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CANONICAL CONNECTIONS AND THEIR CONFORMAL INVARIANTS ON RIEMANNIAN ALMOST-PRODUCT MANIFOLDS

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ABSTRACT. On a Riemannian P-manifold there is no complete analogy with the conformal geometry on a Riemannian manifold. In this paper, we consider a class of Riemannian almost-product manifolds (including Riemannian P-manifold). The general conformal group and its special subgroups are determined. It is shown that the Bochner curvature tensor of the manifold is a conformal invariant. It is proved that the zero Bochner curvature tensor is an integrability condition of a geometrical system of partial differential equations and a characterization condition of a conformally flat manifold, there is a complete analogy with the conformal geometry on a Riemannian manifold. Similar problems are considered in [1] for a complex manifold with B-metric.

1. Riemannian almost-product manifold. Let (M, g, P) be $2n$-dimensional Riemannian almost-product manifold, i.e. P is the almost-product structure and g is the metric on M such that:

$$P^2 x = x, \quad g(Px, Py) = g(x, y)$$

for all vector fields x, y on M. The associated metric $\tilde{g}$ of the manifold is given by $\tilde{g}(x, y) = g(x, P_y)$. In this paper we consider a Riemannian almost-product manifold for which $tr P = 0$. In this case the metric $\tilde{g}$ is necessarily of signature (n, n).

Further, x, y, z, u will stand for arbitrary differentiable vector fields on M. The Levi-Civita connection of g will be denoted by ∇. The tensor field F of type $(0,3)$ on the manifold is defined by $F(x, y, z) = g((\nabla_x P)y, z)$. This tensor has the following symmetry properties:

(1) $$F(x, y, z) = F(x, z, y);$$

(2) $$F(x, Py, Pz) = -F(x, y, z).$$

A classification of the Riemannian almost-product manifolds with respect to the tensor F is given in [2].

Let $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{g}$. Then, $\tilde{\nabla}_x y - \nabla_x y$ is a tensor field of type $(1,2)$ on M. We denote

$$\Phi(x,y) = \tilde{\nabla}_x y - \nabla_x y. \tag{3}$$

This is the fundamental tensor of the manifold.

Since ∇ and $\tilde{\nabla}$ are torsion free, $\Phi(x,y) = \Phi(y,x)$. The corresponding tensor of type $(0,3)$ is denoted by the same letter:

$$\Phi(x,y,z) = g(\Phi(x,y),z).$$

Further, x, y, z, u will stand for arbitrary vectors in the tangential space T_pM to M at an arbitrary point p in M. If $\{e_i\}$ $(i = 1, 2, \ldots, 2n)$ is an arbitrary basis of T_pM and g^{ij} are the components of the inverse matrix of g, then the vector field $tr\Phi$ is defined by $tr\Phi = g^{ij}\Phi(e_i, e_j)$. The form α associated with the tensor F is defined by

$$\alpha(x) = g^{ij}F(e_i, e_j, x).$$

Using (1), (2) and (3), we obtain the relations between the tensors F and Φ:

$$\Phi(x,y,z) = [F(x,y,Pz) + F(y,Pz,x) - F(Pz,x,y)]/2, \tag{4}$$

$$F(x,y,z) = \Phi(x,y,Pz) + \Phi(x,z,Py). \tag{5}$$

The Nijenhuis tensor N of the manifold is given by

$$N(x,y) = [Px,Py] + [x,y] - P[Px,y] - P[x,Py].$$

By using the covariant derivative $(\nabla_x P)y$ of P this tensor is expressed by the equality:

$$N(x,y) = (\nabla_x P)Py - (\nabla_y P)Px + (\nabla_{Px} P)y - (\nabla_{Py} P)x. \tag{6}$$

The associated tensor $\tilde{N}$ with N is defined, by

$$\tilde{N}(x,y) = (\nabla_x P)Py + (\nabla_y P)Px + (\nabla_{Px} P)y + (\nabla_{Py} P)x. \tag{7}$$

Taking into account (4), (5), (6) and (7), we have

$$g(N(x,y),z) = 2\Phi(z,x,y) + 2\Phi(z,Px,Py), \tag{8}$$

$$\tilde{N}(x,y) = -2\Phi(x,y) - 2\Phi(Px,Py). \tag{9}$$

Further we have

Lemma 1. *On a Riemannian almost-product manifold the following conditions are equivalent:*

1) $\Phi(x,y) = \Phi(Px,Py)$, 2) $\Phi(Px,y) = -P\Phi(x,y)$, 3) $N(x,y) = 0$.

