g_{rt}g_{hs}$. Transvecting (2.3) with Q^{lm}_{st} and further with g^{ht}, we have

$$(n-1)R_{sijk}=R_{ik}g_{sj}-R_{ij}g_{sk}+(R_{sijk}+R_{jiks}+R_{kisj}), \tag{2.5}$$

where $R_{ik}=R^{ir}_{kr}$. From the identity (2.2), we get

$$R_{sjk}+R_{jks}+R_{ksj}=0, \tag{2.6}$$

because of $R_{sjk}=R_{rsjk}y^r$. Then, contraction of (2.5) with y^i leads us to

$$(n-1)R_{sjk}=R_{Oj}g_{sk}-R_{Ok}g_{sj}, \tag{2.7}$$

where the index O means contraction with y. By further contraction with y, (2.7) implies $R_{Oj}=L^{-2}R_{OO}y_j$ and therefore

$$R_{sjk}=R'(g_{sk}y_j-g_{sj}y_k), \tag{2.8}$$

with a certain scalar R'. On account of (2.8) the Bianchi identity (2.2) is reduced to $R_{sijk}+R_{jiks}+R_{kisj}=0$. From the equation (2.5) we have

$$(n-1)R_{sijk}=R_{ik}g_{sj}-R_{ij}g_{sk}, \tag{2.9}$$

and consequently

$$R_{sijk}=R(g_{sj}g_{ik}-g_{sk}g_{ij})/n(n-1), \tag{2.10}$$

where $R=g^{ij}R_{ij}$.

Remark. Since Cartan's v-curvature tensor S_{hijk} also satisfies skew-symmetric properties (1.2), the rank of S_{hijk} can be defined. The present authors treated it in the previous paper [5].

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Asahikawa College
Hokkaido Univ. of Education
Asahikawa, Japan

Department of Mathematics I
Polytechnic Institute
Bucharest, Romania

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