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BEST ONESIDED APPROXIMATION AND APPROXIMATION WITH TRIGONOMETRIC OPERATORS IN L_P, $0 < P \leq 1$

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ABSTRACT. V.Hristov (1989) used the locally global norm $\|f\|_{\delta,p}$ for bounded functions and proved that the best onesided approximation of a 2π-periodic bounded function with trigonometric polynomials of degree n in the norm L_p, $1 \leq p \leq \infty$ is equivalent to the best approximation with trigonometric polynomials of degree n in the norm $\|\cdot\|_{\frac{1}{n},p}$, $1 \leq p \leq \infty$. L.Aleksandrov and D.Dryanov (1989), (1991) proved the equivalent proposition for approximation with entire functions from exponential type in the quasi-norm $\|\cdot\|_p$, $0 < p \leq \infty$.

In this paper we prove the equivalent proposition for the best approximation with trigonometric polynomials in the locally global quasi-norm $\|\cdot\|_{\frac{1}{n},p}$, $0 < p \leq 1$ and we express the relationship between the best onesided approximation of the functions and their error with some discrete operators in locally global quasi-norm $\|\cdot\|_{\frac{1}{n},p}$, $0 < p \leq 1$.

1. Assertions. Let f be a 2π-periodic bounded measurable function defined on $\Omega = [-\pi, \pi]$ ($f \in L_\infty$).

We denote by $\|f\|_p$ the L_p- quasi-norm ($0 < p \leq 1$) of $f \in L_p$. For $f \in L_\infty$ we denote by $\|f\|_{\delta,p}$ the locally global quasi-norm of f which is given by ($\delta > 0$, $0 < p \leq 1$):

$$\|f\|_{\delta,p} := \left(\int_{-\pi}^{\pi} (\sup\{|f(t)| : t \in U(\delta, x)\})^p \, dx\right)^{1/p}, \tag{1}$$

where

$$U(\delta, x) := \{y \in [-\pi, \pi] : |x - y| \leq \delta/2\}. \tag{2}$$

We denote by T_n the set of all trigonometric polynomials of degree n. We denote by $E_n^T(f)_p$ the best approximation of a given function $f \in L_\infty(\Omega)$ with trigonometric polynomials from T_n in the metric of the space L_p which is given by:

$$E_n^T(f)_p := \inf\{\|f - T\|_p : T \in T_n\}.$$

The best onesided approximation of a function $f \in L_\infty(\Omega)$ with trigonometric polynomials from T_n in the metric of the space L_p is given by:

$$\tilde{E}_n^T(f)_p := \inf\left\{\|T^+ - T^-\|_p : T^\pm \in T_n,\ T^-(x) \le f(x) \le T^+(x),\ x \in \Omega\right\}.$$

The best (onesided) approximation of a function $f \in L_\infty(\Omega)$ with polynomials from T_n in the metric (1) is given by:

$$E_n^T(f)_{\delta,p} := \inf\{\|f - T\|_{\delta,p} : T \in T_n\},$$

$$\tilde{E}_n^T(f)_{\delta,p} := \inf\left\{\|T^+ - T^-\|_{\delta,p} : T^\pm \in T_n,\ T^-(x) \le f(x) \le T^+(x),\ x \in \Omega\right\}.$$

Let

$$x_{k,m} := 2\pi k/(m+1),\ k = 0, 1, \ldots, m. \tag{3}$$

For $f \in L_\infty[0, 2\pi]$ we define the following discrete norms:

$$\|f\|_{L_m^p} := \left(\frac{2\pi}{m+1}\sum_{k=0}^{m}|f(x_{k,m})|^p\right)^{1/p}, \tag{4}$$

where $x_{k,m}$ are defined in (3).

Lemma 1. *The quasi-norm $\|\cdot\|_{\delta,p}$, $\delta > 0$, $0 < p \le 1$ has the following properties:*

$$\|f + g\|_{\delta,p} \le 2^{1/p-1}\left(\|f\|_{\delta,p} + \|g\|_{\delta,p}\right), \tag{5}$$

$$\|f\|_{\delta,p} \le \|f\|_{\delta',p},\ \delta \le \delta'; \tag{6}$$

$$\|f\|_p \le \|f\|_{\delta,p}; \tag{7}$$

$$\|f(\cdot + x)\|_{\delta,p} = \|f(\cdot)\|_{\delta,p},\ x \in R; \tag{8}$$

$$\|f\|_{m\delta,p} \le m^{1/p}\|f\|_{\delta,p},\quad m \text{ natural}; \tag{9}$$

$$\|f\|_{L_m^p} \le \|f\|_{\pi/(m+1),p}; \tag{10}$$

where $\|f\|_{L_m^p}$ is given in (4).

