Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SERDICA — Bulgaricae
mathematicae publicationes
18 (1992) 186-195

RADICALS OF CROSSED PRODUCTS

J. M. DIMITROVA

ABSTRACT. Let $K_\rho^\sigma G$ be a crossed product of the group G and the ring K with respect to the factor set ρ and the map σ. In this paper, using simple techniques, we prove that if G is an SN-group and K is a central simple F-algebra over an algebraically closed field F of characteristic zero, then the Jacobson radical $J(K_\rho^\sigma G)$ is trivial, i.e. $J(K_\rho^\sigma G) = 0$. Moreover, if H is a normal subgroup of G and G/H is a locally finite group, then $J(K_\rho^\sigma H)$ is contained in $J(K_\rho^\sigma G)$ for every ring K.

Let G be an arbitrary group and K be a ring with an additive group $K(+)$ without G-torsions. If K is a commutative ring with no nilpotent elements, then the upper nilradical $U(K_\rho G)$ is trivial for each twisted group ring $K_\rho G$. If K is a simple ring or a commutative integral domain, then $U(K_\rho^\sigma G) = 0$. Thereofore, if K is a semisimple ring and the order of any torsion element $g \in G$ is invertible in K, then $U(K_\rho G) = 0$.

Let G be a multiplicative group and K be an associative ring with an identity. Suppose that we are given a map $\sigma:\ g \to g\sigma$ from G to the group of automorphisms Aut K of K and a map $\rho:\ (g,h) \to \rho(g,h)$ from $G \times G$ to the group of units K^* of K. The family

$$\rho = \{\rho(g,h) \in K^* \mid g,h \in G\}$$

is called a factor set of G into K under the map σ if the equalities

$$\rho(g,hf)\rho(h,f) = \rho(gh,f)\rho(g,h)^{f\sigma},$$

$$\alpha^{g\sigma.h\sigma} = \rho(g,h)^{-1}\alpha^{(gh)\sigma}\rho(g,h)$$

hold for all $g,h,f \in G$ and $\alpha \in K$, where $\alpha^{g\sigma}$ denotes the image of α under the action of $g\sigma \in$ Aut K.

The ring $K * G = K_\rho^\sigma G$ is a crossed product [1] of the the group G over the ring K with respect to the factor set ρ and the map σ if $K * G$ is a free K-module with a basis $\overline{G} = \{\overline{g} \in K * G \mid g \in G\}$, where

$$\overline{g}\overline{h} = \overline{gh}\rho(g,h), \quad \alpha\overline{g} = \overline{g}\alpha^{g\sigma}$$

for all $g, h \in G$ and $\alpha \in K$. Thus every element $x \in K * G$ is uniquely written as a finite sum of the form

$$x = \overline{g}_1\alpha_1 + \overline{g}_2\alpha_2 + \cdots + \overline{g}_n\alpha_n \quad (\alpha_i \in K). \tag{1}$$

If σ maps G onto the identity automorphism of K, the crossed product $K * G$ is called a twisted group ring, which we denote by $K_\rho G$.

We shall denote by tr x the coefficient of the basis element $\overline{1}$ in the expression (1) of x. If

$$x = \sum_{g \in G} \overline{g}\alpha_g \quad (\alpha_g \in K),$$

then the set

$$\text{Supp } x = \{g \in G \mid \alpha_g \neq 0\}$$

is said to be the support of x. The support subgroup of x is $<$ Supp x $>$, i.e. the subgroup of G generated by the elements of Supp x.

The Jacobson radical of the ring R will be denoted by $J(R)$; the upper nilradical, by $U(R)$; the prime radical, by $P(R)$ and the Brown-McCoy radical, by $B(R)$. If $J(R) = 0$, then the ring R is called semiprimitive and if $B(R) = 0$, then R is said to be semisimple.

Let G_{ker} be the set of all elements of G mapped by σ into the subgroup of inner automorphisms of K. Then G_{ker} is a normal subgroup of G. Moreover, it is known [1] that if H is a subgroup of G, then $K_\rho^\sigma H$ is a subring of $K * G$ where ρ and σ are restricted upon $H \times H$ and H, respectively.

Let I be a totally ordered set. A set $(\wedge_i, \vee_i;\ i \in I)$ of pairs of subgroups of G is called a series of G if

1. $\vee_i$ is a normal subgroup of $\wedge_i$ for all $i \in I$.
2. $\wedge_i$ is a subgroup of $\vee_j$ whenever $i < j$.
3. $G \setminus 1 = \bigcup_{i \in I} (\wedge_i \setminus \vee_i)$.

