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INTEGRALS INVOLVING BESSEL POLYNOMIALS, HYPERGEOMETRIC SERIES AND FOX'S H-FUNCTION, AND FOURIER-BESSEL POLYNOMIAL EXPANSIONS FOR PRODUCTS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

S. MISHRA

ABSTRACT. In this paper, we have evaluated an integral involving Bessel polynomial, generalized hypergeometric series and Fox's H-function, and employed it to evaluate one double integral involving Bessel polynomials generalized hypergeometric series and the H-function. We have further utilized the integral to establish one Fourier-Bessel polynomial expansion and one double Fourier-Bessel polynomial expansion for the products of generalized hypergeometric functions. We also discuss some particular cases of our results and show how our results lead to generalization of many results, some of which are new and two were earlier given by Mathai and Saxena [10].

1. Introduction. The subject of expansion formulae and Fourier series of generalized hypergeometric functions occupies a prominent place in the literature of special functions. Certain expansion formulae and Fourier series of generalized hypergeometric functions play an important role in the development of the theories of special functions and boundary value problems. It is interesting to note that there is a wide scope of applications of the theory of expansion theorems and Fourier series in the field of boundary value problems and applied mathematics. For example, some results of this paper can be used to obtain certain solutions of the classical problem known as the "time-domain synthesis problem", occurring in electrical network theory [5, pp. 139-140].

It is important to note [9, 11, 13] that so far there were only a few attempts to establish single and multiple Fourier-Bessel polynomial expansions of the products of generalized hypergeometric functions.

The Fox's H-function is a generalization of Meijer's G-function [3, pp. 206-222] and therefore on specializing the parameters, the H-function may be reduced to almost all special functions appearing in pure and applied mathematics [10, pp.114-119].

Therefore the results obtained in this paper have a general character and hence they include several interesting particular cases. A large number of results can be derived for the Meijer's G-function, MacRobert's E-function, Hypergeometric functions, Bessel functions, Legendre functions, Whittaker functions, orthogonal polynomials exponential functions, trigonometric functions and other related transcendental functions.

It is very important to note that operations such as differentiation and integration could be performed more easily on the H-function than on the original functions, even though the two are equivalent. Thus the H-function facilitates the analysis by permitting complicated expressions to be represented and handled in a simpler manner.

The H-function introduced by Fox [6, p. 408] is defined as follows:

$$H^{u,v}_{p,q}\left[z\Big|^{(a_p,e_p)}_{(b_q,f_q)}\right] = H^{u,v}_{p,q}\left[z\Big|^{(a_1,e_1),\ldots,(a_p,e_p)}_{(b_1,f_1),\ldots,(b_q,f_q)}\right] = \frac{1}{2\pi i}\int_L X(s)z^s ds, \tag{1.1}$$

where L is a suitable Barnes contour and

$$X(s) = \frac{\prod_{j=1}^{u}\Gamma(b_j - f_j s)\prod_{j=1}^{v}\Gamma(1 - a_j + e_j s)}{\prod_{j=u+1}^{q}\Gamma(1 - b_j + f_j s)\prod_{j=v+1}^{p}\Gamma(a_j - e_j s)}.$$

Asymptotic expansion and analytic continuation of the H-function had been given by Braaksmma [2].

Some preliminary results.

The following formulae are used in the proofs further:

Bessel polynomial [10, p. 134]:

$$Y_n(x,a,t) = \sum_{r=0}^{n}\frac{(-n)_r(a+n-1)_r}{r!}\left(-\frac{x}{t}\right)^r = {}_2F_0\left[{}^{-n,a+n-1;-\frac{x}{t}}_{-}\right]. \tag{1.2}$$

The integral

$$\int_0^\infty e^{-t}t^{\sigma-1}Y_m(1,a,t)dt = \frac{\Gamma(\sigma)\Gamma(1-\sigma)\Gamma(2-a-\sigma)}{\Gamma(1-\sigma+m)\Gamma(2-\sigma-a-m)}, \tag{1.3}$$

where Re $\sigma > 0$.

