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RIEMANNIAN P-MANIFOLDS OF CONSTANT SECTIONAL CURVATURES

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ABSTRACT. An essential problem in the differential geometry of manifolds with metric structures is the investigation of manifolds of constant sectional curvature. In the present paper we study Riemannian P-manifolds of constant sectional curvature of a special kind.

1. Preliminary remarks. The Riemannian almost product manifolds are introduced in [1] and basic classes of such manifolds have been given in [2]. The most important of these classes is the class of Riemannian locally product manifolds with an almost product structure P such that $\nabla P = 0$. We consider manifolds of this class for which $\mathrm{tr}P = 0$. In [5] they have been called Riemannian P-manifolds.

Let (M, g, P), $\dim M = 2n$ be a Riemannian P-manifolds with a Riemannian metric g and an almost product structure P, i.e.

$$g(Px, Py) = g(x, y),\ P^2 = I_{2n},\ \mathrm{tr}P = 0,\ \nabla P = 0, \tag{1}$$

for arbitrary $x,\ y \in \mathcal{X}(\mathcal{M})$, where $\mathcal{X}(\mathcal{M})$ is the algebra of all differentiable vector fields on M and ∇ is the Levi-Civita connection of g. The associated metric $\tilde{g}$ is given by $\tilde{g}(x, y) = g(x, Py)$ and it is necessarily of signature (n, n). The curvature tensor R of g is defined by $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z$ for all $x, y, z \in \mathcal{X}(\mathcal{M})$. The curvature tensor R of type $(0, 4)$ is defined by $R(x, y, z, u) = g(R(x, y)z, u)$ for all x, y, u in the tangent space T_pM, $p \in M$, and it has the property

$$R(x, y, z, u) = R(x, y, Pz, Pu),\ x, y, z, u \in T_pM,\ p \in M. \tag{2}$$

This property implies that the tensor $\tilde{R}$ defined by $\tilde{R}(x, y, z, u) = R(x, y, z, Pu)$ has the property $\tilde{R}(x, y, z, u) = -\tilde{R}(x, y, z, Pu)$.

In the theory of Riemannian P-manifolds the following tensors are essential:

$$\begin{aligned}
\pi_1(x, y, z, u) &= g(y, z)g(x, u) - g(x, z)g(y, u),\\
\pi_2(x, y, z, u) &= \tilde{g}(y, z)\tilde{g}(x, u) - \tilde{g}(x, z)\tilde{g}(y, z),\\
\pi_3(x, y, z, u) &= g(y, z)\tilde{g}(x, u) - g(x, z)\tilde{g}(y, z),\\
&+ \ \tilde{g}(y, z)g(x, u) - \tilde{g}(x, z)g(y, z).
\end{aligned}$$

Every 2-plane (section) in T_pM, $p \in M$, has two sectional curvatures $k(\alpha)$ and $\tilde{k}(\alpha)$.

$$\begin{aligned}
k(\alpha; p) &= R(x, y, z, u)/\pi_1(x, y, z, u),\\
\tilde{k}(\alpha; p) &= \tilde{R}(x, y, z, u)/\pi_1(x, y, z, u),
\end{aligned}$$

where $\{x, y\}$ is a basis of α.

Let T_pM be the tangent space of M at an arbitrary point P in M. Then

$$T_pM = (T_pM)^n \oplus (T_pM)^v,$$

where $(T_pM)^v$ and $(T_pM)^n$ are the eigen-spaces corresponding to the eigen-values ± 1 of the structure P. A classification of the sections in T_pM has been given in [4]. A section α in T_pM is said to be invariant (antiinvariant) if $P\alpha = \alpha$ ($P\alpha \perp \alpha$). We consider the following types of invariant sections:

1) containing a vertical and a horizontal vector;
2) belonging to $(T_pM)^v$ (vertical sections);
3) belonging to $(T_pM)^n$ (horizontal sections).