The series is said to be Abelian if all the factors $\wedge_i / \vee_i$ are Abelian. A group with an Abelian series is called SN-group [8].

It was shown in [8] that every group algebra FG of an SN-group G over a field F of characteristic zero is semiprimitive. Similarly, it can be proved that the latter result can be generalized for crossed products as well. In addition, we prove the following result.

Theorem 1. *If G is an SN-group and K is a central simple F-algebra where F is an algebraically closed field of characteristic zero, then $K * G$ is semiprimitive.*

Proof. Suppose that $J(K * G) \neq 0$. It follows from [2] that

$$J(K * G) \cap K * G_{\text{ker}} \neq 0$$

and we have $J(K * G_{\text{ker}}) \neq 0$ since

$$J(K * G) \cap K * G_{\text{ker}} \subseteq J(K * G_{\text{ker}}).$$

Therefore, in order to establish that $J(K * G) = 0$, it is sufficient to show that $J(K * G_{\text{ker}}) = 0$.

Since the automorphism $g\sigma$ is inner for all $g \in G_{\text{ker}}$, we have $\alpha^{g\sigma} = \alpha_g \alpha \alpha_g^{-1}$ for each $\alpha \in K$ and some $\alpha_g \in K^*$. We set $\tilde{g} = \overline{g}\alpha_g$. Thus, $\alpha\tilde{g} = \tilde{g}\alpha$ yields

$$K_\rho^\sigma G_{\text{ker}} = K_{\tilde{\rho}}^{\tilde{\sigma}} G_{\text{ker}} = K_{\tilde{\rho}} G_{\text{ker}}.$$

So, in order to prove the theorem, it suffices to establish that $J(K_\rho G) = 0$, where G is an SN-group and K is a central simple F-algebra.

Let the element

$$y = \overline{g}_1\gamma_1 + \overline{g}_2\gamma_2 + \cdots + \overline{g}_n\gamma_n$$

from $J(K_\rho G)$ be of minimal length $||y|| = n$. According to [2], we may choose y so that $\gamma_1 = 1$.

Let $\alpha \neq 0$ be an arbitrary element of K. Consider the element

$$\alpha y - y\alpha = \sum_{i=2}^{n} \overline{g}_i(\alpha\gamma_i - \gamma_i\alpha).$$

It belongs to $J(K_\rho G)$ and its length $||\alpha y - y\alpha||$ is smaller than n. Thus, $\alpha y - y\alpha = 0$, i.e. $\alpha y = y\alpha$ which yields $\alpha\gamma_i = \gamma_i\alpha$ for all $\alpha \in K$. Hence, γ_i are elements of the center F. On the other hand, $\sigma = 1$ yields $\rho(g,h)\alpha = \alpha\rho(g,h)$ for all $g, h \in G$ and $\alpha \in K$, i.e. F contains the factor set ρ. So, the twisted group ring $F_\rho G$ exists and y belongs to it. Therefore, according to [6], in order to prove that $J(K_\rho G) = 0$, it is sufficient to show that $J(F_\rho G) = 0$.

Suppose that $J(F_\rho G) \neq 0$ and $x \in J(F_\rho G)$ is a nonzero element. Now we can apply the approach from [8]. Let x be represented in the form

$$x = \overline{g}_1\alpha_1 + \overline{g}_2\alpha_2 + \cdots + \overline{g}_n\alpha_n.$$

As $J(F_\rho G)$ is an ideal of $F_\rho G$, we can assume that $g_1 = 1$. Let $H = <\text{Supp } x>$ be the support subgroup of x. Then x belongs to $J(F_\rho H)$. Furthermore, H is an SN-group and there exists a series $(\wedge_i, \vee_i;\ i \in I)$ of H with cyclic $\wedge_i/\vee_i$ of prime order for each $i \in I$. Since H is finitely generated, there exists $j \in I$ so that $\wedge_j = H$. We set $\vee_j = W$. Then $H = <W, t>$ where $t^p \in W$ for a prime integer p. Thus, each element u of $F_\rho H$ can be written as

$$u = \sum_{i=0}^{p-1} \overline{t^i}\beta_i \quad (\beta_i \in F_\rho W).$$

Let μ be a primitive p-root of unity of F. We define the map

$$\varphi: \sum_{i=0}^{p-1} \overline{t^i}\beta_i \to \sum_{i=0}^{p-1} \overline{t^i}\beta_i\mu^i.$$

It is clear that φ is an automorphism of $F_\rho H$.