To establish (1.3), use the series representation of Bessel polynomial (1.2), interchange the order of integration and summation, evaluate inner-integral [3, p. 1,(1)], put $\Gamma(\sigma - r) = \dfrac{(-1)^r\Gamma(\sigma)}{(1-\sigma)_r}$ and apply Vandermonde's theorem [9, p. 110, (4.1.2)]:

$${}_2F_1\left[{}^{-n,b;1}_{c}\right] = \frac{(c-b)_n}{(c)_n}, \quad c \neq 0, -1, -2, \ldots.$$

The integral

$$\int_0^\infty x^{\sigma-1}e^{-x}Y_m(1,a,x)\,{}_PF_Q\left[{}^{\alpha_P;cx^h}_{\beta_Q}\right]dx \tag{1.4}$$

$$=\sum_{r=0}^{\infty}\frac{(\alpha_P)rc^r\Gamma(\sigma+hr)\Gamma(1-\sigma-hr)\Gamma(2-a-\sigma-hr)}{(\beta_Q)r^{r!}\Gamma(1-\sigma+m-hr)\Gamma(2-\sigma-a-m-hr)},$$

where α_P denotes $\alpha_1,\ldots,\alpha_P$; h is a positive integer; $P<Q$ (or $P=Q+1$ and $|c|<1$); no one of the β_Q is zero or negative integer and Re $\sigma>0$.

The integral (1.4) can easily be established by expressing the generalized hypergeometric series as [3, p. 181, (1)] and interchanging the order of integration and summation, which is justified due to the absolute convergence of the integral and summation involved in the process, and evaluating the integral with the help of (1.3).

The integral

$$\int_0^\infty x^{\sigma-1}e^{-x}Y_m(1,a,x)\,{}_PF_Q\left[{}^{\alpha_P;cx^h}_{\beta_Q}\right]{}_uF_v\left[{}^{v_u;dx^k}_{\delta_v}\right]dx \tag{1.5}$$

$$=\sum_{r,t=0}^{\infty}\frac{(\alpha_P)rc^r(vU)_td^t\Gamma(\sigma+hr+kt)\Gamma(1-\sigma-hr-kt)\Gamma(2-a-\sigma-hr-kt)}{(\beta_Q)r^{r!}(\delta_V)_tt!\Gamma(1-\sigma+m-hr-kt)\Gamma(2-\sigma-a-m-hr-kt)},$$

where in addition to the conditions and notations of (1.2), k is a positive integer; $U<V$ (or $U=V+1$ and $|d|<1$); no one of the δ_V is zero or a negative integer.

To derive (1.5), we use the series representation for ${}_UF_V$ interchange the order of integration and summation and evaluate the resulting integral with the help of (1.4).

Note 1. On applying the above procedure an integral analogous to (1.4) for the procedure of n generalized hypergeometric series can be derived easily.

The orthogonality property of Bessel polynomials [1, p. 76, (2.1)]:

$$\int_0^\infty x^{-a}e^{-x}Y_n(1,a,x)Y_m(1,a,x)dx \tag{1.6}$$

$$=\begin{cases}\frac{n!\Gamma(2-a-n)}{(1-a-2n} & \text{if } m=n;\\ 0 & \text{if } m\neq n.\end{cases}$$

where Re $a<1-m-n$.

In what follows, λ and μ are positive numbers and for brevity:

$$\Phi(r)=\frac{(\alpha_P)_rc^r}{(\beta_Q)_rr!};\quad \Psi(t)=\frac{(\nu_U)_td^t}{(S_\nu)_tt!},$$

$$F_1(x)={}_PF_Q\left[\begin{matrix}\alpha_P;\ cx^h\\ \beta_Q\end{matrix}\right];\quad F_2(x)={}_UF_V\left[\begin{matrix}\nu_U;\ dx^k\\ \delta_V\end{matrix}\right];$$

$$H(x) = H_{p,q}^{u,v}\left[zx^{\lambda}\middle| \begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right];$$

