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ON SOME FORMS OF CLIQUISHNESS ON TOPOLOGICAL SPACES

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ABSTRACT. The notion of cliquishness of maps with values in metric spaces has been studied by many authors including in [3,30,15,21]. In [6,7] J.Ewert investigated cliquishness of maps with values in uniform spaces. In this paper we introduce some forms of cliquishness of maps with values in topological spaces. We shall study fundamental properties of such forms of cliquishness as a natural generalization of weak forms of continuity on topological spaces.

1. Notation and preliminary results.

A subset A of a topological space (X,T) is said to be:

– semi-open [13], if $A \subset Cl(Int(A))$;

– pre-open [18], if $A \subset Int(Cl(A))$;

– an α-set [23], if $A \subset Int(Cl(Int(A)))$;

– pre-semi-open [1], if $A \subset Cl(Int(Cl(A)))$.

The collection of all subsets of a space (X,T) which are semi-open (resp. pre-open, an α-sets, pre-semi-open) is denoted by $SO(X,T)$ (respectively we have $PO(X,T)$, T^α, $PSO(X,T)$). It was observed in [21] that T^α is a topology on X and that $T \subset T^\alpha \subset SO(X,T)$. Moreover $T^\alpha = SO(X,T) \cap PO(X,T)$ [29].

The union of all semi-open (resp. pre-open, pre-semi-open) sets contained in A is called the semi-interior [5] (resp. pre-interior, [19], pre-semi-interior [1]) of A and it is denoted by $sInt(A)$ (resp. $pInt(A)$, $psInt(A)$), $\alpha Int(A)$ denotes the interior of A in (X,T^α).

Semi-closed (resp. pre-closed, pre-semi-closed) sets and semi-closure (resp. pre-closure, pre-semi-closure) are defined in a manner analogous to the corresponding concepts of closed sets and closure [5,19,1]. The semi-closure (resp. pre-closure, pre-semi-closure) of a subset A of (X,T) is denoted by $sCl(A)$ (resp. $pCl(A)$, $psCl(A)$), $\alpha Cl(A)$ denotes the closure of A in (X,T^α). In [1], sets having the property $A \subset Cl(Int(Cl(A)))$

are called semi-pre-open. The usage of the term "pre-semi-open" is motivated by the fact that a subset A is pre-semi-open if and only if $A \subset sInt(sCl(A))$, whereas a subset A satisfying $A \subset pCl(pInt(A))$, according to the term "semi-pre-open", need not be pre-semi-open.

In [1] is proved that

Lemma 1.1. *If A is a subset of a space (X,T), then*

$$\begin{aligned} (i) \quad sInt(A) &= A \cap Cl(Int(A)), \\ (ii) \quad pInt(A) &= A \cap Int(Cl(A)), \\ (iii) \quad \alpha Int(A) &= A \cap Int(Cl(Int(A))), \\ (iv) \quad psInt(A) &= A \cap Cl(Int(Cl(A))). \end{aligned}$$

By $f : X \to Y$ we denote a map f of a topological space X into a topological space Y.

A map $f : X \to Y$ is said to be semi-continuous [13] (resp. α-continuous [29], pre-continuous [18], pre-semi-continuous) if for every open set V of Y, $f^{-1}(V)$ is a semi-open set (resp. an α-set, a pre-open set, a pre-semi-open set) of X.

It is easy to see that a map $f : X \to Y$ is semi-continuous (resp. α-continuous, pre-continuous, pre-semi-continuous) if and only if for each $x \in X$ and each open set V containing $f(X)$, there exists a semi-open (resp. an α-set, a pre-open, a pre-semi-open) set A containing x such that $f(A) \subset V$ i.e., f is semi-continuous (resp. α-continuous, pre-continuous, pre-semi-continuous) at every point $x \in X$.

By $C(f)$ (resp. $sC(f)$, $\alpha C(f)$, $pC(f)$, $psC(f)$) we denote the set of all points at which f is continuous (resp. semi-continuous , α-continuous , pre-continuous , pre-semi-continuous).

In [11] pre-continuous maps are called almost continuous .

A map $f : X \to Y$ is said to be quasi-continuous [12] at $x \in X$ if for each open set V containing $f(X)$ and each open set U containing x, there exists an open set G of X such that $\vee \neq G \subset U$ and $f(G) \subset V$. If f is quasi-continuous at every $x \in X$, then it is called quasi-continuous .

In [20] it is shown that a map is quasi-continuous if and only if it is semi-continuous .

Definition 1.1. *A map $f : X \to Y$ is said to be:*

– basically cliquish (resp. basically pre-cliquish) at a point x of X if for any open set W containing x there exists a point z of X such that $W \cap Int(f^{-1}(V)) \neq \vee$ (resp. $W \cap Int(Cl(f^{-1}(V))) \neq \vee$) for any open set V containing $f(z)$;

– basically s-cliquish (resp. basically pre-s-cliquish) at a point x of X if there exists a point z of X such that $x \in Cl(Int(f^{-1}(V)))$ (resp. $x \in Cl(Int(Cl(f^{-1}(V))))$) for any open set V containing $f(z)$;

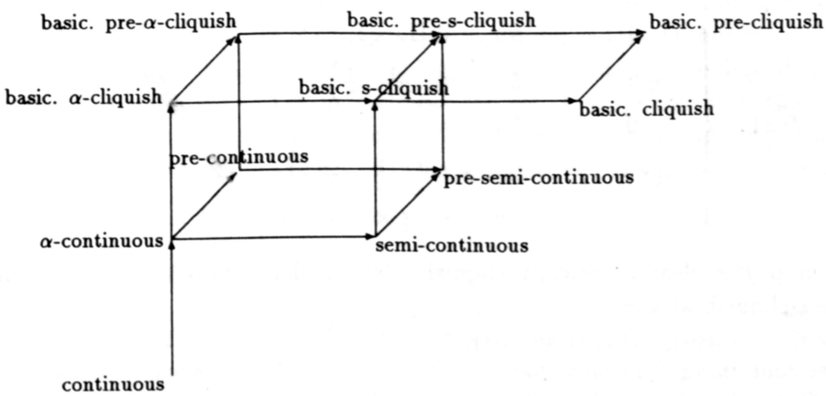
– *basically α-cliquish (resp. basically pre-α-cliquish) at a point x of X if there exists a point z of X such that* $x \in Int(Cl(Int(f^{-1}(V))))$ *(respectively we have* $x \in Cl(Int(Cl(f^{-1}(V)))))$ *for any open set V containing f(z).*

By $BA(f)$ (resp. $pBA(f)$, $sBA(f)$, $psBA(f)$, $\alpha BA(f)$, $p\alpha BA(f)$) we denote the set of all points at which f is basically cliquish (resp. basically pre-cliquish, basically s-cliquish, basically pre-s-cliquish, basically α-cliquish, basically pre α-cliquish).

A map f is called basically cliquish (resp. basically pre-cliquish, basically s-cliquish, basically pre-s-cliquish, basically α-cliquish, basically pre-α-cliquish) if $BA(f) = X$ (resp. $pBA(f) = X$, $sBA(f) = X$, $psBA(f) = X$, $\alpha BA(f) = X$, $p\alpha BA(f) = X$).

From definitions we immediately obtain the following

Diagramm 1.1.



A map f of a topological space X into a uniform space Y with a uniformity $\mathcal{U}$ is said to be cliquish at a point x of X if for every open set W containing x and for every $V \in \mathcal{U}$ there exists an open non-empty set $G \subset W$ such that $f(x'), f(x'')) \in V$ for any x', $x'' \in G$ [6,7]. If a uniformity $\mathcal{U}$ is induced by a metric on Y, then the above definition coincides with the well known definition of the cliquishness [3, 30, 9, 15, 21].