Proof. Using the property $\tilde{N}(Px,y) = -P\tilde{N}(x,y)$, from (9) we get

$$\Phi(x,y) + \Phi(Px,Py) + P\Phi(Px,y) + P\Phi(x,Py) = 0. \tag{10}$$

If $\Phi(x,y) = \Phi(Px,Py)$, then (10) implies $\Phi(Px,y) = -P\Phi(x,y)$. Hence $\Phi(x,Py,Pz) = -\Phi(x,y,z)$. Now, taking into account (8), we find $N = 0$. So, we proved the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3). The implication 2) $\Rightarrow$ 1) is trivial; 3) $\Rightarrow$ 2) follows from (8). □

From (9) one obtains immediately

Lemma 2. *On a Riemannian almost-product manifold the following conditions are equivalent:* 1) $\Phi(x,y) = -\Phi(Px,Py)$, 2) $\tilde{N}(x,y) = 0$.

In [2] the thirty-six different classes of Riemannian almost-product manifolds are characterized by conditions for the tensor F. The fundamental classes in the case $trP = 0$ are:

$$\begin{aligned}
\bar{w}_1 &: \ F(A,A,\xi) = 0, \ F(\xi,\eta,A) = 0;\\
\bar{w}_2 &: \ F(A,B,\xi) = F(B,A,\xi), \ \alpha^v = 0, \ F(\xi,\eta,A) = 0;\\
\bar{w}_3 &: \ F(A,B,\xi) = g(A,B)\alpha^v(\xi)/n, \ F(\xi,\eta,A) = 0;\\
\bar{w}_4 &: \ F(\xi,\xi,A) = 0, \ F(A,B,\xi) = 0;\\
\bar{w}_5 &: \ F(\xi,\eta,A) = F(\eta,\xi,A), \ \alpha^h = 0, \ F(A,B,\xi) = 0;\\
\bar{w}_6 &: \ F(\xi,\eta,A) = g(\xi,\eta)\alpha^h(A)/n, \ F(A,B,\xi) = 0,
\end{aligned}$$

where $PA = A$, $PB = B$, $P\xi = -\xi$, $P\eta = -\eta$, $\alpha^v(x) = (\alpha(x) - \alpha(Px))/2$ and $\alpha^h(x) = (\alpha(x) + \alpha(Px))/2$.

We denote

$$w_1 = \bar{w}_3 \oplus \bar{w}_6, \quad w_2 = \bar{w}_2 \oplus \bar{w}_5, \quad w_3 = \bar{w}_1 \oplus \bar{w}_4.$$

From Lemmas 1 and 2 we obtain characteristics of the eight classes with respect to the fundamental tensor Φ. Below, we give these two types of characterization conditions.