Proof. The inequalities (5) – (7) follow from the definition of the quasi-norm (1); (8) follows from the equality $f_\delta(\cdot + h)(x) = f_\delta(\cdot)(x+h)$, where $f_\delta(x) = \sup\{|f(t)| : t \in U(\delta, x)\}$; (9) follows from (8) and the inequality

$$f_{m\delta}(x) \le \sum_{k=0}^{m-1} f_\delta(x + (2k - (m-1))\delta/2).$$

(10) follows from the inequality $f_\delta(t + x_{k,m}) \ge |f(x_{k,m})|$ for $t \in [-\delta/2, \delta/2]$.

We shall use the properly normalized Jackson kernel:

$$\Phi_{r,n}(t) := \left[\sin\frac{\pi}{4n}\right]^{2r}\left[[\sin nt]/[\sin\frac{t}{2}]\right]^{2r},$$

where r and n are naturals and $\Phi_{r,n}$ is a trigonometric polynomial of degree $r(2n-1)$.

Lemma 2. *The polynomials $\Phi_{r,n}(t)$ have the following properties:*

(11) $$\Phi_{r,n}(t) \ge 1, \ |t| \le \pi/(2n);$$

(12) $$\Phi_{r,n}(t) \le C(r);$$

(13) $$\sup\{\Phi_{r,n}(t) : t \in [m\pi/n, (m+1)\pi/n]\} \le C(r)m^{-2r}, \ m = 1, 2, \cdots, n-1;$$

(14) $$\sup\{\Phi_{r,n}(t) : t \in [(m-1)\pi/n, m\pi/n]\} \le C(r)|m|^{-2r}, \ m = -n+1, \cdots, -1;$$

(15) $$\|\Phi_{r,n}\|_p \le C(r)\left(\frac{1}{n}\right)^{1/p}, \ 0 < p \le 1, \ r > \frac{1}{2p}.$$

Proof. To prove (11) it is clear that $\Phi_{r,n}(\pi/(2n)) = 1$, $\Phi_{r,n}$ is an even polynomial and $\Phi_{r,n}(t)$ is decreasing in $[0, \pi/(2n)]$. Therefore $\Phi_{r,n}(t) \ge 1$, $|t| \le \pi/(2n)$.

Let $t \in \left[\frac{m\pi}{n}, \frac{(m+1)\pi}{n}\right]$. Then

$$\begin{aligned}\Phi_{r,n}(t) &\le [\sin(\pi/(4n))]^{2r} / \left[\sin\frac{t}{2}\right]^{2r} = c(r)/\left[n^{2r}\sin^{2r}\frac{t}{2}\right]\\ &\le c(r)/\left[n^{2r}\sin^{2r}(m\pi/(2n))\right]\\ &\le c(r)m^{-2r},\end{aligned}$$

which proves (13).

Relation (14) follows since $\Phi_{r,n}(t)$ is an even polynomial and from (13). In order to prove (15) take into account that $\Phi_{r,n}(t)$ is even and (12) and (13). We get

$$\begin{aligned}
\|\Phi_{r,n}\|_p^p &= \int_{-\pi/n}^{\pi/n} \Phi_{r,n}(x)^p\,dx + \int_{\pi\geq|x|\geq\pi/n} \Phi_{r,n}(x)^p\,dx \\
&= 2\int_0^{\pi/n} \Phi_{r,n}(x)^p\,dx + 2\int_{\pi/n}^{\pi} \Phi_{r,n}(x)^p\,dx \\
&\leq 2\pi c(r)^p/n + 2\sum_{m=1}^{n-1} c(r)^p \int_{m\pi/n}^{(m+1)\pi/n} 1\,dx \,/\, \left(m^{2rp}\right) \\
&\leq c(r)^p/n + c(r)\pi n^{-1}\sum_{m=1}^{\infty}\frac{1}{m^{2rp}} \\
&\leq c(r,p)/n.
\end{aligned}$$