A map $f : X \to Y$ is said to be T_1-cliquish [27] (resp. feeble T_1-cliquish [28]) at a point x of X if for every open cover $\mathcal{A}$ of Y and for every open set W containing x exists an open non-empty set $G \subset W$ such that $f(G) \subset A$ for some $A \in \mathcal{A}$ (resp. for any points x', $x'' \in G$ there exists a set $A \in \mathcal{A}$ such that $f(x')$, $f(x'') \in A$).

We have the following relations between these types of cliquishness:

T_1-cliquish $\Longrightarrow$ feeble T_1-cliquish $\Longrightarrow$ cliquish.

It is easy to see that any basically cliquish map is T_1-cliquish.

In [28] it is shown that feeble T_1-cliquish map need not be T_1-cliquish and, cliquish map need not be feeble T_1-cliquish.

The following examples show that all the implications in Diagram 1.1 cannot be reversed.

Example 1.1. A semi-continuous map need not be basically pre-α-cliquish. Let X be the space of real numbers with the natural metric, and define $f : X \to X$ by $f(x) = n$ for $x \in [n, n+1)$, where n is an integer.

The map f is clearly semi-continuous. We shall show that for every integer $x \in X$ f is not basically pre-α-cliquish at x.

Let x be an integer. Then for every point z of X there exists an integer k such that $z \in [k,\ k+1)$. Clearly, $f(z) = f(k) = k$ and $V = (k - 1/2, k + 1/2)$ is an open set containing $f(z)$ such that $f^{-1}(V) = [k, k+1)$, $Cl(f^{-1}(V)) = [k, k+1]$ and $Int(Cl(f^{-1}(V))) = (k, k+1)$. Thus $n \notin Int(Cl(f^{-1}(V)))$ for any integer n and, consequently $x \notin Int(Cl(f^{-1}(V)))$. This means that f is not basically pre-α-cliquish.

Example 1.2. A basically cliquish map need not be basically pre-s-cliquish. Let X be the space of real numbers with the natural metric. Define $f : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{for } x \in [1, \infty) \\ n+1 & \text{for } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right) \\ 0 & \text{for } x = 0 \\ -(n+1) & \text{for } x \in \left(-\frac{1}{n}, -\frac{1}{n+1}\right] \\ -1 & \text{for } x \in (-\infty, -1], \quad n = 1, 2, \ldots. \end{cases}$$

The map f is clearly basically cliquish. We shall demonstrate that f is not basically pre-s-cliquish at $x = 0$.

Let $z \in X$. Clearly, $f(z)$ is an integer. At first let $f(z) = 0$. Then $V = (-1, 1)$ is an open set containing $f(z)$ such that $x \notin Cl(Int(Cl(f^{-1}(V))))$ since $f^{-1}(V) = \{0\}$. Secondly, if $f(z) = k \neq 0$, then $V = (k-1,\ k+1)$ is an open set containing $f(z)$ such that $Int(Cl(f^{-1}(V))) \cap U = \vee$ for some open set U containing x; namely $U = (-r,\ r)$, where $\frac{1}{k+1} \notin U$. Thus, $x \notin Cl(Int(Cl(f^{-1}(V))))$ and consequently f is not basically pre-s-cliquish at x.

Example 1.3. A pre-continuous map need not be basically cliquish. Let X be the space of real numbers with the natural metric. The map $f : X \to X$ given by $f(x) = 1$ for $x \in Q$ and $f(x) = 0$ for $x \notin Q$, where Q is the set of rational numbers, is clearly pre-continuous since the sets Q and $X \setminus Q$ are pre-open. We see also that f is not basically cliquish since there exist open sets V, V' such that $V \cap V' = \vee$, $f(X) \subset V \cup V'$ and for every open non-empty set G we have $f(G) \not\subset V$ and $f(G) \not\subset V'$.

Example 1.4. A basically α-cliquish map need not be pre-semi-continuous . Let X be the space of real numbers with natural metric. The map $f : X \to X$ given by $f(x) = 1$ for $x = 1$, and $f(x) = 2$ for $x \neq 1$, is not pre-semi continuous since we have an open set $V = (0, 2)$ such that $f^{-1}(V) = \{1\}$ is not pre-semi-open set. However, f is basically α–cliquish. Indeed, at each $x \neq 1$ the map f is continuous . If $x = 1$, then we have $z \neq 1$ such that for every open set V containing $f(z)$, $Int(Cl(Int(f^{-1}(V)))) = X$. So $x \in Int(Cl(Int(f^{-1}(V))))$.

A map $f : X \to Y$ is said to be barely continuous if for every non-empty closed set $B \subset X$ the restriction f/B has at least one point of the continuity [25].

Proposition 1.1. *Any barely continuous map $f : X \to Y$ is basically cliquish.*

Proof. Let $x \in X$ and let W be an open set containing x. By the barely continuity of f there exists a point $z \in Cl(W)$ of the continuity of the restriction $f/Cl(W)$. Then for every open set V containing $f(z)$ there exists an open set U containing z such that $f(U \cap Cl(W)) \subset V$. The set $G = U \cap W \subset W$ is open, non-empty and $f(G) \subset V$, what implies the basically cliquishness of f at x.

In [7, Examples 1.3 and 1.4] it is shown that semi-continuity and barely continuity are independent properties. We shall show that barely continuity and the four forms of cliquishness (basically s-cliquishness, basically pre-s-cliquishness, basically α-cliquishness, basically pre-α-cliquishness) are independent properties.

Example 1.5 (see [7, Example 1.2]). A basically α-cliquish map need not be barely continuous . Let us consider the set $X = [0, \infty)$ with the topology $T = \{\vee, X\} \cup \{(r, \infty) : r > 0\}$ and let Y be the space of real numbers with the natural metric. By Q we denote the set of rational numbers. The map $f : X \to Y$ given by

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x \in (n-1, n] \cap Q \\ \frac{1}{n+1} & \text{for } x \in (n-1, n] \setminus Q,\ n = 1, 2, \ldots \\ 0 & \text{for } x = 0 \end{cases}$$

is not barely continuous .
We see that for every open set V containing $f(0)$ we have $Int(Cl(Int(f^{-1}(V)))) = X$. So f is basically α-cliquish.

Example 1.6. A barely continuous map need not be basically pre-s-cliquish. Let f be the map in Example 1.2. We shall show that f is barely continuous . Let

$$\begin{aligned} \mathcal{A} &= \left\{\left(-\frac{1}{n}, -\frac{1}{n+1}\right) : n = 1, 2, \ldots\right\} \bigcup \left\{\left(\frac{1}{n+1}, \frac{1}{n}\right) : n = 1, 2, \ldots\right\} \\ &\bigcup \ \{(-\infty, -1),\ (1, \infty)\} \end{aligned}$$

If B is a closed non-empty set in X, then either $B \cap \bigcup \mathcal{A} \neq \vee$ or $B \subset X \setminus \bigcup \mathcal{A}$.

If $B \subset X \setminus \bigcup \mathcal{A}$ and B contains just one element, then the restriction f/B is clearly continuous . On the contrary, if $x' \neq x''$ for some $x', x'' \in B$, then there exists $z \in B$ such that the set $\{z\}$ is open in the subspace B and, consequently, the map f/B is continuous at z.

Now we assume that $B \cap \bigcup \mathcal{A} \neq \vee$ and let $z \in B \cap \bigcup \mathcal{A}$. Without loss of generality we can assume that $z > 0$. Then there exists a number n such that $z \in (1/n+1,\ 1/n)$. Clearly, the set $U \cap B$ is open in the subspace B and, for each $x \in U \cap B$ we have $f(x) = n+1$. It follows that the restriction f/B is continuous at z. Thus, f is barely continuous .

