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EIGENVALUE DISTRIBUTION OF INVARIANT LINEAR ELLIPTIC DIFFERENTIAL OPERATORS OF ARBITRARY ORDER WITH CONSTANT COEFFICIENTS

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ABSTRACT. For a $\mathfrak{G}$-invariant linear elliptic differential operator P with constant coefficients the well-known distribution function $N(\lambda) := \#\{\mu \in \operatorname{spec}_{\mathfrak{G}}(P)\mu \leq \lambda^q\}$ of the $\mathfrak{G}$-automorphic eigenvalue spectrum $\operatorname{spec}_{\mathfrak{G}}(P)$ has the asymptotic estimation $N(\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + O(\lambda^{n-2+2/(n+1)})$ where c_0, c_1 are geometric invariants, q – the order of P, $\mathfrak{G}$ a properly discontinuous group of affine transformations.

Problem and main result. Notations:

$\mathfrak{V}$ n-dimensional real affine space (or its corresponding vector space); bas$\mathfrak{V}=\{0;\mathfrak{b}_1,\ldots,\mathfrak{b}_n\}$ affine base in $\mathfrak{V}$ with the origin 0 and $\mathfrak{x}=x^i\mathfrak{b}_i \in\mathfrak{V}$;

$\mathfrak{V}^*$ its dual; bas$\mathfrak{V}^* = \{0;\mathfrak{b}^1,\ldots,\mathfrak{b}^n\}$ – the dual base, $\mathfrak{v}=v_h\mathfrak{b}^h \in\mathfrak{V}^*$;

$\mathfrak{G}$ properly discontinuous group of affine transformations $(\sigma,\mathfrak{s})$ acting on $\mathfrak{V}$ and having a compact fundamental domain $\mathcal{F}(\mathfrak{G}) \in\mathfrak{V}$;

$L_2(\mathfrak{G})$ the Hilbert space over $\mathbf{C}$ of locally square-integrable $\mathfrak{G}$ – automorphic functions (s.[7], §2);

$$P(D) := \sum_{|\alpha|=q} c_\alpha D^\alpha, \quad \alpha = (\alpha_1,\ldots,\alpha_n) \tag{1}$$

$\mathfrak{G}$-invariant q-th order elliptic differential operator with constant coefficients c_α and the property that its corresponding "gauge domain"

$$\mathbf{D} := \{\mathfrak{v} \in \mathfrak{V}^* | -P(\mathfrak{v}) \leq (1/2\pi)^q\} \tag{2}$$

is a strong convex domain in $\mathfrak{V}^*$ i.e. the principal curvatures of $\partial\mathbf{D}$ are positive;

(1′) $$P(\mathfrak{v}) := \sum_{|\alpha|=q} c_\alpha v^\alpha, \quad v^\alpha = v_1^{\alpha_1} \dots v_n^{\alpha_n}$$

the characteristic polynomial of $P(D)$.

Consider the $\mathfrak{G}$-automorphic eigenvalue problem

(3) $$P(D)[\psi] + \mu\psi = 0, \quad \psi \in L_2(\mathfrak{G}).$$

With respect to the $\mathfrak{G}$-automorphic eigenvalue spectrum $spec_{\mathfrak{G}}(P(D))$ of $P(D)$ we are interested in the asymptotic behavior of the distribution function

(4) $$N(\lambda) := \#\{\mu \in spec_{\mathfrak{G}}(P(D)) \mid \mu \leq \lambda^q\}$$

(eigenvalues counted with regard to their multiplicity).

The paper is divided into 3 sections. Next follows a brief description of each one of them

1. We solve the eigenvalue problem (3) for the so-called f-corresponding functions thus defining $spec_{\mathfrak{G}}(P(D))$ (Proposition 1);

   we interpret $N(\lambda)$ as the number of equivalence classes of the so-called "principal lattice vectors" in the homothetic expansion $\lambda \cdot \mathbf{D} \subset \mathfrak{V}^*$ of $\mathbf{D}$ (Proposition 2);

2. We prove that for $N(\lambda)$ the following asymtotic estimation is valid:

Theorem:

(5) $$N(\lambda) = \frac{1}{\gamma}\mathrm{vol}_n(\mathbf{D})\lambda^n + \frac{1}{\gamma}\sum_{\sigma \in L_{n-1}} \mathrm{vol}_{n-1}(\mathbf{D} \cap \mathfrak{V}^*(\sigma))\delta_\sigma \lambda^{n-1} + O(\lambda^{n-2+2/(n+1)}).$$

(see also[7]);

$\mathfrak{V}^*(\sigma) := \ker(\sigma^T - id) \subset \mathfrak{V}^*$ and $\sigma^T : \mathfrak{V}^* \to \mathfrak{V}^*$ – the adjoined mapping: $\sigma^T\mathfrak{v} = \mathfrak{v} \circ \sigma$ to the "fixed point part" $\sigma : \mathfrak{V} \to \mathfrak{V}$ of the affine transformation $(\sigma, \mathfrak{s})$ with the "translation part" $\mathfrak{s}$;

$\mathfrak{L} := \{\sigma \mid \exists(\sigma, \mathfrak{s} \in \mathfrak{G}\}$ the point group for $\mathfrak{G}$; $r = \mathrm{ord}\mathfrak{L}$ is finite,

$\mathfrak{L}_m := \{\sigma \in \mathfrak{L} \mid \dim\mathfrak{V}^*(\sigma) = m\}$ for $m = 1, 2, \dots, n$, where $\mathfrak{L}_n = \{e\}$ (e is the identity of $\mathfrak{L}$);

$\delta_\sigma := 1$ or $:= 0$ as for the δ-symbol of E. Landau but here it is used in the following sense:

$\Gamma := \mathrm{orb}_{\mathfrak{T}}(\mathfrak{O})$ shall be the orbit to the origin $\mathfrak{O} \in \mathfrak{V}$ under the action of the invariant subgroup $\mathfrak{T} \subset \mathfrak{G}$ of all translations $(e, \mathfrak{t}) \in \mathfrak{G}$, i.e. Γ is the lattice of all vectors $\mathfrak{t}$ fastening to $\mathfrak{O}$ in $\mathfrak{V}$. Then $\delta_\sigma = 1$ if there is a lattice vector $\mathfrak{t}_0 \in \Gamma$ which produces a fixed point element $(\sigma, \mathfrak{s}+\mathfrak{t}_0) \in \mathfrak{G}$, otherwise $\delta_\sigma = 0$ (see also [1], 4.4);

$$(6) \qquad \begin{aligned} \mathrm{vol}_{n-1}(\mathbb{G}) &= \int_{\mathbb{G}} d\mu_\sigma^*(\mathfrak{v}) &&= \int_{\mathbb{G}} d\mathfrak{v} \Big/ \int_{\mathcal{F}(\Gamma^*(\sigma))} d\mathfrak{v}, \\ \mathrm{vol}_n(\mathbb{D}) &= \int_{\mathbb{D}} d\mu^*(\mathfrak{v}) &&= \int_{\mathbb{D}} d\mathfrak{v} \Big/ \int_{\mathcal{F}(\Gamma^*)} d\mathfrak{v}, \end{aligned}$$

where $\mathbb{G} = \mathbb{D} \cap \mathfrak{V}^*(\sigma) \subset \mathfrak{V}^*$ and $\mathfrak{V}^*(\sigma)$ is equipped with a Lebesgue-measure μ_σ^*, normed by $\mu_\sigma^*(\mathcal{F}(\Gamma^*(\sigma))) = 1$ for the fundamental domain $\mathcal{F}(\cdot)$ of the lattice;

$\Gamma^*(\sigma) := \Gamma^* \cup \mathfrak{V}^*(\sigma)$;

$\Gamma^* \subset \mathfrak{V}^*$ is the dual lattice to $\Gamma \subset \mathfrak{V}$. Note that for $\sigma = e$ $\mathfrak{V}^*(e) = \mathfrak{V}^*$, $\Gamma^*(e) = \Gamma^*$, $\mu^* := \mu_e^*$.

Remark 1. The affine volume $\int_G d\mathfrak{v}$ in (6) (also for $\mathbb{G} = \mathbb{D}$) is a relative invariant of the weight -1 and therefore the quotients (6) are absolute invariants. So Theorem (5) is a result in the affine spectral geometry as well.

Remark 2. In general the strong convexity of $\mathbb{D}$ in (2) (which we need so as to prove formula (5)) follows from the ellipticity of $P(D)$ in (1) only for the case $q = 2$. For instance $P(D) = \partial_1^4 + 18\partial_1^2\partial_2^2 + \partial_2^4$, $\partial_i = \partial/\partial x^i$ is an elliptic operator but the corresponding gauge domain $\mathbb{D} := -P(v_1, v_2) \equiv v_1^4 + 18v_1^2v_2^2 + v_2^4 \leq (1/2\pi)^4$ in $\mathfrak{V}^*$ is not convex. Elliptic operators with non-convex (resp. strong convex) gauge domains $\mathbb{D} \subset \mathfrak{V}^*$ are given e.g. by Cassinean curves (resp. e.g. by the powers Δ^k of the Laplacian). It is easier to prove Theorem (5) for $q = 2$ than for $q \geq 4$ because it can be reduced to E. Landau's estimation of the lattice remainder relative to the number of lattice points in the ellipse (resp. ellipsoid) $\mathbb{D}$ (see [1]).

3. We prove the Theorem for an example.

1. f-corresponding functions $\psi_f \in L_2(\mathfrak{G})$ as $\mathfrak{G}$-automorphic eigenfunctions of $P(D)$, $\mathrm{spec}_{\mathfrak{G}}(P(D))$, and $N(\lambda)$ as the number of principal classes in $\lambda \cdot \mathbb{D}$

1.1. The translation lattice $\Gamma \subset \mathfrak{V}$ and its dual $\Gamma^* \subset \mathfrak{V}^*$ for the group $\mathfrak{G}$.

As in [7], §1 we consider the affine transformations $\mathfrak{G} \ni S : \mathfrak{V} \to \mathfrak{V}$ in form of $S = (\sigma, \mathfrak{s})$ with $\mathfrak{z}' = S\mathfrak{z} = \sigma\mathfrak{z} + \mathfrak{s}$; $\mathfrak{z}, \mathfrak{z}' \in \mathfrak{V}$. σ is called a fixed point part, $\mathfrak{s}$ is a translation part of S. The successive application of two elements $R = (\rho, \mathfrak{r})$, $S \in \mathfrak{G}$ follows the composition $R \circ S = (\rho\sigma, \rho\mathfrak{s} + \mathfrak{r})$. Then $S^{-1} = (\sigma^{-1}, \sigma^{-1}\mathfrak{s})$ is the inverse of S, with respect to the identity $E = (e,\mathfrak{O})$ with $e = id$ and $\mathfrak{O}$ is a null vector. Now we consider the invariant subgroup $\mathfrak{T}\subset\mathfrak{G}$ of all translations $(e,\mathfrak{t})$ in $\mathfrak{G}$.

$$\mathfrak{T} := \{(e,\mathfrak{t}) \in \mathfrak{G}\},$$

and also the point group

$$\mathfrak{L} := \{\sigma | \exists \mathfrak{s} \in \mathfrak{V} : (\sigma, \mathfrak{s}) \in \mathfrak{G}\}$$

for $\mathfrak{G}$. The factor group $\mathfrak{G}/\mathfrak{T}$ has a finite order $r = \text{ord}(\mathfrak{G}/\mathfrak{T})$. Then because of the isomorphy $\mathfrak{G}/\mathfrak{T}\cong\mathfrak{L}$ we have

$$r = \text{ord}\mathfrak{L}.$$

Now we introduce the lattice

$$\Gamma := \{\mathfrak{t} = t^k \mathfrak{b}_k \mid t^k \in \mathbf{Z}\} \subset \mathfrak{V},$$