We consider the following types of antiinvariant sections:

4) containing a vertical vector;
5) containing a horizontal vector;
6) containing neither vertical nor horizontal vectors (totally real sections).

For the curvatures of these sections we have [5]:

1. Any section of type 1) has $k = \tilde{k} = 0$;
2. Any section of type 2) or 4) has $k = \tilde{k}$. In this case k is called vertical sectional curvature;
3. Any section of type 3) or 5) has $k = -\tilde{k}$. In this case k is called horizontal sectional curvature;
4. The curvature k and $\tilde{k}$ of a section of type 6) are called totally real sectional curvatures.

Theorem A [4]. *Any Riemannian P-manifold M* (dim$M = 2n \geq 6$) *is of constant totally real sectional curvatures* $k(\alpha;p) = \nu$, $\tilde{k}(\alpha;p) = \tilde{\nu}$ *iff* $R = \nu(\pi_1 + \pi_2) + \tilde{\nu}\pi_3$.

Theorem B [4]. *Any Riemannian P-manifold M* (dim$M = 2n \geq 6$) *is of constant vertical sectional curvature* $k(\alpha;p) = \mu$ *iff* $R + \tilde{R} = \mu(\pi_1 + \pi_2 - \pi_3)$.

Theorem B′ [4]. *Any Riemannian P-manifold M* (dim$M = 2n \geq 6$) *is of constant horizontal sectional curvature* $k(\alpha;p) = \lambda$ *iff* $R - \tilde{R} = \lambda(\pi_1 + \pi_2 - \pi_3)$.

Theorem C [4]. *Any Riemannian P-manifold is of constant vertical sectional curvature* μ *and of constant horizontal sectional curvature* λ *iff the manifold is of constant totally real sectional curvature* ν *and* $\tilde{\nu}$, *where*

$$\nu = (\mu + \lambda)/2, \quad \tilde{\nu} = (\mu - \lambda)/2.$$

Riemannian P-manifolds of constant totally real sectional curvatures were investigated in [5].

Theorem D [5]. *If M is a connected invariant umbilic hypersurface of* (R^{2n+2}, g, P), $(2n+2) \geq 8$), *then M lies on an invariant sphere.*

Now we consider Riemannian P-manifolds of constant vertical (horizontal) sectional curvature.

2. Vertically umbilic and horizontally umbilic hypersurfaces of a Riemannian P-manifold. Let (M', g, P) (dim$M' = 2n+2$) be a Riemannian P-manifold and ∇', R' be the

Levi-Civita connection and its curvature tensor, respectively. A submanifold M ($\dim M = 2n$) is said to be a P-invariant hypersurface of M' if the restriction of g on M has the maximal rank and $P(T_pM) = T_pM$, $p \in M$. We denote the restrictions of g and P on M by the same letters. Then (M, g, P) is also a Riemannian P-manifold [5]. There exist locally vector fields N and PN, normal to M such that

$$g(N,N) = g(PN,PN) = 1, \; g(N,PN) = 0. \tag{3}$$

As in [5] by $A = A_N$ we denote the second fundamental tensor on M, with respect to the vector field N. Using Theorem B, Lemma 1 [5] and Lemma 2 [5] we obtain immediately

Lemma 1. *If M' is a Riemannian P-manifold of constant vertical sectional curvature μ', then*

$$\begin{aligned}\rho(y,z) + \tilde{\rho}(y,z) = {} & 2(n-1)\mu[g(y,z) + \tilde{g}(y,z)] \\ & +[\mathrm{tr}A + \mathrm{tr}(A \circ P)][g(Ay,z) + \tilde{g}(Ay,z)] - 2[g(A^2y,z) + \tilde{g}(A^2y,z)],\end{aligned} \tag{4}$$

where ρ and $\tilde{\rho}$ are the Ricci tensors for R and $\tilde{R}$ respectively.