Let $g_i = t^{m_i}w_i$ where $w_i \in W$ and $0 \leq m_i \leq p-1$. Then $g_1 = 1$ implies $t^{m_1}w_1 = 1$ which yields $m_1 = 0$ and $w_1 = 1$. Since

$$H = < g_1, g_2, \ldots, g_n > = < W, t >,$$

there exists $m_j \neq 0$. Otherwise, all $g_i \in W$ and W contains H which is impossible. Let $m_2 \neq 0$.

The radical $J(F_\rho H)$ is invariant under the automorphisms of $F_\rho H$. Hence,

$$\varphi(x) = \sum_{i=1}^{n} \overline{t^{m_i}w_i}\alpha_i\mu^{m_i} = \sum_{i=1}^{n} \overline{t^{m_i}}\gamma_i\mu^{m_i} \in J(F_\rho H),$$

where $\gamma_i = \overline{w_i}\rho(t^{m_i}, w_i)^{-1}\alpha_i$. Since $J(F_\rho H)$ is an ideal of $F_\rho H$, the product

$$x\mu^{m_2} = \sum_{i=1}^{n} \overline{t^{m_i}}\gamma_i\mu^{m_2}$$

also belongs to $J(F_\rho H)$. Thus,

$$z = \varphi(x) - x\mu^{m_2} \in J(F_\rho H)$$

and $||z|| < ||x||$.

Going on, we come to the conclusion that there exists an SN-group, say M, with an element of length 1 in $J(F_\rho M)$. Then it follows, that $J(F_\rho M) = F_\rho M$ which is impossible since $F_\rho M$ is not a radical ring. The contradiction proves the theorem. □

The approach used for Lemma 4.2 from [11] can be similarly applied to prove the following result.

Lemma. *Let F be a field and x be an element of $F_\rho G$. In the case of* char $F = p > 0$, *we suppose that* Supp x *contains no p-element. Then* tr $x = 0$ *if x is nilpotent.*

We should recall that the additive group $K(+)$ of the ring K has no G-torsions if $n\alpha = 0$ yields $\alpha = 0$ for any $\alpha \in K$ and any integer n which is the order of any torsion element of G.

It was proved in [11] that the group ring KG has no nonzero nil ideals if $K(+)$ has no G-torsions and $U(K) = 0$. Using a similar approach we shall expand this result for crossed products.

Theorem 2. *Let G be an arbitrary group and K be a ring with $K(+)$ having no G-torsions. Then*

1. If K is a commutative ring without nonzero nilpotent elements, then $U(K_\rho G) = 0$ for any twisted group ring $K_\rho G$.

2. If K is a simple ring or a commutative integral domain, then $U(K_\rho^\sigma G) = 0$ for any crossed product $K_\rho^\sigma G$.

Proof. 1. Let K be a commutative ring and $U(K) = 0$. Suppose that $U(K_\rho G) \neq 0$ and let the element

$$x = \sum_{i=1}^{n} \overline{g_i}\alpha_i \in U(K_\rho G)$$

be nonzero. As $U(K_\rho G)$ is an ideal of $K_\rho G$, we can assume that $g_1 = 1$. If Supp x contains elements of prime power order, let m be the product of all prime q such that Supp x contains q-element. Otherwise, we set $m = 1$. So, $K(+)$ has no m-torsion as $K(+)$ has no G-torsions. Thus, $m\alpha_1 \neq 0$.

The ring K is commutative and therefore $P(K) = U(K) = 0$. Thus, $m\alpha_1 \notin P(K)$ and hence, there exists a prime ideal P of K with $m\alpha_1 \notin P$. According to [3], $K_\rho G/(K_\rho G)P \cong (K/P)_{\bar{\rho}}G$ where K/P is a commutative integral domain. Let F be the quotent field of K/P. Then $(K/P)_{\bar{\rho}}G \subseteq F_{\bar{\rho}}G$ and

$$\overline{x} = \sum_{i=1}^{n} \overline{g_i}(\alpha_i + P)$$

is a nilpotent element of $F_{\bar{\rho}}G$. Suppose that char $F = p > 0$. If $m = 1$, then Supp x contains no p-element. If $m > 1$, then m and p are relatively prime since $m(\alpha_1 + P) \neq 0$. So, Supp x contains no p-element again. The Lemma, applied to the nilpotent element $\overline{x}$, results in $\alpha_1 + P = 0$. The latter is a contradiction and we conclude that $U(K_\rho G) = 0$.