where $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ forms a base bas$\mathfrak{V}$ of $\mathfrak{V}$. $(e, \mathfrak{b}_1), \ldots, (e, \mathfrak{b}_n)$ are n generators of $\mathfrak{T}$ with n linear independent translation parts $\mathfrak{b}_k$. Γ is $\mathfrak{L}$-invariant. In the dual space $\mathfrak{V}^*$ let

$$\Gamma^* := \{\mathfrak{u} = u_h \mathfrak{b}^h \mid u_h \in \mathbf{Z}\} \subset \mathfrak{V}^*$$

be the dual lattice to Γ with $\langle \mathfrak{b}^h, \mathfrak{b}_k\rangle = \delta^h_k$; for $\mathfrak{x} \in \mathfrak{V}$ $\langle \mathfrak{v}, \mathfrak{x}\rangle$ is the value of the linear functional $\mathfrak{v} \in \mathfrak{V}$. Further on we shall use the dual base bas$\mathfrak{V}^* = \{\mathfrak{b}^1, \ldots, \mathfrak{b}^n\}$ as a base of $\mathfrak{V}^*$.

In this case $\sigma \in \mathfrak{L}$ should be replaced by the adjoint mapping σ, namely

$$\sigma^T : \mathfrak{V}^* \to \mathfrak{V}^* \quad \text{with} \quad \sigma^T \mathfrak{v} = \mathfrak{v} \circ \sigma.$$

If $(\sigma, \mathfrak{v}) \in \mathfrak{G}$ we also say that "$\mathfrak{a}$ belongs to $\sigma \in \mathfrak{L}$". In this case all the vectors $\mathfrak{a} + \Gamma$ belong to σ. Then moduloΓ exactly one vector $\mathfrak{a}$ can be considered as belonging to σ, and further on we will denote it by $\mathfrak{a} = \mathfrak{s}$.

If $(\sigma_1, \mathfrak{s}_1), (\sigma_2, \mathfrak{s}_2)$, then $(\sigma_1\sigma_2, \mathfrak{s}) \in \mathfrak{G}$. So sometimes it is advantageous to think of the Frobenius-congruence

$$\sigma_1 \mathfrak{s}_2 + \mathfrak{s}_1 \equiv \mathfrak{s} \text{mod}\Gamma.$$

1.2. Decomposition of the group $\mathfrak{G}$ into cosets and decomposition of the dual lattice Γ^* into equivalence classes.

In the coset of decomposition of $\mathfrak{V}$ with respect to $\mathfrak{T}$,

$$\mathfrak{G} = \varkappa(\sigma_1) + \ldots + \varkappa(\sigma_1), \quad \varkappa(\sigma_\nu) = S_\nu \circ \mathfrak{T}, \quad S_\nu = (\sigma_\nu, \mathfrak{s}_\nu)$$

the elements of one and the same coset $\varkappa(\sigma_\nu)$ have the same fixed point part σ_ν but distinct cosets have distinct fixed point parts.

Further we need the decomposition of $\mathfrak{V}^*$ into equivalence classes

$$\mathfrak{k}(\mathfrak{v}_1) = \{\mathfrak{v} = \sigma^T \mathfrak{v}_1 \mid \forall \sigma \in \mathfrak{L}\} = \{\mathfrak{v}_1, \mathfrak{v}_2, \dots, \mathfrak{v}_l\}.$$

For a fixed functional $\mathfrak{v}_1 \in \mathfrak{V}$ and variable $\sigma \in \mathfrak{L}$ the transform $\sigma^T \mathfrak{v}_1$ runs maximally $l \leq r = \mathrm{ord}\mathfrak{L}$ lattice vectors $\mathfrak{v}_1, \mathfrak{v}_2, \dots, \mathfrak{v}_l$. So Γ^* is also decomposed into equivalence classes $\mathfrak{k}(\mathfrak{u}_1) = \{\mathfrak{u}_1, \dots, \mathfrak{u}_l\} \subset \Gamma^*$, $\mathfrak{u}_1 \in \Gamma^*$. We denote the set of all such classes by $\mathfrak{K}$.

1.3. The $\mathfrak{L}$-automorphic P-norm $\|\cdot\|_p$ in $\mathfrak{V}^*$, the class norm $\|\mathfrak{K}\|$.

Consider the Minkowski functional of the convex domain

$$f(\mathfrak{v};\mathbf{D}) := \inf_{\tau}\{\tau > 0 \mid \mathfrak{v} \in \tau \cdot \mathbf{D}\}. \tag{7}$$

Lemma 1. *f has the properties*

[i] $\mathfrak{v} \in \partial(f(\mathfrak{v};\mathbf{D}) \cdot \mathbf{D})\ \forall \mathfrak{v} \in \mathfrak{V}^*$;

[ii] $f(\mathfrak{v};\mathbf{D}) \leq 1$ *if* $\mathfrak{v} \in \mathbf{D}$, *otherwise* > 1;

[iii] $f(\lambda \cdot \mathfrak{v};\mathbf{D}) = \lambda \cdot f(\mathfrak{v};\mathbf{D})\ \ \forall \lambda > 0,\ \mathfrak{v} \in \mathfrak{V}^*$;

[iv] $F(\mathfrak{v} + \mathfrak{w};\mathbf{D}) \leq f(\mathfrak{v};\mathbf{D}) + f(\mathfrak{w};\mathbf{D})\ \forall \mathfrak{v}, \mathfrak{w} \in \mathfrak{V}^*$;

[v] $f(\sigma^T \mathfrak{w};\mathbf{D}) = f(\mathfrak{v};\mathbf{D})\ \ \forall \sigma \in \mathfrak{L}$ *is true iff* $\mathbf{D}$ *is* $\mathfrak{L}$*-invariant.*

