Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SERDICA — Bulgaricae mathematicae publicationes
17 (1991) 243-247

A NOTE ON N–FUNDAMENTAL TENSORS OF A HYPERSURFACE OF A RIEMANNIAN SPACE

S. C. RASTOGI

ABSTRACT. The coefficients of the r-th fundamental form $C_{(r)\alpha\beta}$ of a hypersurface of a Riemannian space were defined and studied by Rund (1971). The purpose of this note is to define the r-th normal curvature and to study some of its properties when the riemannian space of n-dimensions reduces to Euclidean space of n-dimensions.

1. Introduction. Let V_n be an n-dimensional Riemannian space with coordinates x^i, $i = 1, \ldots, n$, metric tensor $g_{ij}(x)$ and curvature tensor R_{hijk}. Let V_{n-1} be an n-dimensional Riemannian space with coordinates U^α, $\alpha = 1, \ldots, n-1$,metric tensor $g_{\alpha\beta}(u)$ and curvature tensor $R_{\alpha\beta\gamma\epsilon}$ and let it be imbedded in V_n so that [1]

$$(1) \qquad g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta \qquad B^i_\alpha = \partial x^i / \partial u^\alpha \, .$$

Let N^i be a unit vector normal to V_{n-1}, then we have

$$(2) \qquad \varepsilon(N) g_{ij} N^i N^j = 1,$$

$$(3) \qquad N_i B^i_\alpha = 0$$

$$(4) \qquad B^i_\alpha B^\beta_i = \delta^\beta_\alpha$$

$$(5) \qquad B^i_\alpha B^\beta_j = \delta^i_j - N^i N_j \, ,$$

where $\varepsilon(N)$ is the indicator of N (which we assume to be 1).

Let $\Omega_{\alpha\beta}$ be the second fundamental tensor of V_{n-1}, then the coefficients of r-th fundamental tensor $C_{(r)\alpha\beta}$ are expressed [2] through

$$(6) \qquad C_{(r)\alpha\beta} = g^{\epsilon\gamma} \Omega_{\alpha\epsilon} C_{(r-1)\gamma\beta}, \qquad r = 2, \ldots, n \, ,$$

where $C_{(1)\alpha\beta} = g_{\alpha\beta}$ and $C_{(2)\alpha\beta} = \Omega_{\alpha\beta}$.

2. The r-th normal curvature. Let $K(u, \acute{u})$ be the normal curvature of V_{n-1} at the point P, in the direction of the unit tangent vector $\acute{u}^\epsilon$. Ten $K(u, \acute{u}) = \Omega_{\alpha\beta} \acute{u}^\alpha \acute{u}^\beta$. If we replace $\acute{u}^\epsilon$, by an arbitrary vector $\dot{u}^\epsilon$, tangent to V_{n_1} at P, then we have

$$(7) \qquad K(u, \dot{u}) = \frac{\Omega_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta}{g_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta} = \frac{\Omega_{\alpha\beta} du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta} \, .$$

Definition 2.1. *The r-th normal curvature in the direction of an arbitrary vector $\dot{u}^\epsilon$ tangent to V_{n-1} at a point P is defined by*

$$K_{(r)}(u,\dot{u}) = \frac{C_{(r)\alpha\beta}\dot{u}^\alpha\dot{u}^\beta}{g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta} \tag{8}$$

From (8) it follows that $K_{(2)}(u,\dot{u}) = K(u,\dot{u})$. For $r=3$, we obtain

$$C_{(3)\alpha\beta}\dot{u}^\alpha\dot{u}^\beta = g^{\epsilon\gamma}\Omega_{\alpha\epsilon}\dot{u}^\alpha\dot{u}^\beta \ . \tag{9}$$