In [9,6] it is shown that a map f of a Baire space X into a metric space or uniform space with a countable base is cliquish if and only if the set $X \setminus C(f)$ is of the first category.

It is easy to see that if the set $C(f)$ is dense, then f is basically cliquish. Hence we have the following

Proposition 1.2. *Let X be a Baire space and let Y be a metric space or uniform space with a countable base. The following statements are equivalent for a map $f : X \to Y$.*

(*i*) *the set $X \setminus C(f)$ is of the first category,*

(*ii*) *f is basically cliquish,*

(*iii*) *f is T_1 – – cliquish ,*

(*iv*) *f is feeble T_1 – cliquish,*

(*v*) *f is cliquish.*

A topological space X is said to be a Baire space in the narrow sense, if every closed subspace of X is a Baire space [8].

In [7] it is shown that a map f of a Baire space in the narrow sense X into a uniform space with a countable base is barely continuous if and only if for every non-empty closed set $B \subset X$ the restriction f/B is cliquish. Therefore, by Proposition 1.1 and Proposition 1.2 we have the following

Proposition 1.3. *Let X be a Baire space in the narrow sense and let Y be a uniform space with a countable base. The following statements are equivalent for a map $f : X \to Y$:*

(*i*) *f is barely continuous ,*

(*ii*) *for every non-empty closed set $B \subset X$ the map f/B is basically cliquish,*

(*iii*) *for every non-empty closed set $B \subset X$ the map f/B is T_1-cliquish,*

(*iv*) *for every non-empty closed set $B \subset X$ the map f/B is feeble T_1-cliquish,*

(*v*) *for every non-empty closed set $B \subset X$ the map f/B is T_1-cliquish.*

2. Characterizations of forms of cliquishness and relationships among them.

Given families $\ddot{U}$ and $\ddot{O}$ of subsets of X, $\ddot{U}$ refines $\ddot{O}$ provided each member of $\ddot{U}$ is contained in some member of $\ddot{O}$ (briefly $\ddot{U} < \ddot{O}$).

For a map $f : X \to Y$ we denote $f(\ddot{U}) = \{f(U) : U \in \ddot{U}\}$.

Proposition 2.1. *The following statements are equivalent for a map $f : X \to Y$.*

(i) The map f is basically cliquish (resp. basically s-cliquish, basically α-cliquish) at a point $x \in X$.

(ii) For every open cover $\ddot{U}$ of $f(X)$ we have $x \in Cl(\bigcup\{Int(f^{-1}(U)) : U \in \ddot{U}\})$ (resp. $x \in \bigcup\{Cl(Int(f^{-1}(U))) : U \in \ddot{U}\}$), $x \in \bigcup\{Int(Cl(Int(f^{-1}(U)))) : U \in \ddot{U}\}$).

(iii) For every open cover $\ddot{U}$ of $f(X)$ there exists a collection $\ddot{O}$ of open subset of X such that $f(\ddot{O}) < \ddot{U}$ and $x \in Cl(\bigcup\{O : O \in \ddot{O}\})$ (resp. $x \in \bigcup\{Cl(O) : O \in \ddot{O}\}$, $x \in \bigcup\{Int(Cl(O) : O \in \ddot{O}\}$).

Proof. (i) implies (ii). Assume that f is basically cliquish at x. Let $\ddot{U}$ be an open cover of $f(X)$ and let W be an open set containing x. Then there exists a point $z \in X$ such that $W \cap Int(f^{-1}(V)) \neq \vee$ for any open set V containing $f(z)$. Clearly, $f(z) \in U$ for some $U \in \ddot{U}$. Thus $W \cap \bigcup\{Int(f^{-1}(U)) : U \in \ddot{U}\} \neq \vee$, what implies $x \in Cl(\bigcup\{Int(f^{-1}(U)) : U \in \ddot{U}\})$.

The implications for basically s-cliquish and basically α-cliquish we immediately obtain from Definition 1.1.

To prove that (ii) implies (iii) it is sufficient to take $\ddot{O} = \left\{Int(f^{-1}(U)) : U \in \ddot{U}\right\}$ where $\ddot{U}$ is an open cover of $f(X)$.

(iii) implies (i). Suppose that f is not basically cliquish at x. Then there exists an open set W containing x such that for every $z \in X$ we have $W \cap Int(f^{-1}(V_z)) = \vee$ for some open set V_z containing $f(z)$. Let us take $\ddot{U} = \{V_z : z \in X\}$. Clearly, $\ddot{U}$ is an open cover of $f(X)$ such that

$$x \notin Cl\left(\cup\left\{Int(f^{-1}(U)) : U \in \ddot{U}\right\}\right). \tag{1}$$

If $\ddot{O}$ is a collection of open subsets of X such that $f(\ddot{O}) < \ddot{U}$ and $x \in Cl(\bigcup\{O : O \in \ddot{O}\})$, then $\ddot{O} < \{Int(f^{-1}(U)) : U \in \ddot{U}\}$ and consequently $x \in Cl(\bigcup\{U : U \in \ddot{U}\})$. This is a contradiction of (1).

Thus, f does not satisfy (iii) and the proof for basically cliquishness is complete.

The proof for basically s-cliquishness and for basically α-cliquishness is analogous and is omitted.

From the above result we have the following

Corollary 2.1. *The following statements are equivalent for a map $f : X \to Y$.*

(i) The map f is basically cliquish (resp. basically s-cliquish, basically α-cliquish).

(ii) For every open cover $\ddot{U}$ of $f(X)$ we have $X = Cl(\bigcup\{Int(f^{-1}(U)) : U \in \ddot{U}\})$ (resp. $X = \bigcup\{Cl(Int(f^{-1}(U))) : U \in \ddot{U}\}$, $X = \bigcup\{Int(Cl(Int(f^{-1}(U)))) : U \in \ddot{U}\}$).

(iii) For every open cover $\ddot{U}$ of $f(X)$ there exists a collection $\ddot{O}$ of open subsets of X such that $f(\ddot{O}) < \ddot{U}$ and $X = Cl(\bigcup\{O : O \in \ddot{O}\})$ (resp. $X = \bigcup\{Cl(O) : O \in \ddot{O}\}$, $X = \bigcup\{Int(Cl(O)) : O \in \ddot{O}\}$).

Proposition 2.2. *The following statements are equivalent for a map $f : X \to Y$.*

(i) The map f is basically pre-cliquish (resp. basically pre-s-cliquish, basically pre-α-cliquish) at point $x \in X$.

(ii) For every open cover $\ddot{U}$ of $f(X)$ we have $x \in Cl(\bigcup\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\})$ (resp. $x \in \bigcup\{Cl(Int(Cl(f^{-1}(U)))) : U \in \ddot{U}\}$, $x \in \bigcup\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\}$).

(iii) For every open cover $\ddot{U}$ of $f(X)$ there exists a collection $\ddot{A}$ of pre-open subsets of X such that $f(\ddot{A}) < \ddot{U}$ and $x \in Cl(\bigcup\{Int(Cl(A)) : A \in \ddot{A}\})$ (resp. $x \in \bigcup\{Cl(Int(Cl(A))) : A \in \ddot{A}\}$, $x \in \bigcup\{Int(Cl(A)) : A \in \ddot{A}\}$).

Proof that (i) implies (ii) is analogous to the proof of (i) implies (ii) of Proposition 2.1.