Proof. [i], [ii] follow immediately from (7); [iii], [iv] are well-known; [v]:

a) $\sigma\mathbf{D} \subseteq \mathbf{D}$ means by (7) $f(\sigma^T \mathfrak{v};\mathbf{D}) \leq f(\mathfrak{v};\mathbf{D})\ \forall \sigma \in \mathfrak{L}$. To prove the inverse we rewrite [i] in the form of $\sigma^T \mathfrak{v} \in \partial(f(\sigma^T \mathfrak{v};\mathbf{D}) \cdot \mathbf{D})$ and obtain $\mathfrak{v} = e\mathfrak{v} = (\sigma^T)^{-1}(\sigma^T \mathfrak{v}) \in f(\sigma^T \mathfrak{v};\mathbf{D}) \cdot \mathbf{D}$.

b) Let $f(\sigma^T \mathfrak{v};\mathbf{D}) = f(\mathfrak{v};\mathbf{D})$, $\mathfrak{v} \in \partial(\lambda \cdot \mathbf{D})$. Then from [i] it follows $\lambda = f(\mathfrak{v};\mathbf{D}) = f(\sigma^T \mathfrak{v};\mathbf{D})$ i.e. $\sigma^T \mathfrak{v} \in \lambda \cdot \mathbf{D}\ \ \forall \sigma \in \mathfrak{L}$. □

Further, because of (2) the homothetic expansion

$$\lambda \cdot \mathbf{D} := \{\mathfrak{v} \in \mathfrak{V}^* \mid -P(\mathfrak{v}) \leq (\lambda/2\pi)^q\},\ \ \lambda > 0 \tag{8}$$

is also strong convex and by Lemma 1, [i] we can write

$$P(\mathfrak{v}) = -(\frac{1}{2\pi} f(\mathfrak{v};\mathbf{D}))^q,\ \ \forall \mathfrak{v} \in \mathfrak{V}^*. \tag{9}$$

Because of Lemma 1, [iii] and [iv] we can introduce in $\mathfrak{V}^*$ (up to the factor $(2\pi)^q$ the P-norm $\|\mathfrak{v}\|_p$ or for short $\|\mathfrak{v}\|$ by

$$\|\mathfrak{v}\|^q = -P(\mathfrak{v}),\ \ \mathfrak{v} \in \mathfrak{V}^*. \tag{10}$$

Lemma 2. *The following conditions are equivalent*

(i) $P(\mathbf{D})$ *is* $\mathfrak{G}$*-invariant,*

(ii) $P(\mathfrak{v})$ *and so also* $\|\mathfrak{v}\|$ *is* $\mathfrak{L}$*-automorphic,*

(iii) $\mathbf{D}$ *and so all* $\lambda \cdot \mathbf{D}$ $(\lambda > 0)$ *are* $\mathfrak{L}$*-invariant.*

((i) $\sim$ (ii) is clear, (ii) $\sim$ (iii) follows from Lemma 1, [v] and (9)).

From Lemma 2, (ii) it follows that $\|\mathfrak{v}\|$ depends on $\mathfrak{v}$ only by classes, i.e. $\|\cdot\|$ is a function on the set of equivalence of $\mathfrak{V}^*$.

Definition. $\|\mathfrak{K}(\mathfrak{v}_1\| := \|\mathfrak{v}\|$ *is called class norm of* $\mathfrak{K}(\mathfrak{v}_1)$ *if* $\mathfrak{v} \in \mathfrak{K}(\mathfrak{v}_1)$.

Remark 3. Since the gauge domain $\mathbf{D}$ is $\mathfrak{L}$-invariant and therefore the same is valid for all $\lambda \cdot \mathbf{D}$, then

$$\text{(11)} \qquad \text{either } \mathfrak{K} \subset \lambda \cdot \mathbf{D} \quad \text{or } \mathfrak{K} \cap \lambda \cdot \mathbf{D} = \emptyset.$$

1.4. f-corresponding functions as $\mathfrak{G}$-automorphic eigenfunctions of $P(D)$ and $\mathrm{spec}_{\mathfrak{G}}(P(D))$.

For a fixed lattice functional $\mathfrak{u} \in \Gamma^*$ we consider the character $\chi(\mathfrak{u}, \cdot)$ of the isotropy group $\mathfrak{R}(\mathfrak{u}) := \{\sigma \in \mathfrak{L} \mid \sigma^T \mathfrak{u} = \mathfrak{u}\}$,

$$\text{(12)} \qquad \chi(\mathfrak{u}, \sigma) := \varphi_{\mathfrak{u}}(\mathfrak{s}) \quad \forall \sigma \in \mathfrak{R}(\mathfrak{u}),\ (\sigma, \mathfrak{s}) \in \mathfrak{G},$$

$$\text{(13)} \qquad \varphi_{\mathfrak{u}}(\mathfrak{x}) := \exp\{2\pi i \langle \mathfrak{u}, \mathfrak{x} \rangle\},\ \mathfrak{x} = x^h \mathfrak{b}_h \in \mathfrak{V}.$$

If $\chi(\mathfrak{u}, \cdot)$ is the principal character of $\mathfrak{R}(\mathfrak{u})$ (i.e. $\chi(\mathfrak{u}, \sigma) = 1\ \forall \sigma \in \mathfrak{R}(\mathfrak{u})$ then $\mathfrak{u}$ is said to be a principal lattice vector and $\mathfrak{k}(\mathfrak{u})$ the principal class $\mathfrak{f}$ because $\mathfrak{f} = \mathfrak{k}(\mathfrak{u})$ contains only such principal vectors.

Remark 4. If $\mathfrak{u} \in \Gamma^*$ is non-principal the same is valid for all vectors from $\mathfrak{k}(\mathfrak{u})$.

Let $\mathfrak{H}$ be the set of all principal classes $\mathfrak{f} \subset \Gamma^*$, $\mathfrak{u} \in \mathfrak{f} = \{\mathfrak{u}_1, \ldots, \mathfrak{u}_l\}$ and let $\{\sigma_1, \ldots, \sigma_l\}$ be a system of representatives of the left coset decomposition $(\mathfrak{L}/\mathfrak{R}(\mathfrak{u}))_L$. For $S_\nu = (\sigma_\nu, \mathfrak{s}_\nu) \in \mathfrak{G}$ $(\nu = 1, \ldots, l)$ we consider the "$\mathfrak{f}$-corresponding" functions

$$\text{(14)} \qquad \psi_{\mathfrak{f}} = \frac{1}{\sqrt{l}} \sum_{\nu=1}^{l} \varphi_{\mathfrak{u}} \circ S_\nu$$

which – normed to 1 – form a complete orthonormal system $\{\psi_{\mathfrak{f}} \mid \mathfrak{f} \in \mathfrak{H}\}$ in $L_2(\mathfrak{G})$ ([7]).