Lemma 3. *Let n, m be natural, $0 < p \leq 1$ and $T \in T_n$ then*

$$\|T\|_{\pi/m,p} \leq C(p)\left(1+\frac{n}{m}\right)^{1/p}\|T\|_p. \tag{16}$$

P r o o f. For $x \in [-\pi,\pi]$ let us denote by ξ_x a point such that

$$T_{\pi/m}(x) = \sup\{|T(y)| : |x-y| \leq \pi/(2m)\} = |T(\xi_x)| \text{ and } |x-\xi_x| \leq \pi/(2m).$$

Using the inequality $|a|^p - |b|^p \leq |a-b|^p$ where a and b are real $(0 < p \leq 1)$, we get

$$\begin{aligned}
\|T\|_{\pi/m,p}^p - \|T\|_p^p &= \int_{-\pi}^{\pi} |T_{\pi/m}(x)|^p\,dx - \int_{-\pi}^{\pi} |T(x)|^p\,dx \\
&= \int_{-\pi}^{\pi} |T(\xi_x)|^p\,dx - \int_{-\pi}^{\pi} |T(x)|^p\,dx \\
&\leq \int_{-\pi}^{\pi} [|T(\xi_x)| - |T(x)|]^p\,dx \leq \int_{-\pi}^{\pi} |\int_x^{\xi_x} T'(t)\,dt|^p\,dx \\
&\leq \int_{-\pi}^{\pi} |\int_x^{\xi_x} |T'(t)|\,dt|^p\,dx \leq \int_{-\pi}^{\pi} [\int_{x-\pi/(2m)}^{x+\pi/(2m)} |T'(t)|\,dt]^p\,dx
\end{aligned}$$

$$= \int_{-\pi}^{\pi}\left[\int_{-\pi/(2m)}^{\pi/(2m)} |T'(x+t)|\,dt\right]^p dx$$

$$= \int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} \chi_{[-\pi/(2m),\pi/(2m)]}(t)|T'(x+t)|\,dt\right]^p dx\ ,$$

where $\chi_{[-\pi/(2m),\pi/(2m)]}(t)$ is the characteristic function of the interval $[-\frac{\pi}{2m},\frac{\pi}{2m}]$.

We shall use (11), (15) and the following Nikolski inequality between different metrics (see [7] theorem 4.9.2 or [4]).

If $P \in T_n,\ 0 < p_2 < p_1 \le \infty$, then

$$\left[\int_{-\pi}^{\pi} |P(t)|^{p_1}\,dt\right]^{1/p_1} \le C(p_1,p_2)n^{1/p_2-1/p_1}\left[\int_{-\pi}^{\pi} |P(t)|^{p_2}\,dt\right]^{1/p_2}. \tag{17}$$

We obtain $(p_1 = 1,\ p_2 = p,\ P(t) = \Phi_{r,m}(t)T'(x+t),\ r > 1/2p)$

$$\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} \chi_{[-\pi/(2m),\pi/(2m)]}(t)|T'(x+t)|\,dt\right]^p dx$$

$$\le \int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} \Phi_{r,m}(t)|T'(x+t)|\,dt\right]^p dx$$

$$\le C(p)[r(2m-1)+n]^{1-p}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi^p_{r,m}(t)|T'(x+t)|^p\,dt\,dx$$

$$\le C(p,r)[r(2m-1)+n]^{1-p}\left(\frac{1}{m}\right)\int_{-\pi}^{\pi} |T'(x)|^p\,dx.$$

Using Bernstein's inequality (see [6] or [5], $0 < p \le 1$), we get

$$\begin{aligned}\|T\|^p_{\pi/m,p} &\le \|T\|^p_p + C(r,p)[r(2m-1)+n]^{1-p}\left(\frac{1}{m}\right)n^p\|T\|^p_p\\ &\le C(r,p)(1+n/m)^{1-p}(n/m)^p\|T\|^p_p\\ &\le C(p)(1+n/m)\|T\|^p_p.\end{aligned}$$

