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AN EXACT ESTIMATE FOR APPROXIMATION OF A CLASS OF CONVEX FUNCTIONS BY BASKAKOV OPERATORS IN L_1

V.A.KOSTOVA

ABSTRACT. The approximation of convex functions by polynomials and polygons in $L_1[a,b]$-metric is investigated in Ivanov (1979). In this paper an exact estimate for approximation of a class of bounded convex monotone decreasing function by Baskakov operators in $L_1(0,\infty]$ is obtained.

An estimate of the approximation of convex function by Bernstein polynomials in $L_1[0,1]$ is given in [3] and [4]. Using the method from [4] we prove the following

Theorem. *If f is a nonnegative, continuous, convex, monotone decreasing function, defined on $[0,\infty]$: $f(0)=M$, $f(x)=0$ for $x\geq a$, $a\in(0,\infty)$ then for every $n\geq 2$ we have:*

$$\int_0^\infty \operatorname{mod}\{B_n(f;x)-f(x)\}\,dx \leq (5+a).M.\{2(n-1)\}^{-1}, \tag{1}$$

where $B_n(f)$ is the Baskakov operator:

$$B_n(f;x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{-n}{k}\frac{(-x)^k}{(1+x)^{n+k}},\quad x\in(0,\infty).$$

The estimate (1) is exact to the order.

First let us mention that the Baskakov operator can be represented in the form

$$B_n(f;x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(-x)^k\phi_n^{(k)}(x)}{k!}, \tag{2}$$

where $\phi_n(x)=1/(1+x)^n$. It is not difficult to show, that the function ϕ_n has the following properties: $(k=1,2,\ldots)$

(a) $$\int_0^\infty \phi_n(x)\,dx=\int_0^\infty (1+x)^{-n}\,dx=(n-1)^{-1}.$$

(b) $$\phi_n^{(k)}(x)=\prod_{i=1}^{k}(-n-k+i)(1+x)^{-n-k}=-n\,\phi_{n+1}^{(k-1)}(x).$$

(c) $\quad (-1)^k \phi_n^{(k)}(x) \geq 0$ for every $x \in [0, \infty]$, $n \geq 2$

(d) $$\int_0^\infty \frac{(-x)^k \phi_n^{(k)}(x)}{k!}\, dx = \int_0^\infty \phi_n(x)\, dx = (n-1)^{-1}.$$

Lemma 1. *Let f be a convex, monotone decreasing function, defined on $[0, \infty]$. Then the following is true:*

(A1) $\quad B_n(f)$ *is a monotone decreasing function on* $[0, \infty]$;

(B1) $\quad B_n(f)$ *is a convex function on* $[0, \infty]$;

(C1) $\quad B_n(g(\lambda); x) \geq g(\lambda; x)$ *for every* $x \in [0, \infty]$,

where $g(\lambda; \cdot)$, $0 < \lambda < a$, $a \in (0, \infty)$, *is defined as* $g(\lambda; x) = \max\{0, M(\lambda - x)/\lambda\}$, $x \in [0, \infty]$.

Proof. First we shall prove (A1). We have for the first derivative:

$$\frac{d\,B_n(f; x)}{dx} = -n \sum_{k=0}^{\infty} \left\{ f\left(\frac{k}{n}\right) - f\left(\frac{k+1}{n}\right) \right\} \frac{(-x)^k \phi_{n+1}^{(k)}(x)}{k!} \tag{3}$$

Further we take into account that f is a monotone decreasing function. Then from (3) follows $dB_n(f; x)/dx \leq 0$ for every $x \in [0, \infty]$. Hence $B_n(f)$ is a monotone decreasing function.

Now we shall prove (B1). We have for the second derivative:

$$\frac{d^2 B_n(f; x)}{dx^2} = n(n+1) \sum_{k=0}^{\infty} \left\{ f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right) \right\} \frac{(-x)^k \phi_{n+2}^{(k)}(x)}{k!}. \tag{4}$$

But the function f is a convex one and in view of (4) we have $d^2 B_n(f; x)/dx^2 \geq 0$ for $x \in [0, \infty]$. Hence $B_n(f)$ is a convex function on $[0, \infty]$.