Proposition 1. *For each principal class* $\mathfrak{f} \in \mathfrak{H}$ *there exists in (3) only one eigenvalue of* $P(D)$*, namely*

$$\text{(15)} \qquad \mu_{\mathfrak{f}} = (2\pi)^q \cdot \|\mathfrak{f}\|^q \text{ with } m_{\mathfrak{G}}(\mu_{\mathfrak{f}}) = \#\{\mathfrak{f}' \in \mathfrak{H} \mid \|\mathfrak{f}'\| = \|\mathfrak{f}\|\}$$

as multiplicity. $\psi_{\mathfrak{f}}$ *is the corresponding eigenfunction (to* $\mu_{\mathfrak{f}}$*) and* $\mathrm{spec}_{\mathfrak{G}}(P(D)) = \{\mu_{\mathfrak{f}} \mid \mathfrak{f} \in \mathfrak{H}\}$ *the complete* $\mathfrak{G}$*-automorphic eigenvalue spectrum of* $P(D)$.

Proof. $P(D)$ is $\mathfrak{G}$-invariant i.e. $P(D)[\varphi \circ S] = P(D)[\varphi] \circ S \quad \forall S \in \mathfrak{G}$. By this the application of $P(D)$ to $\psi_{\mathfrak{f}}$ from (14) yields

$$P(D)[\psi_{\mathfrak{f}}] = \frac{1}{\sqrt{l}} \sum_{\nu=1}^{l} P(D)[\varphi_{\mathrm{u}}] \circ S_\nu. \tag{16}$$

Simple differentiations $\partial/\partial x^i$ of φ_{u} from (13) lead by (1′) to

$$P(D)[\varphi_{\mathrm{u}}] = (2\pi)^q \cdot P(\mathrm{u}) \cdot \varphi_{\mathrm{u}}. \tag{17}$$

Thus (17), (16), (10) and $\|\mathfrak{f}\| = \|\mathrm{u}\|$ if $\mathrm{u} \in \mathfrak{f}$ give

$$p(D)[\psi_{\mathfrak{f}}] = -(2\pi)^q \|\mathfrak{f}\|^q \cdot \psi_{\mathfrak{f}}, \quad \mathrm{u} \in \mathfrak{f}. \tag{18}$$

So we have $\mu_{\mathfrak{f}} = (2\pi)^q \cdot \|\mathfrak{f}\|^q$ and all the principal classes $\mathfrak{f}' \in \mathfrak{H}$ with the same norm as $\mathfrak{f}$ yield the same eigenvalue $\mu = \mu_{\mathfrak{f}}$.

1.5. $N(\lambda)$ as the number of principal classes $\mathfrak{f} \subset \lambda \cdot \mathbf{D}$.

From Proposition 1 it follows: For each $\mathfrak{f} \in \mathfrak{H}$ with $\|\mathfrak{f}\| \leq \lambda/(2\pi)$ there is exactly one $\mu_{\mathfrak{f}} \leq \lambda^q$. Counting all these eigenvalues $\mu_{\mathfrak{f}}$ with regard to their multiplicity we find out: $N(\lambda) = \#\{\mathfrak{f} \in \mathfrak{H} \mid \|\mathfrak{f}\| \leq \lambda/(2\pi)\}$. Because of Remark 3, b) we can read $N(\lambda)$ also in the form of

Proposition 2.

$$N(\lambda) = \#\{\mathfrak{f} \in \mathfrak{H} \mid \mathfrak{f} \subset \lambda \cdot \mathbf{D}\}. \tag{19}$$

2. The asymptotic estimation of the eigenvalue number $N(\lambda)$. We use the method of false position: $N(\lambda)$ will be included by $N_\varepsilon(\lambda - \varepsilon c) \leq N(\lambda) \leq N_\varepsilon(\lambda + \varepsilon c)$ and for both "ε-neighbours" $N_\varepsilon(\lambda \pm \varepsilon c)$ we will prove one and the same asymptotic estimation, the same as (5) in our Theorem.

2.1. $N_\varepsilon(\lambda - \varepsilon c) \leq N(\lambda) \leq N_\varepsilon(\lambda + \varepsilon c)$.

As $\mathfrak{L}$-invariant domain, $\lambda \cdot \mathbf{D}$ has the $\mathfrak{L}$-automorphic characteristic function χ_λ. Therefore χ_λ is a class function and it is well-defined by $\chi_\lambda(\mathfrak{k}) := \chi_\lambda(\mathrm{u})$ if $\mathrm{u} \in \mathfrak{k}$ i.e.

$$\chi_\lambda(\mathfrak{k}) = 1 \text{ if } \mathfrak{k} \subset \lambda \cdot \mathbf{D}, \quad \chi_\lambda(\mathfrak{k}) = 0 \text{ if } \mathfrak{k} \not\subset \lambda \cdot \mathbf{D}. \tag{20}$$

With the help of χ_λ Proposition 2 can be read as

Corollary 1.

$$N(\lambda) = \sum_{\mathfrak{f} \in \mathfrak{H}} \chi_\lambda(\mathfrak{f}). \tag{21}$$