Now using the equation of Gauss [1] in a hypersurface of Riemannian space

$$R_{\alpha\beta\gamma\epsilon} = R_{ijhk}B^{ijhk}_{\alpha\beta\gamma\epsilon} + (\Omega_{\alpha\gamma}\Omega_{\beta\epsilon} - \Omega_{\alpha\epsilon}\Omega_{\beta\gamma}) \tag{10}$$

together with $\dot{x}^i = B^i_\alpha\dot{u}^\alpha$ and $M = g^{\alpha\beta}\Omega_{\alpha\beta}$, we obtain

$$C_{(3)\alpha\beta}\dot{u}^\alpha\dot{u}^\beta = M\Omega_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta + R_{ih}\dot{x}^i\dot{x}^h - R_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta - R_{ijhk}\dot{x}^i\dot{x}^hN^jN^k \ . \tag{11}$$

The equation (11) by virtue of (8) gives

$$K_{(3)}(u,\dot{u}) = MK(u,\dot{u}) + (g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta)^{-1}\left[R_{ih}\dot{x}^i\dot{x}^h - R_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta - R_{ijhk}\dot{x}^i\dot{x}^hN^jN^k\right] \tag{12}$$

equation (12) for $\dot{u}^\epsilon = \acute{u}^\epsilon$ reduces to

$$K_{(3)}(u,\dot{u}) = MK + R_{ih}\acute{x}^i\dot{x}^h - R_{\alpha\beta}\acute{u}^\alpha\dot{u}^\beta - R_{ijhk}\acute{x}^i\dot{x}^hN^jN^k \ . \tag{13}$$

Similar to (12) and (13) for $r=4$, we can obtain

$$\begin{aligned} K_{(3)}(u,\dot{u}) = {} & MK_{(3)}(u,\dot{u}) + (g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta)^{-1}(\Omega_{\alpha\epsilon}\dot{u}^\alpha) \\ & \times\left[R_{ik}B^i_\gamma g^{\epsilon\gamma}\dot{x}^k - R_{\gamma\beta}g^{\epsilon\gamma}\dot{u}^\beta - R_{ijhk}\dot{x}^kg^{\epsilon\gamma}B^l_\gamma N^jN^k\right] \end{aligned} \tag{14}$$

and

$$K_{(4)} = MK_{(3)} + \Omega_{\alpha\epsilon}\acute{u}^\alpha\left[R_{ih}B^i_\gamma g^{\epsilon\gamma}\acute{x}^h - R_{\gamma\beta}g^{\epsilon\gamma}\acute{u}^\beta - R_{ijhk}g^{\epsilon\gamma}B^i_\gamma N^jN^h\acute{x}^k\right] . \tag{15}$$

By an easy calculation it can be proved that

$$C_{(r)\alpha\beta} = g^{\theta\phi}C_{(r-2)\alpha\theta}C_{(3)\alpha\theta}C_{(3)\phi\beta}, \qquad r \geq 3, \tag{16}$$

which gives

$$\begin{aligned} C_{(r)\alpha\beta}\dot{u}^\alpha\dot{u}^\beta = {} & MC_{(r-1)\alpha\beta}\dot{u}^\alpha\dot{u}^\beta \\ & -C_{(r-2)\alpha\theta}\dot{u}^\alpha\dot{u}^\beta R^\theta_\beta + R_{ih}B^i_\phi\dot{x}^hC^\phi_{(r-2)\alpha}\dot{u}^\alpha - R_{ijhk}C^\phi_{(r-2)\alpha}B^i_\phi\dot{x}^hN^jN^k\dot{u}\alpha \ . \end{aligned} \tag{17}$$