Further we shall prove (C1). From the definition of the Baskakov operator (2) it is not difficult to see that $B_n(g(\lambda); 0) = M$, $B_n(g(\lambda); x) > 0$ for $x \in [\lambda, \infty]$. In the case $x \in (0, \lambda)$ we have: $dB_n(g(\lambda); x)/dx|_{x=0} = dg(\lambda; x)/dx|_{x=0} = -M/\lambda$. Then for the convex function $B_n(g(\lambda))$ it is true: $B_n(g(\lambda); x) \geq g(\lambda; x)$ for every $x \in [0, \infty]$.

Thus lemma 1 is proved.

Lemma 2. *Let* $g(\lambda; \cdot)$, $0 < \lambda < a$, $a \in (0, \infty)$, *be a convex, monotone decreasing function, defined as follows:*

$$g(\lambda; x) = \max\{0,\ M(\lambda - x)/\lambda\},\ x \in [0, \infty].$$

Then we have:

(A2) $\quad$ *If* $\lambda \in (0, a]$ *is arbitrary, then*

$$\int_0^\infty mod\{B_n(g(\lambda); x) - g(\lambda; x)\}\, dx \leq (2 + \lambda).M.\{2(n-1)\}^{-1}.$$

(B2) If $\lambda \in (0,a]$ is an integer, then

$$\int_0^\infty mod\{B_n(g(\lambda);x) - g(\lambda;x)\}\,dx = (1+\lambda).M.\{2(n-1)\}^{-1}.$$

P r o o f. Let us mention first, that

$$\int_0^\infty g(\lambda;x)\,dx = \int_0^\lambda g(\lambda;x)\,dx = M\lambda/2. \tag{5}$$

Further we use (d) and the definition of Baskakov operators and we get the following estimate:

$$\int_0^\infty B_n(g(\lambda);x)\,dx = \int_0^\infty \sum_{k=0}^\infty g\left(\lambda;\frac{k}{n}\right)\frac{(-x)^k\phi_n^{(k)}(x)}{k!}\,dx$$

$$= \int_0^\infty \sum_{k=0}^{[n\lambda]} \frac{M}{\lambda}\left(\lambda-\frac{k}{n}\right)\frac{(-x)^k\phi_n^{(k)}(x)}{k!}\,dx$$

$$\begin{aligned} &= \sum_{k=0}^{[n\lambda]} \frac{M}{\lambda}\left(\lambda-\frac{k}{n}\right)\int_0^\infty \phi_n(x)\,dx \\ &= \frac{M}{\lambda(n-1)}\left\{\lambda([n\lambda]+1) - \frac{[n\lambda]([n\lambda]+1)}{2n}\right\} \\ &= \frac{M}{\lambda(n-1)}\frac{n\lambda(n\lambda+1)+\theta(1-\theta)}{2n} \\ &= \frac{M\lambda}{2} + \frac{M(1+\lambda)}{2(n-1)} + \frac{M\theta(1-\theta)}{2\lambda(n-1)n}, \end{aligned} \tag{6}$$

where $\theta = n\lambda - [n\lambda]$. ($[x]$ is the largest integer $\leq x$.)

Now in view of (C1) from Lemma 1 and (6) we get:

$$\int_0^\infty \mathrm{mod}\{B_n(g(\lambda);x) - g(\lambda;x)\}\,dx = \frac{M(1+\lambda)}{2(n-1)} + \frac{M\theta(1-\theta)}{2\lambda(n-1)n} \tag{7}$$

But the result shows that the estimate of the approximation is increasing when $\lambda \to 0$. Let us consider $0 < \lambda < n^{-1}$. In this case $0 < \lambda n < 1$ and then $\theta = n\lambda - [n\lambda] = n\lambda$. Hence it is true

$$\int_0^\infty \mathrm{mod}\{B_n(g(\lambda);x) - g(\lambda;x)\}\,dx = \frac{M(2+\lambda-\theta)}{2(n-1)} \leq \frac{M(2+\lambda)}{2(n-1)},$$

which proves for every $0 < \lambda < a$ the proposition (A2).