Proof that (ii) implies (iii). Let $\ddot{U}$ be an open cover of $f(X)$. Let us take $\ddot{A} = \{f^{-1}(U) \cap Int(Cl(f^{-1}(U))) : U \in \ddot{U}\}$. Clearly, $f(\ddot{A}) < \ddot{U}$ and each member of $\ddot{A}$ is a pre-open subset since for every $U \in \ddot{U}$, $f^{-1}(U) \cap Int(Cl(f^{-1}(U))) = pInt(f^{-1}(U))$ by Lemma 1.1.

It is easy to verify that $Int(Cl(pInt(f^{-1}(U)))) = Int(Cl(f^{-1}(U)))$. Thus we have $Cl(\bigcup\{Int(Cl(A)) : A \in \ddot{A}\}) = Cl(\bigcup\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\})$,

$$\bigcup\{Cl(Int(Cl(A))) : A \in \ddot{A}\} = \bigcup\{Cl(Int(Cl(f^{-1}(U)))) : U \in \ddot{U}\}$$

and

$$\bigcup\{Int(Cl(A)) : A \in \ddot{A}\} = \bigcup\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\}$$

This means that (ii) implies (iii).

(iii) implies (i). Suppose that f is not basically pre-cliquish at x. Then there exists an open set W containing x such that for every $z \in X$ we have $W \cap Int(Cl(f^{-1}(V_z))) = \vee$ for some open set V_z containing $f(z)$. Thus we have an open cover $\ddot{U}$ of $f(X)$, namely $\ddot{U} = \{V_z : z \in X\}$, such that

$$x \notin Cl(\bigcup\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\}). \tag{2}$$

If $\ddot{A}$ is a collection of pre-open subsets of X such that

$$f(\ddot{A}) < \ddot{U} \quad \text{and} \quad x \in Cl(\bigcup\{Int\,(Cl(A)) : A \in \ddot{A}\}),$$

then $\{Int(Cl(A)) : A \in \ddot{A}\} < \{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\}$ and consequently, $x \in Cl(\bigcup\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\}$. This is a contradiction of (2) and thus, f does not satisfy (iii). This completes the proof for basically pre-cliquishness.

The proof for basically pre-s-cliquishness and for basically pre-α-c;iquishness is analogous.

Corollary 2.2. *The following statements are equivalent for a map $f : X \to Y$.*

(i) The map f is basically pre-cliquish (resp. basically pre-s-cliquish, basically pre-α-cliquish).

(ii) For every open cover $\ddot{U}$ of $f(X)$ we have $X = Cl(\bigcup\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\})$ (resp. $X = \bigcup\{Cl(Int(Cl(f^{-1}(U)))) : U \in \ddot{U}\}$, $X = \bigcup\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\}$).

(iii) For every open cover $\ddot{U}$ of $f(X)$ there exists a collection $\ddot{A}$ of pre-open subsets of X such that $f(\ddot{A}) < \ddot{U}$ and $X = Cl(\bigcup\{Int(Cl(A)) : A \in \ddot{A}\})$ (resp. $X = \bigcup\{Cl(Int(Cl(A))) : A \in \ddot{A}\}$, $X = \bigcup\{Int(Cl(A)) : A \in \ddot{A}\}$).

A map $f : X \to Y$ is said to be T_1-continuous if for every open cover $\ddot{U}$ of Y there exists an open cover $\ddot{O}$ of X such that $f(\ddot{O}) < \ddot{U}$ [10].

It is easy to prove that f is T_1-continuous if and only if it is T_1-continuous at each point $x \in X$, i.e., for every open cover $\ddot{U}$ of Y there exists an open set W containing x and $U \in \ddot{U}$ such that $f(W) \subset U$.

We see also that f is T_1-continuous if and only if for every open cover $\ddot{U}$ of Y we have $X = \bigcup\{Int(f^{-1}(U)) : U \in \ddot{U}\}$.

We say that a topological space X is a nearly T_1-space if for every point $x \in X$ and for every open set W containing x there exists an open cover $\ddot{U}$ of X such that $St(x, \ddot{U}) \subset W$ [26], where $S \in (x, \ddot{U})$ denotes the star of the point x with respect to $\ddot{U}$.

It follows from [26, Corollary 3.5] that any T_1-continuous map into a nearly T_1-space is continuous . Thus we have

Proposition 2.3. *A map f of a space X into a nearly T_1-space Y is continuous at a point $x \in X$ if and only if for every open cover $\ddot{U}$ of Y we have $x \in \bigcup\{Int(f^{-1}(U)) : U \in \ddot{U}\}$.*

Since any continuous map $f : X \to Y$ has the following property at each point $x \in X$: for every open cover $\ddot{U}$ of $f(X)$ we have $x \in \bigcup\{Int(f^{-1}(U)) : U \in \ddot{U}\}$; then we have

Corollary 2.3. *A map f of a space X into a nearly T_1-space Y is continuous at a point $x \in X$ if and only if for every open cover $\ddot{U}$ of $f(X)$ we have $x \in \bigcup\{Int(f^{-1}(U)) : U \in \ddot{U}\}$.*

Clearly, by arguments similar to those above we see that the following characterization is easy to verify.

Proposition 2.4. *A map f of a space X into a nearly T_1-space Y is α-continuous (resp. semi-continuous , pre-continuous , pre-semi-continuous) at a point*

$x \in X$ if and only if for every open cover $\ddot{U}$ of $f(X)$ we have $x \in \bigcup\{\alpha Int(f^{-1}(U)) : U \in \ddot{U}\}$ (resp. $x \in \bigcup\{sInt(f^{-1}(U)) : U \in \ddot{U}\}$, $x \in \bigcup\{pInt(f^{-1}(U)) : U \in \ddot{U}\}, x \in \bigcup\{psInt(f^{-1}(U)) : U \in \ddot{U}\}$.

Let $P(X)$ be the power set of X and let $D : P(X) \to P(X)$ be an operator. For $\ddot{U} \subset P(X)$ we denote $D(\ddot{U}) = \{D(U) : U \in \ddot{U}\}$.

The following elementary result will be useful in the sequel.

Lemma 2.1. *For each family $\ddot{U} \subset P(X)$, the following hold:*

(i) $Int(\ddot{U}) = Int(\alpha Int(\ddot{U})) = Int(pInt(\ddot{U}))$,

(ii) $Int(Cl(Int(\ddot{U}))) = Int(Cl(\alpha Int(\ddot{U})))$,

(iii) $Cl(Int(\ddot{U})) = Cl(\alpha Int(\ddot{U}))$,

(iv) $Cl(\cup Int(\ddot{U})) = Cl(\cup \alpha Int(\ddot{U})) = Cl(\cup Int(Cl(Int(\ddot{U})))) = Cl(\cup Cl(Int(\ddot{U})))$,

(v) $Int(Cl(\ddot{U})) = Int(Cl(pInt(\ddot{U}))) = Int(Cl(Int(Cl(\ddot{U}))))$,

(vi) $Cl(Int(Cl(\ddot{U}))) = Cl(pInt(\ddot{U}))$,

(vii) $Cl(\cup Int(Cl(\ddot{U}))) = Cl(\cup pInt(\ddot{U})) = Cl(\cup Cl(Int(Cl(\ddot{U}))))$.

The simple proof is omitted.