2. Let K be a simple ring and suppose that $U(K * G) \neq 0$. According to [2], we have $U(K * G) \cap K * G_{\mathbf{ker}} \neq 0$ and therefore $U(K * G_{\mathbf{ker}}) \neq 0$. Just as in the latter theorem $K_\rho^\sigma G_{\mathbf{ker}} = K_{\bar{\rho}} G_{\mathbf{ker}}$ and we can choose a nonzero element

$$x = \sum_{i=1}^{n} \overline{g_i}\alpha_i \in K_{\bar{\rho}} G_{\mathbf{ker}}$$

of minimal length for which $g_1 = 1$ and $\alpha_1 = 1$. Following the proof of the same theorem, we obtain that there exists the twisted group ring $F_{\bar{\rho}}G_{\mathbf{ker}}$ where F is a field and $x \in F_{\bar{\rho}}G_{\mathbf{ker}}$.

If char $F = p > 0$, Supp x has no p-element, since $K(+)$ has no G-torsions. Thus, according to the Lemma, tr $x = 0$ in contradiction with tr $x = \alpha_1 = 1$. Hence, $U(K * G) = 0$ for each simple ring K.

Let K be a commutative integral domain. Suppose that $U(K * G) \neq 0$. It follows from [6] that $U(K_\rho G_{\mathbf{ker}}) \neq 0$ and it is a contradiction to the first part of the theorem, since $U(K) = 0$. Thus, $U(K * G) = 0$ for each commutative integral domain K. This completes the proof of the theorem. □

Corollary 1. *Let K be a ring with* char $K = m > 0$ *and G be a group containing no p-element for every prime divisor p of the integer m. If K is a commutative ring without nonzero nilpotent elements, then $U(K_\rho G) = 0$. If K is a simple ring or a commutative integral domain, then $U(K * G) = 0$.*

Indeed, in order to apply Theorem 2, it is sufficient to show that $K(+)$ has no G-torsions. Let n be the order of an element of G. Then m and n are relatively prime and there exists integers k and r with $kn + rm = 1$. If $n\alpha = 0$ for some $\alpha \in K$, then $kn\alpha + rm\alpha = \alpha$ yields $\alpha = 0$. Hence, $K(+)$ has no G-torsions.

Corollary 2. *Let K be a semisimple ring and the order of each torsion element of G be invertible in K. If $K_\rho G$ is an arbitrary twisted group ring, then $U(K_\rho G) = 0$.*

Indeed, if P is any maximal ideal of K, then $\overline{K} = K/P$ is a simple ring with a unit and the order of each torsion element of G is invertible in $\overline{K}$. Since $K_\rho G/(K_\rho G)P \cong \overline{K}_{\bar\rho}G$ [3], we obtain $U(\overline{K}_{\bar\rho}G) = 0$, according to Theorem 2. Thus, $U(K_\rho G) \subseteq (K_\rho G)P$ for any maximal ideal P of K. Then

$$U(K_\rho G) \subseteq (K_\rho G)(\cap P) = (K_\rho G).B(K) = 0,$$

since the Brown-McCoy radical $B(K)$ of K is trivial.

We notice that the latter corollary is also valid for crossed products in which each maximal ideal P of K is G-invariant, i.e. $\bar{g}P\bar{g}^{-1} \subseteq P$ for all $g \in G$.

The following theorem was proved by Villamayor in the case of a group ring (see [9]) and it was generalized for crossed products over a field by Kolikov [5]. We shall show that it holds for an arbitrary crossed product. For this purpose we have used Ovsjannikov's approach [7] who proved that if K is a radical ring and G is a finite semigroup, then the semigroup ring KG is also radical, i.e. $J(KG) = KG$. Certainly, the theorem can be proved following Passman's arguments for group rings from [9] or [10]. But the approach we use is more elementary.

Theorem 3. *Let $K * G$ be an arbitrary crossed product of the group G and the ring K and H be a normal subgroup of G of finite index. Then*

$$J(K * H).K * G = K * G.J(K * H) \subseteq J(K * G).$$

Proof. The equality

$$J(K * H).K * G = K * G.J(K * H)$$

is obvious since $J(K*H)$ is an invariant ideal of $K*H$ under the automorphisms of $K*H$ generated by the elements $\overline{g}$ $(g\in G)$.