From (17) we obtain

$$\begin{aligned} K_{(r)}(u,\dot{u}) = {} & MK_{(r-1)} - (g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta)^{-1}\Big[C_{(r-2)\alpha\theta}\dot{u}^\alpha\dot{u}^\beta R^\theta_\beta \\ & +R_{ih}C^\phi_{(r-2)\alpha}\dot{u}^\alpha B^i_\phi\dot{x}^h - R_{ijhk}C^\phi_{(r-2)\alpha}B^i_\phi x^hN^jN^k\dot{u}^\alpha\Big] \end{aligned} \tag{18}$$

and

$$K_{(r)} = MK_{(r-1)} - \Big[C_{(r-2)\alpha\theta}R^{\theta}_{\beta}\acute{u}^{\alpha}\acute{u}^{\beta} + R_{ih}C^{\phi}_{(r-2)\alpha}\acute{u}^{\alpha}B^{i}_{\phi}\acute{x}^{h} - R_{ijhk}C^{\phi}_{(r-2)\alpha}\acute{u}^{\alpha}B^{i}_{\phi}\acute{x}^{h}N^{j}N^{k}\Big] . \tag{19}$$

Hence we have for $r \geq 3$

Theorem 2.1. *The r-th normal curvature in the direction of the unit vector $\acute{u}^{\alpha}$ is expressed by* (19).

3. Codazzi equations. It is well known [1] that the Codazzi equation for a hypersurface V_{n-1} of V_n is given by

$$\Omega_{\alpha\beta\|\gamma} - \Omega_{\alpha\gamma\|\beta} = N_j R^{j}_{ihk} B^{i}_{\alpha} B^{h}_{\beta} B^{k}_{\gamma} , \tag{20}$$

where $\|\gamma$ means a covariant derivative.

With the help of (20) we obtain

$$C_{(3)\alpha\beta\|\gamma} - C_{(3)\alpha\gamma\|\beta} = \Omega_{\alpha\epsilon\|\gamma}\Omega^{\epsilon}_{\beta} - \Omega_{\alpha\epsilon\|\beta}\Omega^{\epsilon}_{\gamma} + \Omega^{\epsilon}_{\alpha}N_j R^{j}_{ihk}B^{i}_{\epsilon}B^{h}_{\beta}B^{k}_{\gamma} . \tag{21}$$

By a similar calculation we get

$$C_{(4)\alpha\beta\|\gamma} - C_{(4)\alpha\gamma\|\beta} = C^{\theta}_{(3)\beta}\Omega_{\alpha\theta\|\gamma} - C^{\theta}_{(3)\gamma}\Omega_{\alpha\theta\|\beta} + \Omega^{\epsilon}_{\alpha}\Big[\Omega_{\epsilon\theta\|\gamma}\Omega^{\theta}_{\beta} - \Omega^{\theta}_{\gamma}\Omega_{\epsilon\theta\|\beta} + \Omega^{\theta}_{\epsilon}N_j R^{j}_{ihk}B^{i}_{\theta}B^{h}_{\beta}B^{k}_{\gamma}\Big] . \tag{22}$$

Using equation (16) together with (21) we can obtain

$$C_{(r)\alpha\beta\|\gamma} - C_{(r)\alpha\gamma\|\beta} = C^{\theta}_{(3)\beta}C_{(r-2)\alpha\theta\|\gamma} - C^{\theta}_{(3)\gamma}C_{(r-2)\alpha\theta\|\beta} + C^{\phi}_{(r-2)\alpha} \cdot\Big[\Omega_{\phi\epsilon\|\gamma}\Omega^{\epsilon}_{\beta} - \Omega_{\phi\epsilon\|\beta}\Omega^{\epsilon}_{\gamma} + \Omega^{\epsilon}_{\phi}N_j R^{j}_{ihk}B^{i}_{\epsilon}B^{h}_{\beta}B^{k}_{\gamma}\Big] , \tag{23}$$

which is an equation analogous to the Codazzi equation for $r \geq 3$.