1. The class w_0 of Riemannian P-manifolds:

I. $F(x,y,z) = 0$;

II. $\Phi(x,y) = 0$.

2. The class w_1:

I. $F(x,y,z) = [g(x,y)\alpha(z) + g(x,z)\alpha(y) - g(x,Py)\alpha(Pz) - g(x,Pz)\alpha(Py)]/2n$;

II. $\Phi(x,y) = [g(x,y)tr\Phi - g(x,Py)Ptr\Phi]/2n$.

3. The class w_2:

I. $F(x,y,Pz) + F(y,z,Px) + F(z,x,Py) = 0$, $\alpha = 0$, $N(x,y) = 0$, $\alpha = 0$;

II. $\Phi(x,y) = \Phi(Px,Py)$, $tr\Phi = 0$, $\Phi(Px,y) = -P\Phi(x,y)$, $tr\Phi = 0$.

4. The class w_3:

I. $F(x,y,z)+F(y,z,x)+F(z,x,y)=0, \qquad \tilde{N}(x,y)=0;$

II. $\Phi(x,y)=-\Phi(Px,Py).$

5. The class $w_1 \oplus w_2$:

I. $F(x,y,Pz)+F(y,z,Px)+F(z,x,Py)=0, \qquad N(x,y)=0;$

II. $\Phi(x,y)=\Phi(Px,Py) \quad \Phi(Px,y)=-P\Phi(x,y).$

6. The class $w_2 \oplus w_3$:

I. $\alpha=0;$

II. $tr\Phi=0.$

7. The class $w_1 \oplus w_3$:

I. $F(x,y,z)+F(y,z,x)+F(z,x,y)=\frac{1}{n}[g(x,y)\alpha(z)+g(x,z)\alpha(y)+g(y,z)\alpha(x)$
$-g(x,Py)\alpha(Pz)-g(x,Pz)\alpha(Py)-g(y,Pz)\alpha(Px)];$

II. $\Phi(x,y)+\Phi(Px,Py)=\frac{1}{n}[g(x,y)tr\Phi-g(x,Py)Ptr\Phi].$

8. The class of Riemannian almost-product manifolds ($trP=0$). No conditions.

2. Conformal transformations. Let (M,g,P) be a Riemannian almost-product manifold. We consider the following conformal transformations of the metric g:

$$\bar{g}=e^{2u}(ch2vg+sh2v\tilde{g}), \tag{11}$$

where u,v are differentiable functions on M. By $v=0$, (11) is the usual conformal change of g. The manifold $(M,\bar{g},P)$ is also a Riemannian almost-product manifold and $trP=0$.

Next, let us consider the local conformal transformations.

Theorem 1. *Let (M,g,P) be a w_1-manifold with form α and $(M,\bar{g},P)$ be conformally related to (M,g,P) by a transformation* (11). *Then $(M,\bar{g},P)$ is a w_1-manifold with form $\bar{\alpha}$, so that*

$$\bar{\alpha}=\alpha+2n(du\circ P-dv). \tag{12}$$

Proof. Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections of g and $\bar{g}$. Then

$$\begin{aligned}
&g(\bar{\nabla}_x y-\nabla_x y,z)=sh4v[F(x,y,z)+F(y,z,x)-F(z,x,y)]/4\\
&-sh^2 2v[F(x,y,Pz)+F(y,Pz,x)-F(Pz,x,y)]/2\\
&+du(x)g(y,z)+du(y)g(x,z)+dv(x)g(y,Pz)+dv(y)g(x,Pz)\\
&+g(x,y)[ch^2 2vdu(z)+sh^2 2vdv(Pz)+sh4vdu(Pz)/2-sh4vdv(z)/2]\\
&+g(x,Py)[-ch^2 2vdv(z)+sh^2 2vdu(Pz)-sh4vdu(z)/2+sh4vdv(Pz)/2].
\end{aligned} \tag{13}$$

Let $\bar{F}(x,y,z) = \bar{g}((\bar{\nabla}_x P)y, z)$. From (13) we obtain

$$(14)\quad \begin{aligned} 2\bar{F}(x,y,z) &= 2e^{2u}ch2vF(x,y,z) + e^{2u}sh2v[F(Py,z,x) \\ &-F(y,Pz,x) - F(z,x,Py) + F(Pz,x,y)] + \omega(y)g(x,z) \\ &+\omega(z)g(x,y) - \omega(Py)g(x,Pz) - \omega(Pz)g(x,Py), \end{aligned}$$

where $\omega = d(e^{2u}ch2v \circ P) - d(e^{2u}sh2v)$.

Now, let (M,g,P) be a w_1-manifold. Taking into account the form of F, from (14) we get

$$\bar{F}(x,y,z) = [\bar{g}(x,y)\bar{\alpha}(z) + \bar{g}(x,z)\bar{\alpha}(y) - \bar{g}(x,Py)\bar{\alpha}(Pz) - \bar{g}(x,Pz)\bar{\alpha}(Py)]/2n,$$

where $\bar{\alpha} = \alpha + 2n(du \circ P - dv)$ which completes the proof. □

Remark. Formulas (12), (13) and (14) are valid for arbitrary Riemannian almost-product manifolds.