Proposition 2.5. *For every map f of a space X into a nearly T_1-space Y there exists a collections*

- $\{\ddot{O}_s : s \in S\}$ *of families of open subsets of X,*
- $\{\ddot{O}'_s : s \in S\}$ *of families of α-sets of X,*
- $\{\ddot{O}''_s : s \in S\}$ *of families of semi-open subsets of X,*
- $\{\ddot{A}_s : s \in S\}$ *of families of regular-open subsets of X and*
- $\{\ddot{A}'_s : s \in S\}$ *of families of regular-closed subsets of X, such that:*

(i) $C(f) = \bigcap\{\bigcup \ddot{O}_s : s \in S\} \subset \alpha BA(f) = \bigcap\{\bigcup Int(Cl(\ddot{O}_s)) : s \in S\} \subset sBA(f) = \bigcap\{\bigcup Cl(\ddot{O}_s) : s \in S\} \subset BA(f) = \bigcap\{Cl(\bigcup \ddot{O}_s) : s \in S\}$;

(ii) $C(f) = \bigcap\{\bigcup Int(\ddot{O}'_s) : s \in S\} \subset \alpha C(f) = \bigcap\{\bigcup \ddot{O}'_s : s \in S\} \subset \alpha BA(f) = \bigcap\{\bigcup Int(Cl(\ddot{O}'_s)) : s \in S\} \subset sBA(f) = \bigcap\{\bigcup Cl(\ddot{O}'_s) : s \in S\} \subset BA(f) = \bigcap\{Cl(\bigcup \ddot{O}'_s) : s \in S\}$;

(iii) $C(f) = \bigcap\{\bigcup Int(\ddot{O}''_s) : s \in S\} \subset sC(f) = \bigcap\{\bigcup \ddot{O}''_s : s \in S\} \subset sBA(f) = \bigcap\{\bigcup Cl(\ddot{O}''_s) : s \in S\} \subset BA(f) = \bigcap\{Cl(\bigcup \ddot{O}''_s) : s \in S\}$;

(iv) $\alpha BA(f) = \bigcap\{\bigcup \ddot{A}_s : s \in S\} \subset sBA(f) = \bigcap\{\bigcup Cl(\ddot{A}_s) : s \in S\} \subset BA(f) = \bigcap\{Cl(\bigcup \ddot{A}_s) : s \in S\}$;

(v) $\alpha BA(f) = \bigcap\{\bigcup(Int(\ddot{A}'_s) : s \in S\} \subset sBA(f) = \bigcap\{\bigcup \ddot{A}'_s : s \in S\} \subset BA(f) = \bigcap\{Cl(\bigcup \ddot{A}'_s) : s \in S\}$.

Proof. The result follows immediately from Proposition 2.1, Corollary 2.3 and from Proposition 2.4. It is sufficient to take:

- $\{\ddot{O}_s : s \in S\} = \Big\{\{Int(f^{-1}(U)) : U \in \ddot{U}\} : \ddot{U}$ is an open cover of $f(X)\Big\}$,
- $\{\ddot{O}'_s : s \in S\} = \Big\{\{\alpha Int(f^{-1}(U)) : U \in \ddot{U}\} : \ddot{U}$ is an open cover of $f(X)\Big\}$,

– $\{\ddot{O}''_s : s \in S\} = \left\{\{sInt(f^{-1}(U)) : U \in \ddot{U}\} : \ddot{U} \text{ is an open cover of } f(X)\right\}$,

– $\{\ddot{A}_s : s \in S\} = \left\{\{Int(Cl(Int(f^{-1}(U)))) : U \in \ddot{U}\} : \ddot{U} \text{ is an open cover of } f(X)\right\}$,

– $\{\ddot{A}'_s : s \in S\} = \left\{\{Cl(Int(f^{-1}(U))) : U \in \ddot{U}\} : \ddot{U} \text{ is an open cover of } f(X)\right\}$, by Lemma 2.1.

Proposition 2.6. *For every map f of a space X into a nearly T_1-space Y there exist the collections*

– $\{\ddot{O}_s : s \in S\}$ *of families of pre-open subsets of X,*

– $\{\ddot{A}_s : s \in S\}$ *of families of regular-open subsets of X and*

– $\{\ddot{A}'_s : s \in S\}$ *of families of regular-closed subsets of X such that:*

(i) $C(f) = \bigcap\{\bigcup Int(\ddot{O}_s) : s \in S\} \subset pC(f) = \bigcap\{\bigcup \ddot{O}_s : s \in S\} \subset p\alpha BA(f) = \bigcap\{\bigcup Int(Cl(\ddot{O}_s)) : s \in S\} \subset psBA(f) = \bigcap\{\bigcup Cl(\ddot{O}_s) : s \in S\} \subset pBA(f) = \bigcap\{Cl(\bigcup \ddot{O}_s) : s \in S\}$;

(ii) $p\alpha BA(f) = \bigcap\{\bigcup \ddot{A}_s : s \in S\} \subset psBA(f) = \bigcap\{\bigcup Cl(\ddot{A}_s : s \in S\} \subset pBA(f) = \bigcap\{Cl(\bigcup \ddot{A}_s) : s \in S\}$;

(iii) $p\alpha BA(f) = \bigcap\{\bigcup Int(\ddot{A}'_s) : s \in S\} \subset psBA(f) = \bigcap\{\bigcup \ddot{A}'_s : s \in S\} \subset pBA(f) = \bigcap\{Cl(\bigcup \ddot{A}'_s) : s \in S\}$.

Proof. By Proposition 2.2, Proposition 2.4 and Lemma 2.1 it is sufficient to take:

– $\{\ddot{O}_s : s \in S\} = \left\{\{pInt(f^{-1}(U)) : U \in \ddot{U}\} : \ddot{U} \text{ is an open cover of } f(X)\right\}$,

– $\{\ddot{A}_s : s \in S\} = \left\{\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\} : \ddot{U} \text{ is an open cover of } f(X)\right\}$,

and

– $\{\ddot{A}'_s : s \in S\} = \left\{\{Cl(Int(Cl(f^{-1}(U)))) : U \in \ddot{U}\} : \ddot{U} \text{ is an open cover of } f(X)\right\}$.

Corollary 2.4. *For every map f of a space X into a nearly T_1-space Y the following hold:*

(i) The sets $BA(f)$ and $pBA(f)$ are closed.

(ii) If at least one of the sets $C(f)$, $\alpha C(f)$, $sC(f)$, $\alpha BA(f)$, $sBA(f)$ (resp. $pC(f)$, $psC(f)$, $p\alpha BA(f)$, $psBA(f)$) is dense, then f is basically cliquish (resp. basically pre-cliquish).

Recall that a topological space (X,T) is said to be extremally disconnected if for each open subset $U \subset X$, $Cl(U)$ is open.

We have, by Propositions 2.1, 2.2 and 2.4, the following

Proposition 2.7. *If X is an extremally disconnected space and Y is a nearly T_1-space, then every semi-continuous (resp. pre-semi-continuous , basically s-cliquish, basically pre-s-cliquish) map $f : X \to Y$ is α-continuous (resp. pre-continuous , basically α-cliquish, basically pre-α-cliquish).*

Proof. It is sufficient to see that extremally disconnectedness of X implies

$$Int(Cl(Int(B))) = Cl(Int(B)),\ Int(Cl(B)) = Cl(Int(Cl(B)))$$

for any $B \subset X$ and, consequently, $sInt(B) = \alpha Int(B)$ and $psInt(B) = pInt(B)$.

Proposition 2.8. *If $f : X \to Y$ is a basically cliquish (resp. basically pre-cliquish) map such that $f(X)$ is compact, then f is basically s-cliquish (resp. basically pre-s-cliquish).*

Proof. Let f be a basically cliquish map such that $f(X)$ is compact and let $\ddot{U}$ be an open cover of $f(X)$. Then there exists a finite subcover $\ddot{U}' \subset \ddot{U}$ of $f(X)$ and, by Corollary 2.1 we have $X = Cl(\bigcup\{Int(f^{-1}(U)) : U \in \ddot{U}'\}$. Thus

$$X = \bigcup\{Cl(Int(f^{-1}(U))) : U \in \ddot{U}'\} \subset \bigcup\{Cl(Int(f^{-1}(U))) : U \in \ddot{U}\}.$$

Then by Corollary 2.1, f is basically s-cliquish.