Let $\Pi(G/H)=\{g_1,g_2,\dots,g_n\}$ be a complete set of coset representatives for H in G. Then each element $a\in K*G$ can be written in the form

$$a=\sum_{i=1}^{n}\overline{g_i}v_i \quad (v_i\in K*H).$$

In order to prove that

$$K*G.J(K*H)\subseteq J(K*G),$$

it suffices to establish that

$$J(K*H)\subseteq J(K*G).$$

For the latter we need to show that the element $1-ax$ is invertible in $K*G$ for each $a\in K*G$ and $x\in J(K*H)$, i.e. for the element $y=ax$ there exists such an element $z\in K*G$ that the equality

$$y+z+yz=0 \tag{2}$$

holds. It means that y has to be quasi-invertible in $K*G$ [4]. We seek z in the form

$$z=\overline{g}_1u_1+\overline{g}_2u_2+\cdots\overline{g}_nu_n,$$

where $u_i\in K*H$ are unknown.

If $a=0$ or $x=0$, then $z=0$. Let a and x be nonzero. Then equality (2) gives

$$\sum_{i=1}^{n}\overline{g}_iv_ix+\sum_{i=1}^{n}\overline{g}_iu_i+\sum_{i=1}^{n}\overline{g}_iv_ix\sum_{j=1}^{n}\overline{g}_ju_j=0 \tag{3}$$

We set $w_i=v_ix\in J(K*H)$ $(i=1,2,\dots,n)$ and write down equality (3) as

$$(\overline{g}_1w_1+\overline{g}_2w_2+\cdots\overline{g}_nw_n)+(\overline{g}_1u_1+\overline{g}_2u_2+\cdots\overline{g}_nu_n)$$

$$+(\overline{g}_1w_1+\cdots\overline{g}_nw_n)\overline{g}_1u_1+\cdots(\overline{g}_1w_1+\cdots\overline{g}_nw_n)\overline{g}_nu_n. \tag{4}$$

Let $g_ig_j=g_{k(i,j)}h_{i,j}$ for all $g_i,g_j\in\Pi(G/H)$, where $h_{i,j}\in H$ $(i,j=1,2,\dots,n)$. In particular, $k(i,1)=k(1,i)=i$.

Since $\overline{g}_1$, $\overline{g}_2$, …, $\overline{g}_n$ are linearly independent over $K*H$, for the terms in (4) containing $\overline{g}_1$, we obtain

$$\overline{g}_1w_1+\overline{g}_1u_1+\overline{g}_1w_1\overline{g}_1u_1+\sum_{\substack{k(i,j)=1\\ i,j>1}}\overline{g}_iw_i\overline{g}_ju_j=0.$$

This implies

$$\overline{g}_1 w_1 + \overline{g}_1 u_1 + \overline{g}_1 w_1' u_1 + \sum_{\substack{k(i,j)=1 \\ i,j>1}} \overline{g}_i w_i' u_j = 0, \tag{5}$$

where

$$w_1' = \rho(g_1, g_1)\overline{g_1}^{-1} w_1 \overline{g}_1, \quad w_i' = h_{i,j}\rho(g_i, h_{ij})^{-1}\rho(g_i, g_j)\overline{g_j}^{-1} w_i \overline{g}_j$$

are elements of $J(K * H)$ as the radical is invariant under the automorphisms of $K * H$. Thus, for the coefficient of $\overline{g}_1$ in (5) we obtain

$$w_1 + u_1 + w_1' u_1 + \sum_{\substack{k(i,j)=1 \\ i,j>1}} w_i' u_j = 0$$

and hence,

$$(1 + w_1')u_1 = -w_1 - \sum_{\substack{k(i,j)=1 \\ i,j>1}} w_i' u_j.$$

The element $1 + w_1'$ is invertible in $K * H$ since $w_1' \in J(K * H)$. So, u_1 can be represented in the form

$$u_1 = \gamma_{11} + \gamma_{12}u_2 + \ldots + \gamma_{1n}u_n \quad (\gamma_{1i} \in K * H). \tag{6}$$