Class w_1 is closed with respect to the conformal transformation according to this theorem.

Corollary 1. *Let (M,g,P) be a P-manifold. Every Riemannian almost-product manifold $(M,\bar{g},P)$ conformally equivalent to (M,g,P) by a transformation (11) is a w_1-manifold of the form $\bar{\alpha} = 2n(du \circ P - dv)$.*

Corollary 2. *Let (M,g,P) be a P-manifold. Every Riemannian almost-product manifold $(M,\bar{g},P)$ conformally equivalent to (M,g,P) by a transformation $\bar{g} = e^{2u}g$ is a w_1-manifold of a closed form $\bar{\alpha} = 2ndu \circ P$.*

Let (M,g,P) be a w_1-manifold. A differentiable function u on M is said to be P-pluriharmonic function if the form $du \circ P$ is closed. The differentiable function (u,v) on M is said to be P-holomorphic function, if $du \circ P = dv$ ($du = dv \circ P$).

Next we consider the following special conformal changes of metric g:

a) Conformal transformations of type I:

$$\bar{g} = e^{2u}g,$$

where u is a P-pluriharmonic function on M.

b) Conformal transformations of type II:

$$\bar{g} = e^{2u}(ch2vg + sh2v\tilde{g}),$$

where (u,v) is a P-holomorphic function on M.

c) Conformal transformations of type III:

$$\bar{g} = e^{2u}(ch2vg + sh2v\tilde{g}),$$

where u and v are P-pluriharmonic functions ($du \circ P \neq dv$) on M.

Corollary 3. *Let (M,g,P) be a w_1-manifold of form α. Then every Riemannian almost-product manifold $(M,\bar{g},P)$ conformally equivalent to (M,g,P) becomes a w_1-manifold of the same form α by a transformation of type II.*

Definition. *A w_1-manifold (M,g,P) of form α is said to be in class CP_0 if both forms α and $\alpha \circ P$ are closed.*

Corollary 4. *Class CP_0 is closed with respect to conformal transformations of type I, II or III.*

Theorem 2. *Let (M,g,P) be a w_1-manifold. The manifold is in class CP_0 iff it is conformally equivalent to a P-manifold by a transformation of type I or III.*

Proof. Let (M,g,P) be a w_1-manifold of form α. If it is conformally equivalent to a P-manifold from Theorem 1 we may obtain $\alpha = 2ndu \circ P$ ($\alpha = 2n(du \circ P - dv)$) by a conformal transformation of type I (type III). Hence, $d\alpha = d(\alpha \circ P) = 0$.

For the converse (α and $\alpha \circ P$ are closed), we solve locally the equation $\alpha \circ P = 2ndu$ ($\alpha = 2n(du \circ P - dv)$, $\alpha \circ P = 2n(du - dv \circ P)$) and find a P-pluriharmonic function u (P-pluriharmonic functions u and v). The conformal transformation $e^{-2u}g$ ($e^{-2u}(ch2vg - sh2v\tilde{g})$) gives rise to a P-manifold. □

3. Canonical connection on a w_1-manifold. Let (M,g,P) be a Riemannian almost-product manifold. A linear connection D on (M,g,P) is said to be natural if $DP = 0$ and $Dg = 0$ (or $Dg = 0$ and $D\tilde{g} = 0$). The natural connection D with torsion tensor T is said to be canonical if

$$T(x,y,z) + T(y,z,x) + T(Px,y,Pz) + T(y,Pz,Px) = 0,$$

where $T(x,y,z) = g(T(x,y),z)$ and $T(x,y) = D_x y - D_y x - [x,y]$.