Now instead of $N(\lambda)$ we investigate its ε-neighbour

$$N_\varepsilon(\lambda) := \sum_{\mathfrak{f}\in\mathfrak{H}} (\chi_\lambda * \rho_\varepsilon)(\mathfrak{f}), \tag{22}$$

where the convolution product $(\chi_\lambda * \rho_\varepsilon)(\mathfrak{f})$ is defined as follows: Consider the measure μ^* on $\mathfrak{V}^*$ which was introduced under (6). $\mathfrak{L}$ is of finite order and so μ^* is a $\mathfrak{L}$-invariant measure. Let $\rho : \mathfrak{V}^* \to \mathbf{R}^+$ be a $\mathfrak{L}$-automorphic function from $\mathcal{S}(\mathfrak{V}^*)$ with $\mathrm{supp}\rho \subseteq \mathbf{D}$ and $\int_{\mathfrak{V}^*} \rho(\mathfrak{v})d\mu^*(\mathfrak{v}) = 1$. Then $\rho_\varepsilon(\mathfrak{v}) := \varepsilon^{-n}\cdot\rho(\varepsilon^{-1}\cdot\mathfrak{v})$ is also a smooth and $\mathfrak{L}$-automorphic function and $\mathrm{supp}\rho_\varepsilon = \varepsilon\cdot\mathrm{supp}\rho \subseteq \varepsilon\cdot\mathbf{D}$. For a fixed $\mathfrak{u}\in\Gamma^*$ we use the function $\rho_{\varepsilon;\mathfrak{u}}(\mathfrak{v}) := \rho_\varepsilon(\mathfrak{u}-\mathfrak{v})$,

$$\mathrm{supp}\rho_{\varepsilon;\mathfrak{u}}(\mathfrak{v}) = \{\mathfrak{u} - \mathrm{supp}\rho_\varepsilon\} \subseteq \{\mathfrak{u} - \varepsilon\cdot\mathbf{D}\}. \tag{23}$$

Lemma 3. *$\tilde{f}(2\pi\mathfrak{v}) := (\chi_\lambda * \rho_\varepsilon)(\mathfrak{v})$ is $\mathfrak{L}$-automorphic and so*

$$\tilde{f}(2\pi\mathfrak{k}) = (\chi_\lambda * \rho_\varepsilon)(\mathfrak{k}) := \tilde{f}(2\pi\mathfrak{u}) \text{ if } \mathfrak{u}\in\mathfrak{k} \tag{24}$$

is a well-defined class function on $\mathfrak{K}$ and naturally also on $\mathfrak{H}\subset\mathfrak{K}$.

Proof. In the convolution definition

$$\tilde{f}(2\pi\mathfrak{v}) = \int_{\mathfrak{V}^*} \chi(\mathfrak{v}-\mathfrak{y})\rho_\varepsilon(\mathfrak{y})d\mu^*(\mathfrak{y}), \quad \mathfrak{v}\in\mathfrak{V}^*$$

we transform $\mathfrak{z} = \sigma^T\mathfrak{y}$; the invariance of μ^* then leads to

$$= \int_{\mathfrak{V}^*} \chi_\lambda(\mathfrak{v} - (\sigma^T)^{-1}\mathfrak{z})\rho_\varepsilon((\sigma^T)^{-1}\mathfrak{z})d\mu^*(\mathfrak{z}).$$

Herein the $\mathfrak{L}$-automorphy of χ_λ and ρ_ε produces those of $\tilde{f}$. □

Lemma 4.

$$\tilde{f}(2\pi\mathfrak{k}) = \chi_\lambda(\mathfrak{k}) \quad \forall\mathfrak{k}\not\subset\mathbf{U}_\lambda(\varepsilon), \tag{25}$$

where $\mathbf{U}_\lambda(\varepsilon) := (\lambda+\varepsilon)\cdot\mathbf{D}\setminus int(\lambda-\varepsilon)\cdot\mathbf{D}$ is a $\mathfrak{L}$-invariant "ε-neighbourhood" of $\partial(\lambda\cdot\mathbf{D})$.

Proof. Let $\mathfrak{u}\notin\mathbf{U}_\lambda(\varepsilon)$. If $\mathfrak{u}\overset{\in}{\underset{\notin}{}}\lambda\cdot\mathbf{D}$ then $\mathrm{supp}\rho_{\varepsilon;\mathfrak{u}}\overset{\subset}{\underset{\not\subset}{}}\lambda\cdot\mathbf{D}$ respectively. This result is geometrically grounded and will be needed further on: For $\mathfrak{u}\in\mathfrak{k}$

$$\begin{aligned}\tilde{f}(2\pi\mathfrak{k}) &= \int_{\mathfrak{V}^*} \chi_\lambda(\mathfrak{v})\rho_\varepsilon(\mathfrak{u}-\mathfrak{v})d\mu^*(\mathfrak{v}) \\ &= \int_{\lambda\cdot\mathbb{D}\cup\mathrm{supp}\rho_{\varepsilon;\mathfrak{u}}} 1\cdot\rho_{\varepsilon;\mathfrak{u}}(\mathfrak{v})d\mu^*(\mathfrak{v}).\end{aligned}$$

Let be $\mathfrak{k} \not\subset \mathbf{U}_\lambda(\varepsilon)$. a) If $\mathfrak{k} \not\subset \lambda \cdot \mathbf{D}$, then $\lambda \cdot \mathbf{D} \cap \operatorname{supp}\rho_{\varepsilon;\mathfrak{u}} = \emptyset$ and $\tilde{f}(2\pi\mathfrak{k}) = 0$. b) If $\mathfrak{k} \subset \lambda \cdot \mathbf{D}$ then $\operatorname{supp}\rho_{\varepsilon;\mathfrak{u}} \subset \lambda \cdot \mathbf{D}$ and

$$\begin{aligned}\tilde{f}(2\pi\mathfrak{k}) &= \int_{\operatorname{supp}\rho_{\varepsilon;\mathfrak{u}}} \rho_{\varepsilon;\mathfrak{u}}(\mathfrak{v}) d\mu^*(\mathfrak{v}) = \varepsilon^{-n} \int_{\{\mathfrak{u}-\varepsilon\cdot\mathbb{D}\}} \rho(\frac{1}{\varepsilon}(\mathfrak{u}-\mathfrak{v}))d\mu^*(\mathfrak{v}) \\ &= \int_{\mathbb{D}} \rho(\mathfrak{y})d\mu^*(\mathfrak{y}) = 1.\end{aligned}$$