The proof of the second part is similar to that of the first part.

Combining Propositions 2.7 and 2.8 we obtain the following

Corollary 2.5. *If f is a basically cliquish (resp. basically pre-cliquish) map of extremally disconnected space X such that $f(X)$ is compact, then f is basically α-cliquish (resp. basically pre-α-cliquish).*

In [14] it is shown that if Y is a second countable infinite Hausdorff space, then X is a Baire space if and only if for every map $f : X \to Y$ the set $pC(f)$ is dense.

Therefore, by Corollary 2.4 we have the following

Corollary 2.6. *If Y is a second countable infinite Hausdorff space and X is a Baire space, then every map $f : X \to Y$ is basically pre-cliquish.*

3. Restrictions.

The restrictions of maps of several types has been investigated by many authors including Long and McGehee [17], Baker [2], Long and Herrington [16], Noiri [24]. In this section we investigate the behavior of the six forms of cliquishness: basically cliquishness, basically pre-cliquishness, basically s-cliquishness, basically pre-s-cliquishness, basically α-cliquishness and basically pre-α-cliquishness under the restriction.

We will denote the closure and the interior of a subset K of a subspace $Z \subset X$ by $Z - Cl(K)$ and $Z - Int(K)$ respectively.

Proposition 3.1. *For a map $f : X \to Y$ and a subspace $Z \subset X$ such that $f(Z)$ is closed, the following hold:*

(i) If $pInt(Z) \subset BA(f)$, then $psInt(Z) \subset BA(f/Z)$.
(ii) If $\alpha Int(Z) \subset pBA(f)$, then $sInt(Z) \subset pBA(f/Z)$.
(iii) If $pInt(Z) \subset sBA(f)$, then $pInt(Z) \subset sBA(f/Z)$.
(iv) If $\alpha Int(Z) \subset psBA(f)$, then $\alpha Int(Z) \subset psBA(f/Z)$.
(v) If Z is pre-semi-open and $Z \subset \alpha BA(f)$, then $Z = \alpha BA(f/Z)$.
(vi) If Z is semi-open and $Z \subset p\alpha BA(f)$, then $Z = p\alpha BA(f/Z)$.

Proof. (i). Assume that $pInt(Z) \subset BA(f)$. Let $x \in psInt(Z)$ and let $\ddot{U}$ be an open cover of $f(Z)$. It suffices to show, by Proposition 2.1 that $x \in Z - Cl(\bigcup\{Z - Int((f/Z)^{-1}(U) : U \in \ddot{U}\})$. Let W be an open set containing x. Then, by Lemma 1.1 (iv) we have $x \in W \cap Z \cap Cl(Int(Cl(Z)))$, what implies $W \cap Z \cap Int(Cl(Z)) \neq \vee$. Take a point $x' \in W \cap Z \cap Int(Cl(Z))$. Obviously $x' \in pInt(Z)$ by Lemma 1.1 (ii), so $x' \in BA(f)$, and thus, by Proposition 2.1 we have

$$x' \in Cl(\bigcup\{Int(f^{-1}(A)) : A \in \ddot{A}\}),$$

where $\ddot{A} = \ddot{U} \cup \{Y \setminus f(Z)\}$. Then there exists $A \in \ddot{A}$ such that the set $G = W \cap Int(Cl(Z)) \cap Int(f^{-1}(A))$ is non-empty, $G \cap Z \neq \vee$, $G \cap Z \subset W \cap Z$ and $G \cap Z \subset f^{-1}(A) \cap Z$. This implies that $G \cap Z \subset Z - Int((f/Z)^{-1}(A)$ and, consequently,

$$W \cap Z \cap \bigcup\{Z - Int((f/Z)^{-1}(U)) : U \in \ddot{U}\} \neq \vee$$

since $A \in \ddot{U}$ by the fact $G \cap Z \neq \vee$. Therefore

$$x \in Z - Cl(\bigcup\{Z - Int((f/Z)^{-1}(U)) : U \in \ddot{U}\})$$

and the proof of (i) is complete.

(ii). Let $\alpha Int(Z) \subset pBA(f)$, $x \in sInt(Z)$ and let $\ddot{U}$ be an open cover of $f(Z)$. If W is an open set containing x, then $x \in W \cap Z \cap Cl(Int(Z))$ by Lemma 1.1 (i), what implies $W \cap Z \cap Int(Cl(Int(Z))) \neq \vee$. Take a point $x' \in W \cap Z \cap Int(Cl(Int(Z)))$. Thus, $x' \in \alpha Int(Z)$ by Lemma 1.1 (iii) and $x' \in pBA(f)$ by the assumption. It follows from Proposition 2.2 that

$$x' \in Cl(\bigcup\{Int(Cl(f^{-1}(A))) : A \in \ddot{A}\}),$$

where $\ddot{A} = \ddot{U} \cup \{Y \setminus f(Z)\}$. Then there exists $A \in \ddot{A}$ such that $W \cap Int(Z) \cap Int(Cl(f^{-1}(A))) \neq \vee$. Clearly, $A \in \ddot{U}$ since $G \subset Z$, where $G = W \cap Int(Z) \cap Int(Cl(f^{-1}(A)))$. We see also that $G \cap Z \subset W \cap Z$ and, for each $z \in G \cap Z$ and every open set V containing z we have $V \cap Z \cap f^{-1}(A) \neq \vee$ since $V \cap G \subset V \cap Z$ and $V \cap G \cap f^{-1}(A)) \neq \vee$. So $G \cap Z \subset Z - Int(Z - Cl((f/Z)^{-1}(A)))$ and, consequently, $x \in Z - Cl(\bigcup\{Z - Int(Z - Cl((f/Z)^{-1}(U))) : U \in \ddot{U}\})$. Therefore, $x \in pBA(f/Z)$ by Proposition 2.2.

(iii). Assume that $pInt(Z) \subset sBA(f)$. Let $x \in pInt(Z)$ and let $\ddot{U}$ be an open cover of $f(Z)$. Take $\ddot{A} = \ddot{U} \cup \{Y \setminus f(Z)\}$. Then, by Lemma 1.1 (ii) and Proposition 2.1, there exists $A \in \ddot{A}$ such that

$$x \in Z \cap Int(Cl(Z)) \cap Cl(Int(f^{-1}(A))).$$

Consequently, for every open set W containing x the set

$$G = W \cap Int(Cl(Z)) \cap Int(f^{-1}(A))$$

is non-empty, $G \cap Z \neq \vee$, $G \cap Z \subset W \cap Z$ and $G \cap Z \subset f^{-1}(A) \cap Z$. So $W \cap Z \cap Z - Int((f/Z)^{-1}(A)) \neq \vee$. Clearly, $A \in \ddot{U}$. Therefore

$$x \in \bigcup\{Z - Cl(Z - Int((f/Z)^{-1}(U))) : U \in \ddot{U}\}$$

and, by Proposition 2.1 we have $x \in sBA(f/Z)$.