Theorem A. [4] *There exists a unique canonical connection on a Riemannian almost-product manifold.*

If ∇ is the Levi-Civita connection on (M,g,P), then

$$\begin{aligned} g(D_x y - \nabla_x y, z) &= [\Phi(x,y,z) - 2\Phi(z,x,y) - \Phi(x,Py,Pz)]/4 \\ T(x,y,z) &= [\Phi(y,z,x) - \Phi(z,x,y) - \Phi(y,Pz,Px) + \Phi(Pz,x,Py)]/4. \end{aligned} \tag{15}$$

Thus, if (M,g,P) is a w_1-manifold, then its canonical connection D is given by

$$D_x y = \nabla_x y + [g(x,y)P\Theta - g(Px,y)\Theta + \alpha(y)Px - \alpha(Py)x]/4n, \tag{16}$$

where $g(\Theta,x) = \alpha(x)$. From this it follows that the torsion tensor T of D is of the form:

$$T(x,y) = [\alpha(Px)y - \alpha(x)Py + \alpha(y)Px - \alpha(Py)x]/4n.$$

For arbitrary vector fields x and y, $\alpha(PT(x,y)) = 0$.

Further, $\mathcal{R}$ will stand for the curvature tensor of ∇, i.e.

$$\mathcal{R}(x,y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z.$$

The corresponding tensor of type $(0,4)$ is denoted by the same letter and it is given by

$$\mathcal{R}(x,y,z,u) = g(\mathcal{R}(x,y)z,u).$$

A tensor L of type $(0,4)$ is said to be a curvature-like tensor if it satisfies the conditions:

i) $L(x,y,z,u) = -L(y,x,z,u)$;

ii) $L(x,y,z,u) + L(y,z,x,u) + L(z,x,y,u) = 0$;

iii) $L(x,y,z,u) = -L(x,y,u,z)$.

Let S be a tensor of type $(0,2)$. We consider the following tensors

$$\begin{aligned}\psi_1(S)(x,y,z,u) &= g(y,z)S(x,u) - g(x,z)S(y,u)\\ &+S(y,z)g(x,u) - S(x,z)g(y,u),\end{aligned}$$

$$\begin{aligned}\psi_2(S)(x,y,z,u) &= g(y,Pz)S(x,Pu) - g(x,Pz)S(y,Pu)\\ &+S(y,Pz)g(x,Pu) - S(x,Pz)g(y,Pu).\end{aligned}$$

We have

Lemma 3. *Let S be a tensor of type $(0,2)$. Then*

a) $\psi_1(S)$ is a curvature-like tensor iff $S(x,y) = S(y,x)$,

b) $\psi_2(S)$ is a curvature-like tensor iff $S(x,Py) = S(y,Px)$.

Tensors π_1, π_2 and π_3 are defined as follows:

$$\pi_1 = \psi_1(g)/2, \quad \pi_2 = \psi_2(g)/2, \quad \pi_3 = \psi_1(\tilde{g}) = \psi_2(\tilde{g}).$$

Tensors $\pi_1 + \pi_2$ and π_3 are curvature-like tensors and they satisfy the conditions

$$(\pi_1+\pi_2)(x,y,z,u) = (\pi_1+\pi_2)(x,y,Pz,Pu), \quad \pi_3(x,y,z,u) = \pi_3(x,y,Pz,Pu).$$

Lemma 4. *Let (M,g,P) be a w_1-manifold. If $\mathcal{R}$ and K are the curvature tensors of ∇ and D, respectively, then*

$$K = \mathcal{R} + \psi(S') - \psi_2(S'') - \alpha(\Theta)(\pi_1+\pi_2)/4n + \alpha(P|\Theta)\pi_3/4n,$$

where

$$S'(x,y) = (\nabla_x\alpha)Py + [\alpha(x)\alpha(y) + \alpha(Px)\alpha(Py)]/4n + \alpha((\nabla_x P)y),$$
$$S''(x,y) = (\nabla_x\alpha)Py + [\alpha(x)\alpha(y) + \alpha(Px)\alpha(Py)]/4n.$$

The statement follows by a direct computation taking into account (16).

Lemma 5. *Let (M, g, P) be a w_1-manifold of form α and Levi-Civita connection ∇. Then*

a) α is closed iff $(\nabla_x \alpha)y = (\nabla_y \alpha)x$;

b) $\alpha \circ P$ is closed iff $(\nabla_x \alpha)Py = (\nabla_y \alpha)Px$.