$$H_1(m,r,t) =$$

$$H_{p+3,q+2}^{u+2,v+1}\left[z\middle| \begin{matrix}(1-\sigma-hr-kt,\lambda),(a_p,e_p),(1-\sigma+m-hr-kt,\lambda),\\(2-\sigma-a-m-hr-kt,\lambda);\\(1-\sigma-hr-kt,\lambda),(2-a-\sigma-hr-kt,\lambda),(b_q,f_q),\end{matrix}\right];$$

$$H_2(m,r,t) =$$

$$(-1)^m H_{p+3,q+2}^{u+1,v+2}\left[z\middle| \begin{matrix}(1-\sigma-hr-kt,\lambda),(2-\sigma-a-m-hr-kt,\lambda),(a_p,e_p)\\(1-\sigma+m-hr-kt,\lambda);\\(1-\sigma-hr-kt,\lambda),(b_q,f_q),(2-\sigma-a-hr-kt,\lambda)\end{matrix}\right];$$

$$H_3(m,r,t) =$$

$$H_{p+2,q+1}^{u,v+2}\left[z\middle| \begin{matrix}(1-\sigma+m-hr-kt,\lambda),(2-\sigma-a-m-hr-kt,\lambda),(a_p,e_p)\\(b_q,f_q),(2-\sigma-a-hr-kt,\lambda)\end{matrix}\right];$$

$$H_4(xy) = H_{p,q}^{u,v}\left[zx^{\lambda}y^{\mu}\middle| \begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right];$$

$$H_5(m_1,m_2,r_1,t_1,r_2,t_2) =$$

$$H_{p+6,q+4}^{u+4,v+2}\left[z\middle| \begin{matrix}(1-\sigma_1-hr_1-kt_1,\lambda),(1-\sigma_2-hr_2-kt_2,\mu),(a_p,e_p),\\(1-\sigma_1+m_1-hr_1-kt_1,\lambda),(2-\sigma_1-a-m_1-hr_1-kt_1,\lambda);\\(1-\sigma_2+m_2-hr_2-kt_2,\mu),(2-\sigma_2-b-m_2-hr_2-kt_2,\mu);\\(1-\sigma_1-hr_1-kt_1,\lambda),(2-a-\sigma_1-hr_1-kt_1,\lambda),\\(1-\sigma_2-hr_2-kt_2,\mu),(2-b-\sigma_2-hr_2-kt_2,\mu),(b_q,f_q)\end{matrix}\right];$$

$$A = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j;$$

$$B = \sum_{j=1}^{v} e_j - \sum_{j=v+1}^{p} e_j + \sum_{j=1}^{u} f_j - \sum_{j=u+1}^{q} f_j.$$

Single Integral.

(i) Theorem.

(2.1)
$$\int_0^{\infty} x^{\sigma-1}e^{-x}Y_m(1,a,x)F_1(x)F_2(x)H(x)dx$$

$$= \sum_{r,t=0}^{\infty} \Phi(r)\Psi(t)H_1(m,r,t),$$

where $A \leq 0,\ B > 0,\ |arg z| < \frac{1}{2}B\pi$, Re $(\sigma + \lambda b_j/f_j) > 0\ (j = 1, \ldots, u)$, *together with the conditions given in (1.4) and (1.5).*

Proof. To establish (2.1), expressing the H-function in the integrand as Mellin-Barnes type integral (1.1) and interchanging the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L X(s)z^s \int_0^\infty x^{\sigma+\lambda s-1}e^{-x}Y_m(1,a,x)F_1(x)F_2(x)dxds.$$

Evaluating the inner-integral with the help of (1.5) we get

$$\sum_{r,t=0}^{\infty}\Phi(r)\Psi(t)\frac{1}{2\pi i}\int_L X(s)$$

$$\times\frac{\Gamma(\sigma+hr+kt+\lambda s)\Gamma(1-\sigma-hr-kt-\lambda s)\Gamma(2-a-\sigma-hr-kt-\lambda s)z^s ds}{\Gamma(1-\sigma+m-hr-kt-\lambda s)\Gamma(2-\sigma-a-m-hr-kt-\lambda s)}$$

Now, using (1.1), the value of the integral (2.1) is obtained.