(iv). Assume that $\alpha Int(Z) \subset psBA(f)$. Let $x \in \alpha Int(Z)$ and let $\ddot{U}$ be an open cover of $f(Z)$. Take $\ddot{A} = \ddot{U} \cup \{Y \setminus f(z)\}$. By Lemma 1.1 (iii) and Proposition 2.2, there exists $A \in \ddot{A}$ such that $x \in Z \cap Int(Cl(Int(Z))) \cap Cl(Int(Cl(f^{-1}))))$. Thus for every open set W containing x the set $G = W \cap Int(Z) \cap Int(Cl(f^{-1}(A)))$ is non-empty, $G \cap Z \neq \vee$, $G \cap Z \subset W \cap Z$ and, for each $z \in G \cap Z$ and every open set V containing z we have $V \cap Z \cap f^{-1}(A) \neq \vee$. It implies

$$x \in Z - Cl(Z - Int(Z - Cl((f/Z)^{-1}(A))))$$

and, consequently,

$$x \in \bigcup\{Z - Cl(Z - Int(Z - Cl((f/Z)^{-1}(U)))) : U \in \ddot{U}\}$$

since $A \in \ddot{U}$. Therefore $x \in psBA(f/Z)$ by Proposition 2.2.

(v). Assume that Z is pre-semi-open and $Z \subset \alpha BA(f)$. Let $x \in Z$ and $\ddot{U}$ be an open cover of $f(Z)$. Take $\ddot{A} = \ddot{U} \cup \{Y \setminus f(Z)\}$. Then, by Proposition 2.1, there exists $A \in \ddot{A}$ such that $x \in Z \cap Int(Cl(Int(f^{-1}(A))))$. The set $Z \cap Int(Cl(Int(f^{-1}(A))))$ is open in the subspace Z and, for each $z \in Z \cap Int(Cl(Int(f^{-1}(A))))$ and for every open set V containing z the set $G = V \cap Int(Cl(Z)) \cap Int(f^{-1}(A))$ is non-empty, $G \cap Z \neq \vee$, $G \cap Z \subset V \cap Z$ and $G \cap Z \subset f^{-1}(A) \cap Z$. It implies

$$Z \cap Int(Cl(Int(f^{-1}(A)))) \subset Z - Cl(Z - Int((f/Z)^{-1}(A))),$$

so

$$x \in \bigcup\left\{Z - Int(Z - Cl(Z - Int((f/Z)^{-1}(U)))) : U \in \ddot{U}\right\}$$

since $A \in \ddot{U}$. Thus, by Proposition 2.1, $x \in \alpha BA(f/Z)$.

(vi). Assume that Z is semi-open and $Z \subset p\alpha BA(f)$. Let $x \in Z$ and let $\ddot{U}$ be an open cover of $f(Z)$. Take $\ddot{A} = \ddot{U} \cup \{Y \setminus f(Z)\}$. Then, by Proposition 2.2, there exists $A \in \ddot{A}$ such that $x \in Z \cap Int(Cl(f^{-1}(A)))$. It is easy to see that the set $Z \cap Int(Cl(f^{-1}(A)))$ is open in the subspace Z and, for each $z \in Z \cap Int(Cl(f^{-1}(A)))$ and every open set V containing z we have $V \cap Int(Z) \cap Int(Cl(f^{-1}(A))) \neq \vee$, so $V \cap Z \cap f^{-1}(A) \neq \vee$. It implies

$$Z \cap Int(Cl(f^{-1}(A))) \subset Z - Int(Z - Cl((f/Z)^{-1}(A)))$$

and, consequently,

$$x \in \bigcup \left\{ Z - Int(Z - Cl((f/Z)^{-1}(U))) : \ U \in \ddot{U} \right\}$$

since $A \in \ddot{U}$. Thus, by Proposition 2.2, $x \in p\alpha BA(f/Z)$. The proof of Proposition 3.1 is complete.

Corollary 3.1. *For a map $f : X \to Y$ and a subspace $Z \subset X$ such that $f(Z)$ is closed, the following hold:*

(i) If Z is pre-semi-open and f is basically cliquish (resp. basically α-cliquish), then the restriction f/Z is basically cliquish (resp. basically α-cliquish).

(ii) If Z is pre-open and f is basically s-cliquish, then the restriction f/Z is basically s-cliquish.

(iii) If Z is semi-open and f is basically pre-cliquish (resp. basically pre-α-cliquish), then the restriction f/Z is basically pre-cliquish (resp. basically pre-α-cliquish).

(iv) If Z is an α-set and f is basically pre-s-cliquish, then the restriction f/Z is basically pre-s-cliquish.

Proposition 3.2. *For the map $f : X \to Y$ and a subspace $Z \subset X$, the following hold:*

(i) If the restriction f/Z is basically α-cliquish (resp. basically s-cliquish), then $\alpha Int(Z) \subset \alpha BA(f))$ (resp. $\alpha Int(Z) \subset sBA(f)$).

(ii) If the restriction f/Z is basically pre-α-cliquish (resp. basically pre-s-cliquish), then $pInt(Z) \subset p\alpha BA(f)$ (resp. $pInt(Z) \subset psBA(f)$).

(iii) If the restriction f/Z is basically cliquish, then $sInt(Z) \subset BA(f)$.

(iv) If the restriction f/Z is basically pre-cliquish, then $psInt(Z) \subset pBA(f)$.

Proof. (i). Assume that f/Z is basically α-cliquish. Let $x \in \alpha Int(Z)$ and let $\ddot{U}$ be an open cover of $f(X)$. Then, by Lemma 1.1 (iii) and Proposition 2.1, there exists $U \in \ddot{U}$ such that $x \in Int(Cl(Int(Z))) \cap Z - Int(Z - Cl(Z - Int(f^{-1}(U) \cap Z)))$. Hence there exists an open set V such that

$$x \in V \cap Z \subset Int(Cl(Int(Z))) \cap Z - Cl(Z - Int(f^{-1}(U) \cap Z)).$$

It is easy to see that for each $z \in V \cap Int(Cl(Int(Z)))$ and for every open set W containing z we have $V \cap Int(Z) \cap W \subset Z - Cl(Z - Int(f^{-1}(U) \cap Z))$, so there exists an open non-empty set G such that $G \cap Z \neq \vee$, $G \cap Z \subset V \cap Int(Z) \cap W$ and $G \cap Z \subset f^{-1}(U) \cap Z$. Thus, the set $G' = G \cap V \cap Int(Z) \cap W$ is open, non-empty, $G' \subset W$ and $G' \subset f^{-1}(U)$. Therefore, $x \in \bigcup\{Int(Cl(Int(f^{-1}(U)))) : \ U \in \ddot{U}\}$ and, by Proposition 2.1, $x \in \alpha BA(f)$.

Now we assume that f/Z is basically s-cliquish. Let $x \in \alpha Int(Z)$ and let $\ddot{U}$ be an open cover of $f(X)$. By Lemma 1.1 (iii) and Proposition 2.1, there exists $U \in \ddot{U}$such that

$$x \in Int(Cl(Int(Z))) \cap Z - Cl(Z - Int(f^{-1}(U) \cap Z))$$

and, for every open set V containing x we have

$$V \cap Int(Cl(Int(Z))) \cap Z - Int(f^{-1}(U) \cap Z) \neq \vee ,$$

what is equivalent to the existence of open non-empty set G such that $G \cap Z \neq \vee$, and $G \cap Z \subset f^{-1}(U) \cap Z$. Thus, the set $G' = G \cap V \cap Int(Z)$ is open, non-empty $G' \subset V$ and $G' \subset f^{-1}(U)$. so $x \in \bigcup\{Cl(Int(f^{-1}(U)) : U \in \ddot{U}\}$ and, by Proposition 2.1, $x \in sBA(f)$.