Theorem 3. *Let (M, g, P) be a CP_0-manifold with canonical connection D. If K is the curvature tensor of D, then K is a curvature-like tensor and satisfies the condition*

$$K(x, y, z, u) = K(x, y, Pz, Pu). \tag{17}$$

Proof. Using Lemma 5, we obtain that tensors S' and S'' in Lemma 4 satisfy the conditions of Lemma 3. Thus, Lemma 4 implies that K is a curvature-like tensor. From the condition $DP = 0$ it follows immediately (17). □

4. Conformal invariants. In this section we consider conformal transformations of type I and II of the metric and we find the groups of conformal transformations of the canonical connection. The Bochner curvature tensor of the canonical connection on a CP_0-manifold is shown to be a conformal invariant of type I, II or III.

Lemma 6. *Let (M, g, P) and $(M, \bar{g}, P)$ be conformally related w_1-manifolds by a transformation $\bar{g} = e^{2u}(ch2vg + sh2v\tilde{g})$ with differentiable functions u, v. The corresponding canonical connections D and $\bar{D}$ are related as follows:*

$$\begin{aligned} 2\bar{D}_x y &= 2D_x y + 2du(x)y + 2dv(x)Py \\ &+[du(y) + dv(Dy)]x + [du(Py) + dv(y)]Px \\ &-g(x, y)(\mathrm{grad}u + P\mathrm{grad}v) - g(x, Py)(P\mathrm{grad}u + \mathrm{grad}v). \end{aligned} \tag{18}$$

The proof is straightforward calculation from formulas (13) and (16).

The transformations $\bar{g} = e^{2u}(ch2vg + sh2v\tilde{g})$ with differentiable functions u, v on M form the (general) conformal group on M and give rise to the conformal group of transformations of the canonical connection on M. The formula (18) is an analytic expression of a conformal transformation D [4].

Lemma 6 implies:

Corollary 5. *Let (M, g, P) and $(M, \bar{g}, P)$ be conformally related w_1-manifolds by transformation (11) and let D, $\bar{D}$ be their canonical connections.*

If transformation (11) is of type I, then

$$2\bar{D}_x y = 2D_x y + 2\sigma(x)y + \sigma(y)x + \sigma(Py)Px - g(x, y)S - g(x, Py)PS. \tag{19}$$

If transformation (11) is of type II, then

(20) $\bar{D}_x y = D_x y + \sigma(x)y + \sigma(y)x + \sigma(Px)Py + \sigma(Py)Px - g(x,y)S - g(x,Py)PS.$

Here $\sigma = du$, $g(S,x) = du(x)$.

Formulas (19) and (20) express analytically the conformal groups of conformal transformations of the canonical connection of type I and II, respectively.

Using (19) and (20) we obtain

Lemma 7. *Let (M,g,P) and $(M,\bar{g},P)$ be conformally related CP_0-manifolds by transformation of type I. If K and $\bar{K}$ are the curvature tensors of the corresponding canonical connections, then*

$$\bar{K} = K - \psi_1(L) - \psi_2(L), \tag{21}$$

where

$$\begin{aligned} L(x,y) &= (\nabla_x \sigma)y - [\sigma(x)\sigma(y) + \sigma(Px)\sigma(Py) - g(x,y)\sigma(S)/2 - g(x,Py)\sigma(PS)/2]/2 \\ &\quad +[g(x,Py)\alpha(S) - g(x,y)\alpha(PS)]/4n \end{aligned}$$

and α is of the form (M,g,P).

Lemma 8. *Let (M,g,P) and $(M,\bar{g},P)$ be conformally related CP_0-manifolds by transformation of type II. If K and $\bar{K}$ are the curvature tensors of the corresponding canonical connections, then*

$$\bar{K} = K - \psi_1(L) - \psi_2(L), \tag{22}$$

where

$$\begin{aligned} L(x,y) &= (\nabla_x(y) - \sigma(x)\sigma(y) - \sigma(Px)\sigma(Py) + g(x,y)\sigma(S)/2 + g(x,Py)\sigma(PS)/2] \\ &\quad +[g(x,Py)\alpha(S) - g(x,y)\alpha(PS)]/4n. \end{aligned}$$