Note 2. An integral analogous to (2.1), involving the product of n generalized series, Bessel polynomial and the H-function can be evaluated easily with the help of the result mentioned in Note 1.

(ii) Particular Cases: By virtue of the following identities, apparent from the definition of the H-function and [3, p. 3, (3) & (4)]:

$$H_1(m,r,t) = H_2(m,r,t), \tag{2.2}$$

$$H_1(m,r,t) = H_3(m,r,t), \tag{2.3}$$

the integral (2.1) reduces to the following special forms:

$$\int_0^\infty x^{\sigma-1}e^{-x}Y_m(1,a,x)F_1(x)F_2(x)H(x)dx = \sum_{r,t=0}^{\infty}\Phi(r)\Psi(t)H_2(m,r,t), \tag{2.4}$$

$$\int_0^\infty x^{\sigma-1}e^{-x}Y_m(1,a,x)F_1(x)F_2(x)H(x)dx = \sum_{r,t=0}^{\infty}\Phi(r)\Psi(t)H_3(m,r,t), \tag{2.5}$$

valid under the conditions of (2.1).

In (2.1), (2.4) and (2.5), putting $d = 0$, we get the representations:

$$\int_0^\infty x^{\sigma-1}e^{-x}Y_m(1,a,x)F_1(x)H(x)dx = \sum_{r=0}^{\infty}\Phi(r)H_1(m,r,0), \tag{2.6}$$

$$\int_0^\infty x^{\sigma-1}e^{-x}Y_m(1,a,x)F_1(x)H(x)dx = \sum_{r=0}^{\infty}\Phi(r)H_2(m,r,0), \tag{2.7}$$

$$\int_0^\infty x^{\sigma-1}e^{-x}Y_m(1,a,x)F_1(x)H(x)dx = \sum_{r=0}^{\infty}\Phi(r)H_3(m,r,0), \tag{2.8}$$

valid under the conditions of (2.1) with $d = 0$.

It is interesting to note that Singh and Varma [12] evaluated an integral involving the product of an associated Legendre function, a generalized hypergeometric series and the H-function [10, p. 40, (2.9.4)] on making use of finite difference operator E [11, p. 33 with $W = 1$]. It is also interesting to note that Gupta and Olkha [8] evaluated an integral involving the product of a generalized hypergeometric series and the H-function using an integral due to Goyal [7, p. 202].

Srivastava, Gupta and Goyal [13, pp. 61 - 63] presented some integrals based on the technique of Gupta and Olkha.

In view of the above discussion and [9, 10, 13] it appears that our integrals are more general and new in addition to the new and simple technique of evaluating such integrals.

In (2.6), (2.7) and (2.8), putting $c = 0$, we obtain

$$\int_0^\infty x^{\sigma-1}e^{-x}Y_m(1,a,x)H(x)dx = H_1(m,0,0), \tag{2.9}$$

$$\int_0^\infty x^{\sigma-1}e^{-x}Y_m(1,a,x)H(x)dx = H_2(m,0,0), \tag{2.10}$$

$$\int_0^\infty x^{\sigma-1}e^{-x}Y_m(1,a,x)H(x)dx = H_3(m,0,0), \tag{2.11}$$

valid under the conditions of (2.1) with $c = d = 0$.

In (2.11), if we take $\lambda = 1$, it reduces to a known result due to Mathai and Saxena [10, p.80, (3.8)].