(ii). Assume that f/Z is basically pre-α-cliquish. Let $x \in pInt(Z)$ and let $\ddot{U}$ be an open cover of $f(X)$. Then, by Lemma 1.1 (ii) and Proposition 2.2, there exists $U \in \ddot{U}$ such that $x \in Int(Cl(Z)) \cap Z - Int(Z - Cl(f^{-1}(U) \cap Z))$ and consequently, there exists an open set V such that $x \in V \cap Z \subset Int(Cl(Z)) \cap Z - Cl(f^{-1}(U) \cap Z)$. Let us put $V' = V \cap Int(Cl(Z))$. Clearly, V' is open, $x \in V'$ and, for each $z \in V'$ and every open set W containing z we have $W \cap f^{-1}(U) \neq \vee$. Indeed, because $z \in W \cap V \cap Int(Cl(Z))$, the set $K = W \cap V \cap Z$ is open in the subspace Z, non-empty and $K \subset Z - Cl(f^{-1}(U) \cap Z)$, what implies $W \cap f^{-1}(U) \neq \vee$. So $x \in \bigcup\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\}$ and, by Proposition 2.2, $x \in p\alpha BA(f)$.

We assume that f/Z is basically pre-s-cliquish. Let $x \in pInt(Z)$ and let $\ddot{U}$ be an open cover of $f(X)$. By Lemma 1.1 (ii) and Proposition 2.2, there exists $U \in \ddot{U}$ such that $x \in Int(Cl(Z)) \cap Z - Cl(Z - Int(Z - Cl(f^{-1}(U) \cap Z)))$. This implies that for every open set V containing x there exists an open non-empty set G such that $G \cap Z \neq \vee$, $G \cap Z \subset V \cap Int(Cl(Z))$ and $G \cap Z \subset Z - Cl(f^{-1}(U) \cap Z)$. Observe that the set $G' = G \cap V \cap Int(Cl(Z))$ is open, non-empty, $G' \subset V$ and, for each $z \in G'$ and for every open set W containing z we have $W \cap f^{-1}(U) \neq \vee$. Indeed, it holds $W \cap G' \cap Z \subset W$, $W \cap G' \cap Z \neq \vee$ and $W \cap G' \cap Z \cap f^{-1}(U) \neq \vee$ since $W \cap G' \cap Z \subset G \cap Z \subset Z - Cl(f^{-1}(U) \cap Z)$. So $W \cap f^{-1}(U) \neq \vee$. Therefore, $x \in \bigcup\{Cl(Int(Cl(f^{-1}(U)))) : U \in \ddot{U}\}$, what implies, by Proposition 2.2, $x \in psBA(f)$.

(iii). Assume that f/Z is basically cliquish. Let $x \in sInt(Z)$ and let $\ddot{U}$ be an open cover of $f(X)$. If V is an open set containing x, then $x \in V \cap Cl(Int(Z))$ and $\vee \neq V \cap Int(Z) \subset Z - Cl(\bigcup\{Z - Int((f/Z)^{-1}(U)) : U \in \ddot{U}\})$ by Lemma 1.1 (i) and Proposition 2.2. Then there exists $U \in \ddot{U}$ such that $V \cap Int(Z) \cap Z - Int(f^{-1}(U) \cap Z) \neq \vee$, what implies the existence of an open non-empty set G such that $G \cap Z \neq \vee$, $G \cap Z \subset V \cap Int(Z)$ and $G \cap Z \subset f^{-1}(U) \cap Z$. Observe that the set $G' = G \cap V \cap Int(Z)$ is open, non-empty $G' \subset f^{-1}(U)$ and $G' \subset V$. Thus $x \in Cl(\bigcup\{Int(f^{-1}(U)) : U \in \ddot{U}\})$ and, by Proposition 2.1, $x \in BA(f)$.

(iv). Assume that f/Z is basically pre-cliquish. Let $x \in psInt(Z)$ and let $\ddot{U}$ be an open cover of $f(X)$. If V is an open set containing x, then $x \in V \cap Cl(Int(Cl(Z)))$

and $\vee \neq V \cap Int(Cl(Z)) \cap Z \subset Z - Cl(\bigcup\{Z - Int(Z - Cl((f/Z)^{-1}(U))) : U \in \ddot{U}\})$ by Lemma 1.1 (iv) and Proposition 2.2. Then there exists $U \in \ddot{U}$ such that $\vee \neq V \cap Int(Cl(Z)) \cap Z \cap Z - Int(Z - Cl(f^{-1}(U) \cap Z))$. Consequently, there exists an open non-empty set G such that $G \cap Z \neq \vee$ and $G \cap Z \subset V \cap Int(Cl(Z)) \cap Z - Cl(f^{-1}(U) \cap Z)$. Observe that the set $G' = G \cap V \cap Int(Cl(Z))$ is open, non-empty, $G' \subset V$ and, for each $z \in G'$ and for every open set W containing z we have $W \cap f-1(U) \neq \vee$. Indeed, we have $G' \cap W \cap Z \neq \vee$, $G' \cap W \cap Z \subset W$, $G' \cap W \cap Z \subset G \cap Z \subset Z - Cl(f^{-1}(U) \cap Z)$ and $G' \cap W \cap Z$ is open in the subspace Z, so $G' \cap W \cap Z \cap f^{-1}(U) \neq \vee$. Thus $W \cap f^{-1}(U) \neq \vee$. From it follows $x \in Cl(\bigcup\{Int(Cl(f^{-1}(U))) : U \in \ddot{U}\})$ and, by Proposition 2.2, $x \in pBA(f)$. This completes the proof of proposition 3.2.

Corollary 3.2. *For a map $f : X \to Y$ the following hold:*

(i) If the restriction f/D is basically pre-cliquish (resp. basically pre-s-cliquish, basically pre-α-cliquish) for some dense subset $D \subset X$, then f is basically pre-cliquish (resp. the set $psBA(f)$ is dense, the set $p\alpha BA(f)$ is dense).

(ii) If the restriction f/B is basically cliquish (resp. basically s-cliquish, basically α-cliquish) for some semi-open dense subset $B \subset X$, then f is basically cliquish (resp. the set $sBA(f)$ is semi-open and dense, the set $\alpha BA(f)$ is semi-open and dense).

Proof. It follows from the Corollary 2.4, from the above result and from the fact that for every dence subset $D \subset X$ (resp. semi-open dense subset $B \subset X$) we have $pInt(D) = psInt(D) = D$ (resp. $\alpha Int(B) = sInt(B) = B$).

Let $f : X \to Y$ be a map. A subset $D \subset X$ is called a Blumberg set [4] (resp. quasi Blumberg set [22]) of f if D is dense and the restriction f/D is continuous (resp. semi-continuous).

It follows from Corollary 3.2 that

Corollary 3.3. *If $f : X \to Y$ possesses a Blumberg set (resp. a quasi Blumberg set), then the set $p\alpha BA(f)$ (resp. $psBA(f)$) is dense.*

From Proposition 3.2 we also have the following

Corollary 3.4. *For the map $f : X \to Y$ the following hold:*

(i) If there exists a cover $\ddot{A}$ of X by α-sets such that for every $A \in \ddot{A}$, the restriction f/A is basically α-qliquish (resp. basically s-cliquish), then f is basically α-qliquish (resp. basically s-qliquish).

(ii) If there exists a cover $\ddot{A}$ of X by pre-open sets such that for every $A \in \ddot{A}$, the restriction f/A is basically pre-α-cliquish (resp. basically pre-s-cliquish), then f is basically pre-α-cliquish (resp. basically pre-s-qliquish).

(iii) If there exists a cover $\ddot{A}$ of X by semi-open sets such that for every $A \in \ddot{A}$, the restriction f/A is basically cliquish, then f is basically cliquish.

(iv) If there exists a cover $\ddot{A}$ of X by pre-semi-open sets such that for every $A \in \ddot{A}$, the restriction f/A is basically pre-cliquish, then f is basically pre-cliquish.

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