Let (M,g,P) be an arbitrary Riemannian almost-product manifold and $\{e_1, e_2, \dots, e_{2n}\}$ be a basis of T_pM, $p \in M$. If K is a curvature-like tensor and if it satisfies condition (17), then the Ricci tensor ρ and the scalar curvatures τ and $\tilde{\tau}$ of K are given by

$$\rho(x,y) = g^{ij}K(e_i,x,y,e_j), \quad \tau = g^{ij}\rho(e_i,e_j), \quad \tilde{\tau} = g^{ij}\tilde{\rho}(e_i,e_j).$$

The associated Bochner curvature tensor $\mathcal{B}(K)$ is defined by

(23) $\mathcal{B}(K) = K - (\psi_1 + \psi_2)(\rho)/2(n-2) + [\tau(\pi_1 + \pi_2) + \tilde{\tau}\pi_3]/4(n-1)(n-2).$

Theorem 4. *Let (M,g,P) be a CP_0-manifold with canonical connection D and corresponding curvature tensor K. Then the Bochner curvature tensor $\mathcal{B}(K)$ is a conformal invariant of type I or II.*

Proof. Let $(M, \bar{g}, P)$ be conformally related to (M, g, P) by transformation (11) of type I or II. If $\bar{D}$ and $\bar{K}$ are the canonical connection and its curvature tensor of $(M, \bar{g}, P)$, then (21) and (22) imply

$$L = (\rho - \bar{\rho})/2(n-2) - (\tau g + \tilde{\tau}\tilde{g} - \bar{\tau}\bar{g} - \bar{\tilde{\tau}}\bar{\tilde{g}})/8(n-1)(n-2),$$

where $\bar{\rho}$, $\bar{\tau}$, $\bar{\tilde{\tau}}$ are the associated Ricci tensor and the scalar curvatures of $\bar{K}$. Substituting L into (21), respectively (22), and taking into account (23), we obtain

$$\mathcal{B}(\bar{K}) = e^{2u}\mathcal{B}(K), \tag{24}$$

respectively

$$\mathcal{B}(\bar{K}) = e^{2u}[ch2v\mathcal{B}(K) + sh2v\mathcal{B}(\tilde{K})], \tag{24'}$$

where $\tilde{K}$ is the curvature tensor, given by $\tilde{K}(x,y,z,u) = K(x,y,z,Pu)$. Thus, if $\mathcal{B}(K)$ and $\mathcal{B}(\bar{K})$ are the corresponding tensors of type $(1,3)$, then (24) and (24′) imply $\mathcal{B}(K) = \mathcal{B}(\bar{K})$. □

5. The Bochner curvature tensor and integrability conditions. In this section we show that the zero Bochner curvature tensor of the canonical connection is an integrability condition for a system of PDE describing conformally flat (of type I, II or III) CP_0-manifolds.

Theorem B. [3]. *Let the Riemannian P-manifolds (M, g, P) and $(M, \bar{g}, P)$ ($\dim M = 2n \geq 8$) be conformally related by a transformation of type II. The connection $\bar{\nabla}$ of $\bar{g}$ is flat iff $\mathcal{B}(R) = 0$.*

Theorem 5. *Let (M, g, P) ($\dim M \geq 8$) be a P-manifold and vanishing Bochner curvature tensor of the Levi-Civita connection ∇. Then (M, g, P) is conformally related to a CP_0-manifold $(M, \bar{g}, P)$ by transformation of type I so that the canonical connection $\bar{D}$ of $(M, \bar{g}, P)$ is flat.*

Proof. Let $\bar{g} = e^{2u}g$, with an unknown P-pluriharmonic function u on M. Using (21) we obtain that the canonical connection $\bar{D}$ on $(M, \bar{g}, P)$ is flat iff

$$\begin{aligned}(\nabla_x\sigma)y - [\sigma(x)\sigma(y) + \sigma(Px)\sigma(Py) - g(x,y)\sigma(S)/2 - g(x,Py)\sigma(PS)/2]/2 \\ = \rho(x,y)/(n-2) - [\tau g(x,y) + \tilde{\tau} g(x,Py)]/4(n-1)(n-2),\end{aligned} \tag{25}$$

where $\sigma = du$, $g(S,x) = du(x)$, $\rho, \tau, \tilde{\tau}$ are the Ricci tensor and the scalar curvatures of the curvature tensor R of ∇.