3. Double Integral.

(i) Theorem.

$$\int_0^\infty \int_0^\infty x^{\sigma_1-1}y^{\sigma_2-1}e^{-(x+y)}$$

$$\times Y_{m_1}(1,a,x)Y_{m_2}(1,b,y)F_1(x)F_2(x)F_1(y)F_2(y)H_4(xy)dxdy \tag{3.1}$$

$$= \sum_{r_1,t_1=0}^{\infty} \sum_{r_2,t_2=0}^{\infty} \Phi(r_1)\Psi(t_1)\Phi(r_2)\Psi(t_2)H_5(m_1,m_2,r_1,t_1,r_2,t_2),$$

where in addition to the conditions stated in (2.1) Re $(\sigma_1 + \lambda b_j/f_j) > 0$, Re $(\sigma_2 + \mu b_j/f_j) > 0$, $j = 1,\ldots,u$.

P r o o f. To establish (3.1), evaluating x-integral with the help of (2.1) and interchanging the order of integration and summation, we get

$$\sum_{r_1,t_1=0}^{\infty} \Phi(r_1)\Psi(t_1) \int_0^\infty y^{\sigma_2-1}e^{-y}Y_{m_2}(1,a_2,y)F_2(y)$$

$$\times H^{u+2,v+1}_{p+3,q+2}\left[zy^\mu \left| \begin{matrix} (1-\sigma_1-hr_1-kt_1,\lambda),(a_p,e_p),(1-\sigma_1+m_1-hr_1-kt_1,\lambda), \\ (2-\sigma_1-a-m_1-hr_1-kt_1,\lambda); \\ (1-\sigma_1-hr_1-kt_1,\lambda),(2-a-\sigma_1-hr_1-kt_1,\lambda),(b_q,f_q), \end{matrix} \right.\right] dy.$$

Now applying (2.1) to evaluate the y-integral, the value of (3.1) is obtained.

Note 3. A multiple-integral analogous to (3.1) can be established easily on applying the above technique $(n-1)$ times.

Note 4. Double integrals analogous to (2.5) and (2.6) can also be derived on applying the above procedure.

(ii) Particular Cases:

Putting $d = 0$ in (3.1), we get

$$\int_0^\infty \int_0^\infty x^{\sigma_1-1}y^{\sigma_2-1}e^{-(x+y)}Y_{m_1}(1,a,x)Y_{m_2}(1,b,y)F_1(x)F_2(x)H_4(xy)dxdy \tag{3.2}$$

$$= \sum_{r_1,r_2=0}^{\infty} \Phi(r_1)\Phi(r_2)H_5(m_1,m_2,r_1,0,r_2,0),$$

valid under the conditions of (3.1) with $d = 0$.

Putting $c = 0$ in (3.2), we obtain

$$\int_0^\infty \int_0^\infty x^{\sigma_1-1}y^{\sigma_2-1}e^{-(x+y)}Y_{m_1}(1,a,x)Y_{m_2}(1,b,y)H_3(xy)dxdy \tag{3.3}$$

$$= H_5(m_1,m_2,0,0,0,0),$$

valid under the conditions of (3.2) with $c = 0$.

Note 5. The integrals of this section can be useful for establishing double and multiple Fourier-Bessel polynomial expansions for the products of generalized hypergeometric series and the H-function.

4. Fourier-Bessel Polynomial Expansion.

(i) Theorem.

$$x^{\sigma+a-1}F_1(x)F_2(x)H(x) \tag{4.1}$$

$$= \sum_{n=0}^{\infty} \frac{(1-a-2n)}{n!\Gamma(2-a-n)!} Y_n(1,a,x) \sum_{r,t=0}^{\infty} \Phi(r)\Psi(t)H_1(n,r,t)$$

valid under the conditions of (2.1) and Re $a < 1 - 2n$.

Proof. To prove (4.1), let

$$f(x) = x^{\sigma+a-1}F_1(x)F_2(x)H(x) = \sum_{n=0}^{\infty} C_n Y_n(1,a,x). \tag{4.2}$$

Equation (4.2) is valid, since $f(x)$ is continuous and of bounded variation in the open interval $(0,\infty)$.