If $0 < \lambda < a$ is integer, then $\theta = n\lambda - [n\lambda] = 0$ and in view of (7) follows (B2). Lemma 2 is proved.

Lemma 3. *Let f be nonnegative, continuous, convex, monotone, decreasing function, defined on $(0, \infty]$: $f(x) = M$, $f(x) = 0$ for $x \geq a$, $a \in (0, \infty)$. Then for every $\varepsilon > 0$ there exists a function such that:*

$$Q(x) = \sum_{i=1}^{m(\varepsilon)} \mu_i . g_i(x_i; x) \tag{8}$$

where:

1. $0 = x_0 < x_1 < \cdots < x_i < \cdots < x_m = a, \quad (m = m(\varepsilon)).$
2. $g_i(x_i; x) = \max\{0, M(x_i - x)/x_i\}, \ x \in [0, \infty]; \ i = 1, 2, \ldots, m.$
3. $\mu_i = i\{f(x_{i+1}) - 2.f(x_i) + f(x_{i-1})\}/M, \ \ \mu_i \geq 0, \ (i = 1, 2, \ldots, m-1),$

$$\mu_m = m\{f(x_{m-1}) - f(x_m)\}/M, \quad \mu_m \geq 0, \quad \sum_{i=1}^{m} \mu_i \leq 1,$$

which has the property $\mathrm{mod}\{f(x) - Q(x)\} < \varepsilon$ for every $x \geq 0$.

Proof. Let $\varepsilon > 0$ be arbitrarily small. The function f is continuous on the bounded interval [0,a] and therefore it is uniform continuous. Then there exists $\delta(\varepsilon) > 0$ such that for every choice of $x_1, \ x_2 \in [0, a]$ for which $\mathrm{mod}\{x_1 - x_2\} < \delta(\varepsilon)$ it holds $\mathrm{mod}\{f(x_1) - f(x_2)\} < \varepsilon$.

We set:

$k(\varepsilon) = [a/\delta(\varepsilon)]$

$m = m(\varepsilon) = k(\varepsilon) + 1; \ \ \delta = a/m; \ \ x_i = x_0 + i\delta, \ \ i = 1, 2, \ldots, m.$

and we define:

$$q_i(x) \ = \ \begin{cases} \delta^{-1}\{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\}(x_i - x), & x \in [x_0, x_i] \\ 0, \quad x > x_i; & \end{cases} \ , \quad i = 1, 2, \ldots, m-1$$

$$q_m(x) \ = \ \begin{cases} \delta^{-1}\{f(x_{m-1}) - f(x_m)\}(x_m - x), & x \in [x_0, x_m] \\ 0, \quad x > x_m; & \end{cases}$$

Now we consider the function $Q(x) = \sum_{i=1}^{m} q_i(x)$, defined for every $x \in [0, \infty]$. In view of the properties of f and the type of q_i- construction, it follows that for $Q(\cdot)$ the following properties hold:

$$\begin{aligned} & Q(x_i) = f(x_i), \quad i = 0, 1, \ldots, m \\ & Q(x) = f(x), \quad \text{for } x > x_m; \\ & \mathrm{mod}\{Q(x) - f(x)\} < \varepsilon \quad \text{for every } x \in [0, \infty]; \end{aligned} \tag{9}$$

Now we shall show that the function $Q(\cdot)$ can be expressed as in (8). To this end we transform $q_i(\cdot), \ i = 1, 2, \ldots m-1$ as follows:

$$q_i(x) = \begin{cases} \dfrac{x_i}{M}\dfrac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{\delta}\dfrac{M}{x_i}(x_i - x), & x \in [x_0, x_i]; \\ 0, \quad x > x_i; \end{cases}$$

$$= \mu_i g_i(x_i; x),$$

where

$$g_i(x_i; x) = \max\{0, M(x_i - x)/x_i\},$$

$$\mu_i = i\{f(x_{i+1}) \; - \; 2.f(x_i) \; + \; f(x_{i-1})\}/M,$$

and

$$q_m(x) = \begin{cases} \dfrac{x_m}{M}\dfrac{f(x_{m-1}) - f(x_m))}{\delta}\dfrac{M}{x_m}(x_m - x), & x \in [x_0, x_m]; \\ 0, \quad x > x_m; \end{cases}$$