Similarly we have

Lemma 1′. *If M' is a Riemannian P-manifold of constant horizontal sectional curvature λ', then*

$$\begin{aligned}\rho(y,z) + \tilde{\rho}(y,z) = {} & 2(n-1)\lambda'[g(y,z) + \tilde{g}(y,z)] \\ & +[\mathrm{tr}A - \mathrm{tr}(A \circ P)][g(Ay,z) - \tilde{g}(Ay,z)] - 2[g(A^2y,z) + \tilde{g}(A^2y,z)].\end{aligned}$$

Let H be the mean curvature vector on M. Then we have

$$H = \mathrm{tr}\sigma/2n = [\mathrm{tr}AN + \mathrm{tr}(A \circ P)PN]/2n, \tag{5}$$

where σ is the second fundamental form on M.

Definition 1. *A P-invariant hypersurface M is said to be vertically umbilic if in every point on M the equality*

$$\sigma + P\sigma = [\mathrm{tr}\sigma + \mathrm{tr}(P \circ \sigma)](g + \tilde{g})/2n = (H + PH)(g + \tilde{g}) \tag{6}$$

holds.

Definition 2. *A P-invariant hypersurface M is said to be horizontally umbilic if in every point on M the equality*

$$\sigma - P\sigma = [\mathrm{tr}\sigma - \mathrm{tr}(P \circ \sigma)](g - \tilde{g})/2n = (H - PH)(g - \tilde{g}) \tag{6'}$$

holds.

From these definitions it follows that M is simultaneously vertically and horizontally umbilic, iff

$$\sigma = [\mathrm{tr}\sigma g + \mathrm{tr}(\sigma \circ P)\tilde{g}]/2n = Hg + PH\tilde{g}.$$

In this case, M is said to be P-invariant umbilic [5].

Lemma 2. *A P-invariant hypersurface M is vertically umbilic iff*

$$A + A \circ P = [\mathrm{tr}A + \mathrm{tr}(A \circ P)](I + P)/2n. \tag{7}$$

Lemma 2′. *A P-invariant hypersurface M is horizontally umbilic iff*

$$A - A \circ P = [\mathrm{tr}A - \mathrm{tr}(A \circ P)](I - P)/2n. \tag{7'}$$

Theorem 1. *Let M' ($\dim M' = 2n + 2 \geq 8$) be of constant vertical curvature μ'. If M is vertically umbilic, then M is of constant sectional curvature μ and*

$$\mu = \mu' + g(H, H) + \tilde{g}(H, H) \tag{8}$$

Proof. Using (7), we have for $x \in T_pM$

$$(A + A \circ P)x = c(x + Px), \ c = [\mathrm{tr}A + \mathrm{tr}(A \circ P)]/2n \tag{9}$$

Applying Theorem B for M' we have $R' + \tilde{R}' = \mu'(\pi_1 + \pi_2 + \pi_3)$. On the other hand, according to Lemma 1, we have

$$R'(x, y, z, u) = r(x, y, z, u) - (pi_1 + pi_2)(Ax, Ay, z, u).$$

From (2) we obtain $R + \tilde{R} = (\mu' + c^2)(\pi_1 + \pi_2 + \pi_3)$ and Theorem B implies that M is of constant vertical sectional curvature $\mu = \mu' + c^2$. Now (8) follows form (5) and (9).

Analogously we have

Theorem 1′. *Let M' ($\dim M = 2n + 2 \geq 8$) be of constant horizontal curvature λ'. If M is vertically umbilic, then M is of constant horizontal sectional curvature λ and*

$$\lambda = \lambda' + g(H, H) + \tilde{g}(H, H). \tag{8'}$$

Corollary 1. *Let M' ($\dim M' = 2n+2 \geq 8$) be of constant vertical sectional curvature. If M is connected and vertically umbilic, then we have on M:*

1) $\mathrm{tr}A + \mathrm{tr}(A \circ P) = \mathrm{const}$;