Multiplying both sides of (4.2) by $x^{-a}e^{-x}Y_m(1,a,x)$ and integrating with respect to x from 0 to ∞, we have

$$\int_0^{\infty} x^{\sigma-1}e^{-x}Y_m(1,a,x)F_1(x)F_2(x)H(x)dx$$

$$= \sum_{n=0}^{\infty} c_n \int_0^{\infty} x^{-a}e^{-x}Y_m(1,a,x)Y_n(1,a,x)dx.$$

Now using (2.1) and (1.6) we obtain

$$C_m = \frac{(1-a-2m)}{m!\Gamma(2-a-m)!} \sum_{r,t=0}^{\infty} \Phi(r)\Psi(t)H_1(m,r,t). \tag{4.3}$$

From (4.2) and (4.3) the Fourier-Bessel polynomial expansion (4.1) is obtained.

(ii) Particular Cases.

By virtue of the identities (2.2) and (2.3), the Fourier-Bessel polynomial expansion (4.1) is reduced to the following special forms:

$$x^{\sigma+a-1}F_1(x)F_2(x)H(x) \tag{4.4}$$

$$= \sum_{n=0}^{\infty} \frac{(1-a-2n)}{n!\Gamma(2-a-n)} Y_n(1,a,x) \sum_{r,t=0}^{\infty} \Phi(r)\Psi(t) H_2(n,r,t),$$

(4.5)
$$x^{\sigma+a-1} F_1(x) F_2(x) H(x)$$

$$= \sum_{n=0}^{\infty} \frac{(1-a-2n)}{n!\Gamma(2-a-n)} Y_n(1,a,x) \sum_{r,t=0}^{\infty} \Phi(r)\Psi(t) H_3(n,r,t),$$

valid under the conditions of (4.1).

In (4.1), (4.2) and (4.5), putting $c = d = 0$, we have

(4.6)
$$x^{\sigma+a-1} H(x) = \sum_{n=0}^{\infty} \frac{(1-a-2n)}{n!\Gamma(2-a-n)} Y_n(1,a,x) H_1(n,0,0),$$

(4.7)
$$x^{\sigma+a-1} H(x) = \sum_{n=0}^{\infty} \frac{(1-a-2n)}{n!\Gamma(2-a-n)} Y_n(1,a,x) H_2(n,0,0),$$

(4.8)
$$x^{\sigma+a-1} H(x) = \sum_{n=0}^{\infty} \frac{(1-a-2n)}{n!\Gamma(2-a-n)} Y_n(1,a,x) H_3(n,0,0),$$

valid under the conditions of (4.1) with $c = d = 0$.

In (4.8), setting $\sigma + a - 1 = \omega$ and $\lambda = 1$, it reduces to a known result due to Mathai and Saxena [10, p. 81].

5. Double Fourier-Bessel polynomial expansion.

(i) Theorem.

(5.1)
$$x^{\sigma_1+a-1} y^{\sigma_2-b-1} F_1(x) F_2(x) F_1(y) F_2(y) H_4(xy)$$

$$= \sum_{n_1,n_2=0}^{\infty} \frac{(1-a-2n_1)(1-b-n_2)}{n_1!\Gamma(2-a-n_1)n_2!\Gamma(2-b-n_2)} Y_{n_1}(1,a,x) Y_{n_2}(1,b,y)$$

$$\times \sum_{r_1,t_1=0}^{\infty} \sum_{r_2,t_2=0}^{\infty} \Phi(r_1)\Psi(t_1)\Phi(r_2)\Psi(t_2) H_5(n_1,n_2,r_1,t_1,r_2,t_2),$$

valid under the conditions of (3.1) and Re $a < 1 - 2n_1$, Re $b < 1 - 2n_2$.

P r o o f. To establish (5.1), let

(5.2)
$$f(x,y) = x^{\sigma_1+a-1} y^{\sigma_2+b-1} F_1(x) F_2(x) F_1(y) F_2(y) H_4(xy)$$

$$\sum_{n_1,n_2=0}^{\infty} A_{n_1,n_2} Y_{n_1}(1,a,x) Y_{n_2}(1,b,y).$$

Equation (5.2) is valid, since $f(x,y)$ is continuous and of bounded variation in the open interval $(0,\infty)$.