Proposition 3. *For $c > 2$ we have*

$$N_\varepsilon(\lambda - \varepsilon c) \leq N(\lambda) \leq N_\varepsilon(\lambda + \varepsilon c), \quad \varepsilon > 0. \tag{26}$$

Proof. We restrict the proof so as to show the left hand side of (26). From (22), (24) it follows that $N_\varepsilon(\lambda) = \sum \tilde{f}(2\pi\mathfrak{f})$ where the summation over all $\mathfrak{f} \in \mathfrak{H}$ can be distributed into two parts by (25):

$$N_\varepsilon(\lambda) = \sum_{\mathfrak{H}\ni\mathfrak{f}\subset\mathbb{U}_\lambda(\varepsilon)} \tilde{f}(2\pi\mathfrak{f}) + \sum_{\mathfrak{H}\ni\mathfrak{f}\not\subset\mathbb{U}_\lambda(\varepsilon)} \chi_\lambda(\mathfrak{f}).$$

Obviously for all $\mathfrak{k} \in \mathfrak{K}$ we have $\chi_{\lambda-c\varepsilon}(\mathfrak{k}) \lneq \chi_\lambda(\mathfrak{k})$ and $(\chi_{\lambda-c\varepsilon} * (\rho_\varepsilon)(\mathfrak{K}) \leq (\chi_\lambda * \rho_\varepsilon)(\mathfrak{k})$. Therefore

$$N_\varepsilon(\lambda - c\varepsilon) \leq \sum_{\mathfrak{H}\ni\mathfrak{f}\subset\mathbb{U}_{\lambda-c\varepsilon}(\varepsilon)} \tilde{f}(2\pi\mathfrak{f}) + \sum_{\mathfrak{H}\ni\mathfrak{f}\not\subset\mathbb{U}_{\lambda-c\varepsilon}(\varepsilon)} \chi_\lambda(\mathfrak{f}). \tag{27}$$

Because of $c > 2$ it follows $\mathbf{U}_{\lambda-c\varepsilon}(\varepsilon) \cap \mathbf{U}_\lambda(\varepsilon) = \emptyset$. Thus from $\mathfrak{f} \subset \mathbf{U}_{\lambda-c\varepsilon}(\varepsilon)$ we have $\mathfrak{f} \not\subset \mathbf{U}_\lambda(\varepsilon)$ and then $\tilde{f}(2\pi\mathfrak{f}) = \chi_\lambda(\mathfrak{f})$ from (25). Therefore the right hand side of (27) is, according to (21), equal to $N(\lambda)$. □

2.2. Reformation of $N_\varepsilon(\lambda)$ by Günther's Poisson formula. Although used later, we introduce here the vector spaces resp.– modules and the difference module ([7])

$$\begin{aligned}&\mathfrak{V}(\sigma) := \ker(\sigma - id), \quad \mathfrak{V}^*(\sigma) := \ker(\sigma^T - id), \\ &\mathfrak{V}^\perp(\sigma) := \operatorname{im}(\sigma - id), \quad \mathfrak{V}^{*\perp}(\sigma) := \operatorname{im}(\sigma^T - id); \\ &\Gamma(\sigma) := \Gamma \cap \mathfrak{V}(\sigma), \quad \Gamma^*(\sigma) := \Gamma^* \cap \mathfrak{V}^*(\sigma), \\ &\Gamma^\perp(\sigma) := \Gamma \cap \mathfrak{V}^\perp(\sigma), \quad \Gamma^{*\perp}(\sigma) := \Gamma^* \cap \mathfrak{V}^{*\perp}(\sigma); \\ &\Gamma^\perp(\sigma) - \Gamma_e^\perp(\sigma) \text{ for } \Gamma_e^\perp(\sigma) := (\sigma - id)(\Gamma) \subseteq \Gamma^\perp(\sigma)\end{aligned} \tag{28}$$

according to the decomposition $\mathfrak{V} = \mathfrak{V}(\sigma) \oplus \mathfrak{V}^\perp(\sigma)$, $\mathfrak{V}^* = \mathfrak{V}^*(\sigma) \oplus \mathfrak{V}^{*\perp}(\sigma)$.

Let $m = n(\sigma) = \dim \mathfrak{V}(\sigma)$ $(= \dim \mathfrak{V}^*(\sigma), \Gamma(\sigma), \Gamma^*(\sigma))$ and $n - n(\sigma)$ be the dimension of the belonging complements, $e(\sigma) :=\operatorname{card}(\Gamma^\perp(\sigma) - \Gamma_e^\perp(\sigma))$. Now μ, μ_σ^*, $\mu_\sigma^\perp$

denote the Lebesgue measures on $\mathfrak{V}$, $\mathfrak{V}^*(\sigma)$, $\mathfrak{V}^\perp(\sigma)$, normed to 1 with respect to the belonging fundamental domains $\mathcal{F}(\Gamma)$, $\mathcal{F}(\Gamma^*(\sigma))$, $\mathcal{F}(\Gamma^\perp(\sigma))$.

Günter's Poisson formula for space groups ([7], Theorem, parts (3.13), (3.14)) says: For a Schwartz function $f \in \mathcal{S}(\mathfrak{V})$ and its Fourier transform $\tilde{f}$, for a $\mathfrak{T}$-conjugate class τ of $\mathfrak{G}$ and for the translation part $\mathfrak{b}$ of $S = (\sigma, \mathfrak{b}) \in \tau$ it is valid

$$I_\tau(f) := \frac{1}{e(\sigma)} \int_{\mathfrak{V}^\perp} f(\mathfrak{y} + \mathfrak{b}) d\mu_\sigma^\perp(\mathfrak{y}) \tag{29}$$

$$= (1/(2\pi)^{n(\sigma)} e(\sigma)) \int_{\mathfrak{V}^*(\sigma)} \exp\{i\langle \mathfrak{v}, \mathfrak{b}\rangle \tilde{f}(\mathfrak{v}) d\mu_\sigma^*(\mathfrak{v}); \tag{30}$$