Now we shall show $\mathcal{B}(R) = 0$ is an integrability condition for the system (25). Denoting the right hand side of (25) by $L(x,y)$, we have $L(x,y) = L(y,x)$ and $L(x,y) = L(Px,Py)$. Applying the Ricci identity

$$(\nabla_x\nabla_y\sigma)z - (\nabla_y\nabla_x\sigma)z = -\sigma(R(x,y)z)$$

to the left hand side of (25) and using $\mathcal{B}(R) = 0$, we find that the system is integrable iff

$$(\nabla_x L)(y,z) = (\nabla_y L)(x,z). \tag{26}$$

To prove (26) we use the equality $R = (\psi_1 + \psi_2)(L)$ and apply the second Bianchi identity for R. After a contraction we obtain

$$(2n-5)[\nabla_{Px} L)(Py,z) - (\nabla_{Py} L)(Px,z)] - [(\nabla_x L)(y,z) - (\nabla_y L)(x,z)] = 0$$

and by the substitution $x \mapsto Px$, $y \mapsto Py$

$$(2n-5)[(\nabla_x L)(y,z) - (\nabla_y L)(x,z)] - [(\nabla_{Px} L)(Py,z) - (\nabla_{Py} L)(Px,z)] = 0.$$

From the last equalities it follows that

$$(n-2)(n-3)[(\nabla_x L)(y,z) - (\nabla_y L)(x,z)] = 0.$$

Hence (26) is a consequence of $\mathcal{B}(R) = 0$ and (25) is integrable. It follows immediately that every solution u ($du = \sigma$) of (25) is a P-pluriharmonic function. The change $\bar{g} = e^{2u} g$ (u - a solution of (25)) gives rise to a CP_0-manifold $(M, \bar{g}, P)$ with flat canonical connection $\bar{D}$ and this completes the proof. □

If $2n = 6$ and $\mathcal{B}(R) = 0$, the integrability condition (26) is not a consequence of $\mathcal{B}(R) = 0$. In this case $\mathcal{B}(R) = 0$ implies only

$$(\nabla_x L)(y,z) - (\nabla_y L)(x,z) = (\nabla_{Px} L)(Py,z) - (\nabla_{Py} L)(Px,z).$$

Theorem 6. *Let (M, g, P) ($\dim M \geq 8$) be a CP_0-manifold with canonical connection D and vanishing Bochner curvature tensor of D. Then there exist P-pluriharmonic functions u and v on M, such that the conformal transformation* (11) *of type III gives rise to a P-manifold (M, g, P) with flat Levi-Civita connection $\bar{\nabla}$.*

Proof. Let α be the form of (M, g, P). Solving (locally) the equation $du' = \alpha/2n$, we obtain a Riemannian P-manifold (M, g', P) with $g' = e^{-2u'} g$. From Theorem 4 it follows that the Bochner curvature tensor $\mathcal{B}(R') = 0$, R' being the curvature tensor of the Levi-Civita connection ∇' of (M, g', P). Let the conformal transformation $\bar{g} = e^{2u''}(ch2vg' + sh2v\tilde{g}')$ be of type II with unknown function u'' and v. Applying Theorem B we obtain that the Riemannian P-manifold $(M, \bar{g}, P)$ is of flat Levi-Civita connection $\bar{\nabla}$. It follows immediately that the function $u = u'' - u'$ is a P-pluriharmonic function. Then the conformal transformation $\bar{g} = e^{2u}(ch2vg + sh2v\tilde{g})$ is of type III and the manifold $(M, \bar{g}, P)$ is of flat Levi-Civita connection $\bar{\nabla}$. □

Applying Theorems 2, 4 and Corollary 3, we check

Theorem 7. *Let (M, g, P) ($\dim M \geq 8$) be a CP_0-manifold with canonical connection D and vanishing Bochner curvature tensor of D. Then, (M, g, P) is conformally related to a CP_0-manifold $(M, \bar{g}, P)$ by a conformal transformation of type II, so that the canonical connection $\bar{D}$ of $(M, \bar{g}, P)$ is flat.*

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