2) $g(H, H) + \tilde{g}(H, H) = \mathrm{const}$;

Corollary 1′. *Let M' ($\dim M' = 2n + 2 \geq 8$) be of constant horizontal sectional curvature. If M is connected and vertically umbilic, then we have on M:*

1) $\mathrm{tr}A + \mathrm{tr}(A \circ P) = \mathrm{const}$;

2) $g(H, H) + \tilde{g}(H, H) = \mathrm{const}$;

3. Examples of Riemannian P-manifolds of constant vertically and horizontally curvatures. Let $M' = R^{2n+2}$ be equipped with the canonical structure P and the metric g, given in [4]. Then (M', g, P) is a Riemannian P-manifold. In this case, the curvature tensor R' of M' is zero. We identify an arbitrary point $z = (x^1, \ldots, x^{n+1}; y^1, \ldots, y^{n+1})$ in M' with the position vector Z.

Definition 2. *Let $a \in R$ ($a > 0$) and $z_0 \in R^{2n+2}$. Every P-invariant hypersurface $S^v(z_0; a)$ or R^{2n+2} for which*

$$g(z - z_0, z - z_0) + \tilde{g}(z - z_0, z - z_0) = a \tag{10}$$

is said to be vertical sphere with a center z_0 and parameter a.

Definition 2'. *Let $b \in R$ $(b > 0)$ and $z_0 \in R^{2n+2}$. Every P-invariant hypersurface $S^h(z_0; b)$ or R^{2n+2} for which*

$$g(z - z_0, z - z_0) + \tilde{g}(z - z_0, z - z_0) = b \tag{10'}$$

is said to be horizontal sphere with a center z_0 and parameter b.

Obviously, there exist an infinite number of vertical (respectively horizontal) spheres with given center and parameter. It is clear that a P-invariant hypersurface of R^{2n+2} is a vertical sphere $S^v(z_0; a)$ and a horizontal sphere $S^h(z_0; a)$ in the same iff is the P-invariant sphere $S^{2n}(z_0; (a+b)/2, (a-b)/2)$ [5].

Let N and PN be vector fields normal to $S^v(z_0; a)$ satisfying (3). Since ∇' is flat, then from (10) we obtain that $z + Pz$ is a vector normal to $S^v(z_0; a)$. Then there exists a real l such that

$$N + PN = l(z + Pz), \; l^2 = 1/a. \tag{11}$$

It is known [5] that

$$A_{PN} = P \circ A = A \circ P, \; \nabla'_x N = -Ax, \; x \in T_z S^v \tag{12}$$

Taking into account (11) and (12), we have $-(A + A \circ P)x = \nabla'_x(N + PN) = l(\nabla'_x z + P\nabla'_x z)$. Since ∇' is flat, then $\nabla'_x z = x$, where z is the position vector field. Therefore we obtain

$$A + A \circ P = -l(I + P). \tag{13}$$

Applying Lemma 1, Theorem 1, and the equalities (11), (13), we obtain

Theorem 2. *Every vertical sphere $S^v(z_0; a)$ in R^{2n+2} $(2n + 2 \geq 6)$ is a vertically umbilic hypersurface of constant vertical sectional curvature $\mu = 1/a$.*

In a similar way we obtain

Theorem 2'. *Every vertical sphere $S^h(z_0; a)$ in R^{2n+2} $(2n + 2 \geq 6)$ is a horizontally umbilic hypersurface of constant horizontal sectional curvature $\lambda = 1/b$.*

Thus, we obtain P-invariant hypersurfaces of M with prescribed constant vertical (horizontal) sectional curvatures.