Corollary 1. *If $T \in T_n$ and $n/m \le C$, then*

$$\|T\|_p \le \|T\|_{\pi/m,p} \le C(p)\|T\|_p,\quad 0 < p \le 1. \tag{18}$$

Let $f \in L_\infty[-\pi,\pi]$, r, n be natural, $G \in T_m$. Define

$$(19)\ S^{\pm}(f,x) := G(x) \pm n\pi^{-1}\int_{-\pi}^{\pi}\Phi_{r,n}(x-t).\sup\{|(f-G)(y)| :\ y\in U(\frac{2\pi}{n},t\}\,dt.$$

Lemma 4. [3] *The polynomials* $S^{\pm}(f,x)\in T_{\max(m,N)}$, $N=r(2n-1)$, r, n *are natural and*

$$S^{-}(f,x)\le f(x)\le S^{+}(f,x). \tag{20}$$

Let $F_{r,n}(x) := [[\sin nx/2]/[n\sin x/2]]^{2r}$, be the general Jackson kernel.
We have $F_{r,n}(x)\in T_{r(n-1)}$.
Now let

$$D_m(x) := \left[[\sin\frac{2m+1}{2}x]/[2\sin\frac{x}{2}]\right]$$

be the Dirichlet kernel.
We have that

$$D_m(x_{k,2m}) := \begin{cases}\frac{2m+1}{2} & \text{if } k=0\\ 0 & \text{if } k=1,2,\ldots,2m.\end{cases}$$

Let us define ($N=r(n-1)$)

$$P_{r,n}(x) := D_{2N}(x)F_{r,n}(x),$$

$$L_{3N}(f,x) := \frac{2}{4N+1}\sum_{k=0}^{4N} f(x_{k,4N})P_{r,n}(x-x_{k,4N}).$$

Because of $P_{r,n}\in T_{3N}$, we have that $L_{3N}\in T_{3N}$.
Clear that $L_{3N}(f,x_{k,4N})=f(x_{k,4N})$, $k=0,1,\ldots,4N$.

Lemma 5. *Let* $T\in T_N$, $N=r(n-1)$, *then*

$$L_{3N}(T,x)=T(x). \tag{21}$$

P r o o f. Let us recall the interpolation polynomial

$$I_m(f,x) := \frac{2}{2m+1}\sum_{k=0}^{2m} f(x_{k,2m})D_m(x-x_{k,2m}).$$

For every $R \in T_m$, we have $R(x) = I_m(R, x)$.
Then in case $R(t,x) = T(x)F_{r,n}(t-x)$, $T \in T_{r(n-1)}$, we have $(N = r(n-1),\ R \in T_{2N})$

$$R(t,x) = T(x)F_{r,n}(t-x) = \frac{2}{4N+1}\sum_{k=0}^{4N} T(x_k)F_{r,n}(t-x_k)D_{2r(n-1)}(x-x_k),\ x_k = x_{k,4N}.$$

For $t = x$, we get $(F_{r,n}(0) = 1)$

$$\begin{aligned} R(x,x) = T(x) &= \frac{2}{4N+1}\sum_{k=0}^{4N} T(x_k)\left[F_{r,n}(x-x_k)D_{2r(n-1)}(x-x_k)\right] \\ &= \frac{2}{4N+1}\sum_{k=0}^{4N} T(x_k)P_{r,n}(x-x_k) = L_{3N}(T,x). \end{aligned}$$

Lemma 6. *For* $f \in L_\infty[-\pi,\pi]$, $N = r(n-1)$, $0 < p \le 1$, *we have* $(r > \frac{1}{2p})$

$$\|L_{3N}(f)\|_p^p \le C(p,r)\|f\|_{L^p_{4N}}. \tag{22}$$

Proof. From the definition of $L_{3N}(f,x)$ and $(a+b)^p \le a^p + b^p$ for every $a,\ b \ge 0$, we get

$$\|L_{3N}(f)\|_p^p \le \frac{2}{4N+1}\sum_{k=0}^{4N} |f(x_k)|^p \int_{-\pi}^{\pi} |P_{r,n}(x)|^p\,dx. \tag{23}$$