4. Some special Cases.

Case I. Space of Constant curvature. If V_n is a space of constant curvature R_{ijhk} is expressed [1] by

$$R_{ijhk} = k(x)(g_{ih}g_{jk} - g_{ik}g_{jh}) , \tag{24}$$

which gives

$$R_{ih} = k(x)(n-1)(g_{ih} . \tag{25}$$

From equation (9) and (24) we can easily obtain

$$R^{\theta}_{\alpha} = k(x)(n-2)\delta^{\theta}_{\alpha} + (M\Omega^{\theta}_{\alpha} - C^{\theta}_{(3)\alpha}) . \tag{26}$$

Substituting from (24), (25) and (26) in (19) we obtain after simplification

(27) $$K_{(r)} = MK_{(r-1)} - 2(n-2)k(x)K_{(r-2)} - C_{(r-2)\alpha\theta}\acute{u}^{\alpha}\acute{u}^{\beta}(M\delta^{\epsilon}_{\beta} - \Omega^{\epsilon}_{\beta})\Omega^{\theta}_{\epsilon} ,$$

which gives

Theorem 4.1. *The r-th normal curvature in the direction of the unit tangent vector $\acute{u}\alpha$ for a hypersurface V_{n-1} of a space V_n of constant curvature is given by* (27).

Substituting in (23) from (24) we obtain

(28) $$\begin{aligned} &C_{(r)\alpha\beta\|\gamma} - C_{(r)\alpha\gamma\|\beta} \\ &= C^{\theta}_{(3)\beta}C_{(r-2)\alpha\theta\|\gamma} - C^{\theta}_{(3)\gamma}C_{(r-2)\alpha\theta\|\beta} \\ &+C^{\phi}_{(r-2)\alpha}(\Omega_{\phi\epsilon\|\gamma}\Omega^{\epsilon}_{\beta} - \Omega_{\phi\epsilon\|\beta}\Omega^{\epsilon}_{\gamma}) , \end{aligned}$$

which is an equation analogous to the Codazzi equation in a hypersurface V_{n-1} of a space of constant curvature V_n.

Case II. V_n is E_n. In this equation (10) gives

(29) $$R_{\alpha\beta\gamma\epsilon} = \Omega_{\alpha\gamma}\Omega_{\beta\epsilon} - \Omega_{\alpha\epsilon}\Omega_{\beta\gamma} ,$$

where (19) gives

(30) $$K_{(r)} = MK_{(r-1)} - C_{(r-2)\alpha\theta}R^{\theta}_{\beta}\acute{u}^{\alpha}\acute{u}\beta .$$

From (29) and (30) we can easy obtain

(31) $$K_{(r)} = MK_{(r-1)} - C_{(r-2)\alpha\theta}\acute{u}^{\alpha}\acute{u}\beta(M\Omega^{\theta}_{\beta} - \Omega^{\gamma}_{\beta}\Omega^{\theta}_{\gamma}) ,$$

which implies

Theorem 4.2. *The n-th normal curvature $K_{(r)}$ of a Riemannian space V_{n-1} imbedded in the Euclidean space E_n is given by* (31).

Remark. In this case equation (23) again reduces to (28).

Case III. Both V_n and V_{n-1} are spaces of constant curvature. In this case the equation (19) reduces to

(32) $$K_{(r)} = MK_{(r-1)} - (n-2)(k(u) + k(x))K_{(r-2)}$$

where $k(u)$ is a term in V_{n-1} similar to $k(x)$.

Substituting the value of $k(u)$ in (32) we get

(33) $$K_{(r)} = MK_{(r-1)} - (n-2)\left[2k(u) + (M^2 - \Omega^{\beta}_{\alpha}\Omega^{\alpha}_{\beta})/(n-1)(n-2)\right]K_{(r-2)} .$$

Hence we have

Theorem 4.3. *The recurrence relation for the r-th normal curvature of a hypersurface of constant curvature imbedded in a space of constant curvature is given by* (33).

If $k(u) = k(x)$, i.e., if $\Omega_{\alpha\gamma}\Omega_{\beta\epsilon} = \Omega_{\alpha\epsilon}\Omega_{\beta\gamma}$, then (33) reduces to

(34) $$K_{(r)} = MK_{(r-1)} - 2(n-2)k(x) .$$

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Sanjay Gandhi Post-Graduate
Institute of Medical Sciences
P. O. Box 375
RAEBARELI ROAD
LUCKNOW -226001, INDIA

Received 16.01.1991