Theorem 3. *Let μ $(\mu > 0)$ be a real number. Then $S^v(z_0; 1/\mu)$ is a vertical sphere in R^{2n+2} $(n \geq 2)$ of constant vertical sectional curvature μ.*

Theorem 3'. *Let λ $(\lambda > 0)$ be a real number. Then $S^h(z_0; 1/\lambda)$ is a horizontal sphere in R^{2n+2} $(n \geq 2)$ of constant horizontal sectional curvature λ.*

4. A classification of P-invariant hypersurfaces of constant vertical and horizontal sectional curvature in R^{2n+2}. In the previous section we proved that the vertical (horizontal) spheres have constant vertical (horizontal) sectional curvatures. In this section, we consider the inverse question.

Theorem 4. *Let M' $(\dim M' = 2n + 2 \geq 8)$ be a Riemannian P-manifold of constant vertical sectional curvature μ'. If M is a P-invariant hypersurface of M' with constant vertical sectional curvature μ, then M is vertically umbilic.*

Proof. Since M' and M are of constant vertical sectional curvatures, Theorem B and Lemma 1 imply

$$(14)\qquad 2(n-1)(\mu-\mu')(y+Py) = [\mathrm{tr}A + \mathrm{tr}(A\circ P)](Ay + A\circ Py) - 2(A^2y + A^2\circ Py)$$

for arbitrary $y \in T_pM,\ p \in M$.

Applying Lemma A [5] to the invariant - symmetric operator A, we get

$$(15)\qquad Al_i = \lambda_i l_i + \mu_i Pl_i \ (\lambda_i^2 - \mu_i^2 \neq 0,\ i = 1, 2, \dots, n),$$

where $\{l_1, l_2, \dots, l_n; Pl_1, Pl_2, \dots, Pl_n\}$ is an adapted basis for T_pM, $P \in M$ of invariant eigen vectors of A. Using (14) and (15) we find

$$\begin{aligned}\mathrm{tr}A + \mathrm{tr}(A\circ P) &= 2(n-1)(\lambda_j + \mu_j) + 2(\lambda_k + \mu_k)\\ &= 2(n-1)(\lambda_k + \mu_k) + 2(\lambda_j + \mu_j),\ j \neq k.\end{aligned}$$

Since $n \geq 3$, these equalities imply $\lambda_j + \mu_j = m = const$ for all $j = 1, \dots, n$. Hence

$$(16)\qquad A + A\circ P = m(I + P),\ m = [\mathrm{tr}A + \mathrm{tr}(A\circ P)]/2n$$

because of (15). Lemma 2 implies, that M is vertically umbilic.

In a similar way we obtain

Theorem 4'. *Let M' ($\dim M' = 2n+2 \geq 8$) be a Riemannian P-manifold of constant horizontal sectional curvature λ'. If M is a P-invariant hypersurface of M' with constant horizontal sectional curvature λ, then M is horizontally umbilic.*

Applying Lemma 2 and Lemma 3, we obtain following propositions.

Corollary 3. *Let M be a P-invariant hypersurface of R^{2n+2} ($2n+2 \geq 8$) with constant vertical sectional curvature μ. Then M is vertically umbilic and*

$$\mu = g(H,H) + \tilde{g}(H,H).$$

Corollary 3'. *Let M be a P-invariant hypersurface of R^{2n+2} ($2n+2 \geq 8$) with constant horizontal sectional curvature λ. Then M is horizontally umbilic and*

$$\lambda = g(H,H) + \tilde{g}(H,H).$$

Theorem 5. *Let M be a connected vertically umbilic hypersurface in R^{2n+2} ($2n+2 \geq 8$). Then M lies on a vertical sphere.*

Proof. Let U be a coordinate neighbourhood on M and $\{N, PN\}$ be normal vector fields on U, satisfying (3). From the condition of the theorem, we have $A + A\circ P = q(I+P)$ on U, where $q = [\mathrm{tr}A + \mathrm{tr}(A\circ P)]/2n$. Theorem 1 implies that M is of constant vertical sectional curvature $\mu = q^2$. Hence, $q = \mathrm{const}$ on U.