Now we have

$$\int_{-\pi}^{\pi} |P_{r,n}(x)|^p\,dx = 2\left(\int_0^{\pi/(4N+1)} + \int_{\pi/(4N+1)}^{\pi}\right)|P_{r,n}(x)|^p\,dx = A_1 + A_2\ . \tag{24}$$

$$\begin{aligned} A_1 &\le 2\int_0^{\pi/(4N+1)} [(4N+1)x/2]^p/[2(2/\pi)(x/2)]^p.[(nx/2)/[n(2/\pi)(x/2)]]^{2rp}\,dx \\ &\le C(r,p)(4N+1)^{p-1}. \qquad (25) \end{aligned}$$

$$A_2 \le 2\int_{\pi/(4N+1)}^{\pi} [1/[2(2/\pi)(x/2)]]^p[1/[n(2/\pi)(x/2)]]^{2rp}\,dx$$

Let $v = (4N+1)x/\pi$. Then

$$
\begin{aligned}
A_2 &\le C(r,p)\frac{(4N+1)^{2rp+p}}{(4N+1)n^{2rp}}\int_1^{4N+1}\frac{dv}{v^{2rp+p}}\\
&\le C(r,p)(4N+1)^{p-1}\int_1^{\infty}\frac{dv}{v^{2rp+p}} \qquad (26)\\
&\le C(r,p)(4N+1)^{p-1}
\end{aligned}
$$

From (23), (24), (25) and (26), we obtain

$$
\|L_{3N}(f)\|_p^p \le C(r,p)\left[\frac{2\pi}{4N+1}\sum_{k=0}^{4N}|f(x_k)|^p\right] = C(r,p)\|f\|_{4N}^p.
$$

Lemma 7. *Let $f \in L_\infty[-\pi,\pi]$, then*

$$
\|L_{3N}(f)\|_{1/N,p} \le C(p,r)\|f\|_{1/N,p}. \tag{27}
$$

Proof. Using (18), (22) for $T = L_{3N}(f)$, $L_{3N}(f, x_{k,4N}) = f(x_{k,4N})$ and (10), we get

$$
\begin{aligned}
\|L_{3N}(f)\|_{1/N,p} &\le C(p,r)\|L_{3N}(f)\|_p\\
&\le C(p)\|L_{3N}(f)\|_{L_{4N}^p}\\
&\le C(p,r)\|f\|_{L_{4N}^p}\\
&\le C(p,r)\|f\|_{1/N,p}.
\end{aligned}
$$

2. Main results. We shall prove the equivalence between the best onesided approximation of a 2π -periodic bounded measurable function with the trigonometric polynomials of order n in the quasi-norm L_p and the best approximation of this function with trigonometric polynomials of order n in the quasi-norm $\|\cdot\|_{\frac{1}{n},p}$, $0 < p \le 1$.

We shall express the relationship between the best onesided approximation of the functions and their error in the interpolation with operator $L_{3N}(f,x)$ in locally global quasi-norm $\|\cdot\|_{\frac{1}{n},p}$, $0 < p \le 1$.

Theorem 1. *Let $f \in L_\infty[-\pi,\pi]$. Then (n natural and $0 < p \le 1$)*

$$
\tilde{E}_n^T(f)_p \le C(p)E_n^T(f)_{1/n,p} \le \tilde{E}_n^T(f)_p. \tag{28}
$$

Proof. Let $T^-(x) \le f(x) \le T^+(x)$, $T^\pm \in T_n$ such that $\tilde{E}_n^T(f)_p = \|T^+ - T^-\|_p$. Using (18) with $m = n$, we get

$$\begin{aligned} E_n^T(f)_{1/n,p} &\le \tilde{E}_n^T(f)_{1/n,p} \le \|T^+ - T^-\|_{1/n,p} \\ &\le C(p)\|T^+ - T^-\|_p \\ &\le C(p)\tilde{E}_n^T(f)_p. \end{aligned}$$

For the proof of the other inequality we shall use the polynomials $S^\pm$ which are given in (19) with $G \in T_n$ such that

$$E_n^T(f)_{1/n,p} = \|f - G\|_{1/n,p}.$$

We will prove $\|S^+ - S^-\|_p \le C(p)E_n^T(f)_{1/n,p}$, $0 < p \le 1$. From (12), (13) and (14), we obtain