$$= \mu_m g_m(x_m; x),$$

where

$$g_m(x_m; x) = \max\{0, \; M(x_m - x)/x_m\},$$

$$\mu_m = m\{f(x_{m-1}) \; - \; f(x_m)\}/M.$$

It is not difficult to show that if f is a convex function, then $\mu_i \geq 0$ for $i = 1, 2, \ldots, m$. One can show directly that

$$\sum_{i=1}^{m} \mu_i = \sum_{i=1}^{m} i\{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})/M \leq 1.$$

Lemma 3 is proved.

P r o o f o f t h e T h e o r e m.

In view of Lemma 3 for every choice of $\varepsilon > 0$ in particular for $\varepsilon = M/na$, there exists a function $Q(\cdot)$ such that for every $x \in [0, \infty]$ it holds

$$\text{mod}\{f(x) \; - \; Q(x)\} < \varepsilon,$$

where

$$Q(x) = \sum_{i=1}^{m} \mu_i g_i(x_i; x);$$

$$g_i(x_i; x) = \max\{0, \; M(x_i - x)/x_i\},$$

$$\mu_i \geq 0, \quad (i = 1, 2, \ldots, m), \quad \sum_{i=1}^{m} \mu_i \leq 1.$$

Now applying Lemma 2 we have:

$$\begin{aligned} &\int_0^\infty \operatorname{mod}\{B_n(Q;x) - Q(x)\}\,dx \\ (10)\qquad &= \int_0^\infty \operatorname{mod}\left\{\sum_{i=1}^m \mu_i\{B_n(g_i(x_i);x) - g_i(x_i;x)\}\right\} dx \\ &\le \sum_{i=1}^m \mu_i \int_0^\infty \operatorname{mod}\{B_n(g_i(x_i);x) - g_i(x_i;x)\}\,dx \le \frac{M(2+a)}{2(n-1)} \end{aligned}$$

Further using (9) we get:

$$\begin{aligned} &\int_0^\infty \operatorname{mod}\{B_n(Q;x) - B_n(f;x)\}\,dx \\ (11)\qquad &\le \sum_{k=0}^{[n\alpha]} \operatorname{mod}\{f(k/n) - Q(k/n)\} \int_0^\infty \frac{(-x)^k \phi_n^{(k)}(x)}{k!}\,dx \\ &\le \frac{1}{(n-1)} \sum_{k=0}^{[n\alpha]} \operatorname{mod}\{f(k/n) - Q(k/n)\} \le \frac{M}{n-1} \end{aligned}$$

and

$$(12)\qquad \int_0^\infty \operatorname{mod}\{f(x) - Q(x)\}\,dx = \int_0^a \operatorname{mod}\{f(x) - Q(x)\}\,dx \le \frac{m}{n}.$$

Estimates (10), (11) and (12) lead us to the conclusion:

$$\begin{aligned} &\textstyle\int_0^\infty \{\operatorname{mod}\{B_n(f;x) - f(x)\}\,dx \\ &\le \textstyle\int_0^\infty \operatorname{mod}\{B_n(Q;x) - B_n(f;x)\}\,dx \\ (13)\qquad & \\ &+ \textstyle\int_0^\infty \operatorname{mod}\{B_n(Q;x) - Q(x)\}\,dx \\ &+ \textstyle\int_0^\infty \operatorname{mod}\{f(x) - Q(x)\}\,dx \ \le\ (5+a).M.\{2(n-1)\}^{-1}. \end{aligned}$$

The result (13) cannot be improved. The estimate is exact to the order, as according to Lemma 2 there exists a function $g(\lambda)$, $0 \le \lambda \le \infty$, λ- integer, for which (B2) holds.

The proof of the theorem is completed.

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Angel Kanchev Polytechnic
Departement of Mathematics
7004 Russe, Bulgaria

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