Identifying $z \in U$ with the position vector $Z \in R^{2n+2}$, we consider the vector field $N + PN + q(Z + PZ)$ on u. Using the Weingarten formula (12) and taking into account that ∇' is flat, we obtain

$$\nabla'_x[N + PN + q(z + Pz)] = -q(I+P)x + q(I+P)x = 0$$

for an arbitrary vector field x on U, tangent to M. The last equality implies

$$N + PN + q(z + Pz) = q(z_0 + Pz_0), \; z_0 = \text{const}. \tag{17}$$

If $q \neq 0$ from (17) we find

$$z - z_0 + P(z - z_0) = -(N + PN)/q.$$

Thus, for every $z \in U$ we have

$$g(z - z_0, z - z_0) + \tilde{g}(z - z_0, z - z_0) = 1/q^2.$$

Hence, U lies on a vertical sphere $S^v(z_0; 1/q^2)$ with center z_0 and parameter $1/q^2$. So we proved that for every $z \in M$, there exist $U \ni z$, such that U lies on a vertical sphere S^v.

Let U be a fixed coordinate neighbourhood and S^v be the vertical sphere, such that $U \in S^v$. By M_0 we shall denote the set of points z in M belonging to S^v together with a neighbourhood $V_z \ni z$. Obviously, $M_0 \neq \emptyset$ and M_0 is open.

Let W be in the closure $\bar{M}_0$ of M_0. Then, there exists $W \ni w$ and W lies on a vertical sphere S^v_1. The open set $W \cap M_0 \neq \emptyset$ lies on S^v_1 and S^v. Hence, $S^v_1 = S^v$ and $M_0 = M$ because of the connectedness of M.

In a similar way we obtain

Theorem 5′. *Let M be a connected horizontally umbilic hypersurface in R^{2n+2} $(2n + 2 \geq 8)$. Then M lies on a horizontal sphere.*

Using theorems 4, 4′, 5 and 5′, we obtain the following classification theorems.

Theorem 6. *Let M be a connected invariant hypersurface of (R^{2n+2}, g, P) $(2n + 2 \geq 8)$. If M is of constant vertical sectional curvature $\mu > 0$, then M lies on a vertical sphere $S^v(z_0; 1/\mu)$.*

Theorem 6′. *Let M be a connected invariant hypersurface of (R^{2n+2}, g, P) $(2n + 2 \geq 8)$. If M is of constant horizontal sectional curvature $\lambda > 0$, then M lies on a horizontal sphere $S^v(z_0; 1/\lambda)$.*

From Theorem 6, Theorem 6′ and Theorem D we obtain

Theorem 7. *Let M be a connected invariant hypersurface of (R^{2n+2}, g, P) $(2n+2 \geq 8)$. If M is of constant vertical and horizontal sectional curvatures μ and λ respectively $(\lambda > 0, \; \mu > 0)$, then M lies on an invariant sphere $S^{2n}(z_0; (\lambda + \mu)/\lambda\mu, (\lambda - \mu)/\lambda\mu)$.*

REFERENCES

[1] K. YANO. Differential geometry of complex and almost complex spaces. New York. 1965.

[2] A.M.NAVIERA. A classification of Riemannian almost-product manifolds. *Rendiconti di Mat. di Roma (7)* **3** (1983) 577–592.

[3] Г. ГАНЧЕВ, К. ГРИБАЧЕВ, В. МИХОВА. Холоморфни хиперповърхнини на келерови многообразия с норденова метрика. *Научни тр. Пловдивски унив.* **23** *2* (1985) 221–237.

[4] М. СТАЙКОВА. Свойства на тензора на кривина на риманови P-многообразия. *Научни тр. Пловдивски унив.* **25** *3* (1987) 241–251.

[5] М. Стайкова, К. Грибачев, Д. Мекеров. Инвариантни хиперповърхнини на римanови P-многообразия. *Научни тр. Пловдивски унив.* **25** *3* (1987) 254–266.

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