$$\begin{aligned} S^+(f,x) - S^-(f,x) &= \frac{2n}{\pi}\int_{-\pi}^{\pi} \Phi_{r,n}(t)(f-G)_{2\pi/n}(x-t)\,dt \\ &= \frac{2n}{\pi}\left[\sum_{m=1}^{n-1}\int_{m\pi/n}^{\frac{(m+1)\pi}{n}} + \int_{-\pi/n}^{\pi/n} + \sum_{m=-n+1}^{-1}\int_{\frac{(m-1)\pi}{n}}^{\frac{m\pi}{n}}\right] \\ &\qquad .\Phi_{r,n}(t)(f-G)_{2\pi/n}(x-t)\,dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

$$\begin{aligned} I_1 &\le C(r)\sum_{m=1}^{n-1} m^{-2r} n \int_{m\pi/n}^{\frac{(m+1)\pi}{n}} (f-G)_{2\pi/n}(x-t)\,dt \\ &\le C(r)\sum_{m=1}^{n-1} m^{-2r} n \int_{m\pi/n}^{\frac{(m+1)\pi}{n}} (f-G)_{4\pi/n}(x-m\pi/n)\,dt \\ &\le C(r)\sum_{m=1}^{n-1} m^{-2r}(f-G)_{4\pi/n}(x-m\pi/n)\,dt. \end{aligned}$$

Analogously we can get

$$I_3 \le C(r)\sum_{m=-n+1}^{-1} |m|^{-2r}(f-G)_{4\pi/n}(x-m\pi/n).$$

$$I_2 \leq 2n \int_{-\pi/n}^{\pi/n} C(r)(f-G)_{4\pi/n}(x)\, dt \leq C(r)(f-G)_{4\pi/n}(x).$$

From the estimations of I_1, I_2 and I_3 we get

$$\begin{aligned} S^+(f,x) - S^-(f,x) &\leq \sum_{m=1}^{n} m^{-2r}[(f-G)_{4\pi/n}(x - m\pi/n) \\ &+ (f-G)_{4\pi/n}(x + m\pi/n)] + (f-G)_{4\pi/n}(x) \end{aligned}$$

From the last inequality and

$$\left[\sum_{m=1}^{n} a_m\right]^p \leq \sum_{m=1}^{n} a_m^p \quad \text{for} \quad a_m \geq 0, \quad 0 < p \leq 1,$$

we obtain

$$|S^+(f,x) - S^-(f,x)|^p \leq C(p,r) \sum_{m=0}^{\infty} (m+1)^{-2rp} |(f-G)_{4\pi/n}(x - 2m\pi/n)|^p.$$

If we integrate the last inequality and using (8) and (9), we get ($r > 1/(2p)$).

$$\begin{aligned} \|S^+ - S^-\|_p &\leq C(p,r)\|f-G\|_{4\pi/n,p} \\ &\leq C(p,r)\|f-G\|_{1/n,p} \leq C(r,p) E_n^T(f)_{1/n,p}. \end{aligned}$$

Therefore

$$\tilde{E}_n^T(f)_p \leq \|S^+ - S^-\|_p \leq C(p) E_n^T(f)_{1/n,p}.$$

Theorem 2. *Let $f \in L_\infty[-\pi,\pi]$. Then ($0 < p \leq 1$, n natural, $N = r(n-1)$ and $r > 1/2p$)*

$$\tilde{E}_{3N}^T(f)_p \leq C(p)\|f - L_{3N}(f)\|_{1/N,p} \leq C(p)\tilde{E}_N^T(f)_p. \tag{29}$$

P r o o f. From Theorem 1, we get ($L_{3N}(f) \in T_{3N}$)

$$\tilde{E}_{3N}^T(f)_p \leq C(p) E_{3N}^T(f)_{1/N,p} \leq C(p)\|f - L_{3N}(f)\|_{1/N,p} \tag{30}$$

The inverse.

Let $T \in T_N$ be such that $\|f-T\|_{1/N,p} = E_N^T(f)_{1/N,p}$. Using (5), (21) and (27), we obtain

$$\begin{aligned}\|f - L_{3N}(f)\|_{1/N,p} &\le 2^{1/p-1}\left\{\|f-T\|_{1/N,p} + \|L_{3N}(f-T)\|_{1/N,p}\right\}\\ &\le C(p)E_N^T(f)_{1/N,p} + C(p)\|f-T\|_{1/N,p}\\ &\le C(p)E_N^T(f)_{1/N,p} \le C(p)\tilde{E}_N^T(f)_{1/N,p}.\end{aligned}$$

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