We determine the coefficient of $\overline{g}_2$ in (4) by the equality

$$\overline{g}_2 w_2 + \overline{g}_2 u_2 + \overline{g}_1 w_1 \overline{g}_2 u_2 + \overline{g}_2 w_2 \overline{g}_1 u_1 + \sum_{\substack{k(i,j)=2 \\ ,j>2}} \overline{g}_i u_i \overline{g}_j u_j = 0,$$

after substituting u_1 by its representation (6). So, we have

$$w_2 + u_2 + w_1'' u_2 + w_2''(\gamma_{11} + \gamma_{12}u_2 + \ldots + \gamma_{1n}u_n) + \sum_{\substack{k(i,j)=2 \\ i,j>2}} w_i'' u_j = 0,$$

where the elements $w_1'' = \rho(g_1, g_2)\overline{g}_2^{-1} w_1 \overline{g}_2$ $w_2'' = \rho(g_2, g_1)\overline{g}_1^{-1} w_2 \overline{g}_1$,

$$w_i'' = \overline{h_{i,j}}\rho(g_2, h_{i,j})^{-1}\rho(g_i, g_j)\overline{g}_j^{-1} w_i \overline{g}_j$$

belong to $J(K * H)$ again. The latter equality can be written for suitable $\beta_i \in K * H$ as

$$u_2 + w_1'' u_2 + w_2'' \gamma_{12} u_2 = \beta_1 + \beta_3 u_3 + \ldots + \beta_n u_n.$$

As the elements w_1'', w_2'' belong to $J(K * H)$, the element $1 + w_1'' + w_2''\gamma_{12}$ is invertible in $K * H$. So, we can express u_2 in the form

$$u_2 = \gamma_{21} + \gamma_{23}u_3 + \ldots + \gamma_{2n}u_n,$$

where $\gamma_{2i} \in K * H$.

Proceeding further, we obtain the system

$$\begin{aligned} u_1 &= \gamma_{11} + \gamma_{12}u_2 + \ldots + \gamma_{1n}u_n, \\ u_2 &= \gamma_{21} + \gamma_{23}u_3 + \ldots + \gamma_{2n}u_n, \\ \ldots \\ u_{n-1} &= \gamma_{n-1,1} + \gamma_{n-1,n}u_n, \\ u_n &= \gamma_{n,1}, \end{aligned}$$

where $\gamma_{i,j} \in K * H \ (i, j = 1, 2, \ldots, n)$ are known. Now we can determine consecutively $u_n, u_{n-1}, \ldots, u_1$. Hence, there exists the element

$$z = \sum_{i=1}^{n} \overline{g}_i u_i \in K * G$$

for which

$$y + z + yz = 0$$

and therefore $y = ax$ is quasi-invertible in $K * G$ for any $a \in K * G$ and $x \in J(K * H)$, i.e.

$$K * G.J(K * H) \subseteq J(K * G).$$

The theorem is proved.

Corollary. *If H is a normal subgroup of G and G/H is a locally finite group, then*

$$K * G.J(K * H) \subseteq J(K * G).$$

We shall show again that the element $1 - ax$ is invertible in $K * G$ for each $a \in K * G$ and $x \in J(K * H)$.

Indeed, if $G_0 =< H, \text{Supp } a >$ is the subgroup of G generated by H and Supp a, then $|G_0/H| < \infty$ and therefore $x \in J(K * G_0)$. The latter implies that $1 - ax$ is invertible in $K * G_0$. So, $1 - ax$ is invertible in $K * G$.

REFERENCES

[1] Бовди,А.А. Скрещенные произведения полугруппы и кольца. *Сибирский математический журнал*, **4** (1963) 481-499.

[2] Бовди, А.А. Скрещенные произведения полугруппы и простого кольца. *Сибирский математический журнал*, **5** (1964) 465-467.

[3] Бовди, А.А., С. В. Миховски. Идемпотенты скрещенных произведений. *Известия математического института, БАН,* **13** (1971) 247-263.

[4] Джекобсон, Н. Строение колец. Москва, 1961.

[5] Коликов, К.Х. Фундаментальный идеал и радикал Джекобсона скрещенных произведений (to appear).

[6] Миховски, С.В. Скрещенные произведения групп и простых колец (to appear).

[7] Овсянников, А.Я. О радикальных полугрупповых кольцах. *Математические заметки,* **37** (1985) 452-455.

[8] Green, J.A., S. E. Stonehewer. The radical of some group rings. *J. Algebra,* **13** (1969) 137-142.

[9] Passman, D.S. The Algebraic Structure of Group Rings. New York, 1977.

[10] Passman, D.S. Infinite Group Rings. New York, 1971.

[11] Schneider, H., J. Weissglass. Group rings, semigroup rings and their radicals. *J. Algebra,* **5** (1967) 1-15.

Bourgas University of Chemical Technology
"A. Zlatarov"
8010 Bourgas
BULGARIA

Received 29.07.1991