The series (5.2) is an example of what is called a double Fourier-Bessel polynomial expansion. Instead of discussing the theory, we show a method to find formally A_{n_1,n_2} from (5.2). For fixed x, we note that $\sum_{n_1=0}^{\infty} A_{n_1,n_2} Y_{n_1}(1,a,x)$ depends only on n_2, furthermore, it must be the coefficient of Fourier-Bessel polynomial expansion in y of $f(x,y)$ over $0 < y < \infty$.

Multiplying both sides of (5.2) by $y^{-b}e^{-y}Y_{m_2}(1,b,y)$, integrating with respect to y from 0 to ∞ and using (2.1) and (1.6), we get

$$x^{\sigma_1+a-1} F_1(x) F_2(x) \sum_{r_2,t_2=0}^{\infty} \Phi(r_2)\Psi(t_2) \tag{5.3}$$

$$\times H^{u+2,v+1}_{p+3,q+2}\left[zx^{\lambda} \middle| \begin{matrix} (1-\sigma_2-hr_2-kt_1,\mu),(a_p,e_p),(1-\sigma_2+m_2-hr_2-kt_2,\mu) \\ (2-\sigma_2-b-m_2-hr_2-kt_2,\mu) \\ (1-\sigma_2-hr_2-kt_2,\mu),(2-b-\sigma_2-hr_2-kt_2,\mu),(b_q,f_q) \end{matrix} \right];$$

$$= \sum_{n_1=0}^{\infty} A_{n_1,m_2} m_2! \frac{\Gamma(2-b-m_2)}{(1-b-2m_2)} Y_{n_1}(1,a,x).$$

Multiplying both sides of (5.3) by $x^{-a}e^{-x}Y_{m_1}(1,a,x)$, integrating with respect to x from 0 to ∞, and using (2.1) and (1.6), we obtain

$$A_{m_1,m_2} = \frac{(1-a-2m_1)(1-b-2m_2)}{m_1!\Gamma(2-a-m_1)m_2!\Gamma(2-b-m_2)} \tag{5.4}$$

$$\times \sum_{r_1,t_1=0}^{\infty} \sum_{r_2,t_2=0}^{\infty} \Phi(r_1)\Psi(t_1)\Phi(r_2)\Psi(t_2) H_5(m_1,m_2,r_1,t_1,r_2,t_2).$$

From (5.2) and (5.4), the double Fourier-Bessel polynomial expansion (1.5) is obtained.

(ii) Particular Cases.

In (5.1), putting $c = d = 0$, we get

$$x^{\sigma_1+a-1} y^{\sigma_2-b-1} H_3(xy) \tag{5.5}$$

$$= \sum_{n_1,n_2=0}^{\infty} \frac{(1-a-2n_1)(1-b-2n_2)}{n_1!\Gamma(2-a-n_1)n_2!\Gamma(2-b-n_2)}$$

$$\times H_5(n_1, n_2, 0, 0, 0, 0) Y_{n_1}(1, a, x) Y_{n_2}(1, b, y),$$

valid under the conditions of (5.1) with $c = d = 0$.

Note 6. A double Fourier-Bessel polynomial expansions analogous to (4.4) and (4.5) can be derived with the help of (2.4) and (2.5) respectively.

Note 7. Multiple Fourier-Bessel polynomial expansion analogous to (5.1) can be established on applying the above technique repeatedly.

Note 8. The Fourier-Bessel polynomial expansion and the double Fourier-Bessel polynomial expansion given in section 4 and 5 may further be generalized by using Laplace transform technique given by Wimp and Luke [14].

Note 9. Results analogous to our main results (2.1), (3.1), (4.1) and (5.1) involving the H-function of several complex variables [13, pp. 251-254] can be derived on following the techniques given in this paper.

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Department of Civil Engineering
Vidya-Bhawan Rural Institute,
UDAIPUR,
INDIA

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