$I_\tau(f)$ depends only on τ, i.e. it is independent of the choice of $S \in \tau$ and

$$\sum_{\mathfrak{f} \in \mathfrak{H}} \frac{1}{\operatorname{card} \mathfrak{f}} \sum_{\mathfrak{u} \in \mathfrak{f}} \tilde{f}(2\pi \mathfrak{u}) = \frac{1}{r} \sum_{\Theta \in \Omega} \sum_{\tau \in \Theta} I_\tau(f). \tag{31}$$

The summation relative to Θ, τ is to be understood from the decomposition of $\mathfrak{G}$ into the set Ω resp. $\mathcal{T}$ of the $\mathfrak{G}$-conjugate resp. $\mathfrak{T}$-conjugate classes Θ resp. τ of $\mathfrak{G}$ and the additional decomposition of each Θ into a finite number of τ. Of course we can decompose also $\mathfrak{G} = S_1 \circ \mathfrak{T} + \ldots + S_r \circ \mathfrak{T}$ into its cosets $\varkappa(\sigma_\nu) = S_\nu \circ \mathfrak{T}$, $S_\nu = (\sigma_\nu, \mathfrak{s}_\nu)$ and furthermore $\varkappa(\sigma_\nu)$ in classes τ. Then we can perform the summation over all $\tau \in \mathcal{T}$ on the right of (31) also by running through $\tau \in \varkappa(\sigma_\nu)$ $\forall \nu = 1, \ldots, r$, i.e. $\tau \in \varkappa(\sigma)$ $\forall \sigma \in \mathfrak{L}$. From (22) and (24) on the left of (31) after the summation is $N_\epsilon(\lambda)$ and therefore finally we obtain:

Corollary 2.

$$N_\epsilon(\lambda) = \frac{1}{r} \sum_{\sigma \in \mathfrak{L}} \sum_{\tau \subseteq \varkappa(\sigma)} I_\tau(f). \tag{32}$$

2.3. Decomposition of $N_\epsilon(\lambda)$ according to the fixed point behaviour of $(\sigma, \mathfrak{s}) \in \mathfrak{G}$. Each $S \in \varkappa(\sigma)$ is of the form $S = (\sigma, \mathfrak{s} + \mathfrak{t})$ with one and the same $\sigma \in \mathfrak{L}$, where $\mathfrak{s} \in \mathfrak{V}$ is well-defined modulo Γ and $\mathfrak{t} \in \Gamma$. So the summation over $\tau \subseteq \varkappa(\sigma)$ in (32) means with regard to (29) only a summation relative to $\mathfrak{b}$, namely: The decomposition of the coset $\varkappa(\sigma) = \cup_{\iota \in \mathcal{J}_\sigma} \tau_\iota$ into $\mathfrak{T}$-conjugate classes τ_ι with representatives $S_\iota = (\sigma, \mathfrak{b}_\iota) \in \tau_\iota$ ($\mathcal{J}_\sigma$: appropriate index set) results into

$$\begin{aligned} \sum_{\tau \subseteq \varkappa(\sigma)} I_\tau(f) \quad & \sum_{\iota \in \mathcal{J}_\sigma} \frac{1}{e(\sigma)} \int_{\mathfrak{V}^\perp(\sigma)} f(\mathfrak{y} + \mathfrak{b}_\iota) d\mu_\sigma^\perp(\mathfrak{y}) \\ & \sum_{\iota \in \operatorname{rep}(\Gamma - \Gamma_\epsilon^\perp(\sigma))} \frac{1}{e(\sigma)} \int_{\mathfrak{V}^\perp(\sigma)} f(\mathfrak{y} + \mathfrak{a} + \mathfrak{t}) d\mu_\sigma^\perp(\mathfrak{y}) \end{aligned} \tag{33}$$

where (33) may be explained by the easy-to-prove

Lemma 5. $(\sigma, \mathfrak{a} + \mathfrak{t}'), (\sigma, \mathfrak{a} + \mathfrak{t}'')$ *are* $\mathfrak{T}$*-conjugate iff* $\mathfrak{t}' - \mathfrak{t}'' \in \Gamma_e^{\perp}(\sigma)$.

Now in (33) $S_\iota = (\sigma, \mathfrak{b}_\iota)$, where $\mathfrak{b}_\iota = \mathfrak{a} + \mathfrak{t}_\iota$ runs through a complete system of representatives of the decomposition of $\varkappa(\sigma)$, so $\mathfrak{t}_\iota$ runs through the complete system of representatives $\mathrm{rep}(\Gamma - \Gamma_e^{\perp}(\sigma))$ and vice versa.

Next (33) can be written as

$$(34) \qquad \sum_{\tau \subseteq \varkappa(\sigma)} I(f) = \sum_{\mathfrak{t} \in \mathrm{rep}(\Gamma - \Gamma_e^{\perp}(\sigma))} \int_{\mathfrak{V}^{\perp}(\sigma)} f(\mathfrak{y} + \mathfrak{a} + \mathfrak{t}) d\mu_e^{\perp}(\mathfrak{y}).$$

We know that $\Gamma_e^{\perp}(\sigma) \subseteq \Gamma^{\perp}(\sigma) \subseteq \Gamma$ for these three $\mathbb{Z}$-modules. $\Gamma_e^{\perp}(\sigma)$ is an invariant subgroup in Γ as well as $\Gamma^{\perp}(\sigma)$. So by the second isomorphism theorem $\Gamma - \Gamma^{\perp}(\sigma) \cong [\Gamma - \Gamma_e^{\perp}(\sigma)] - [\Gamma^{\perp}(\sigma) - \Gamma_e^{\perp}(\sigma)]$ and by the fundamental theorem for abelian groups $\Gamma - \Gamma_e^{\perp}(\sigma) \cong [\Gamma - \Gamma^{\perp}(\sigma)] - [\Gamma^{\perp}(\sigma) - \Gamma_e^{\perp}(\sigma)]$. The difference module $\Gamma^{\perp}(\sigma) - \Gamma_e^{\perp}(\sigma)$ contains $e(\sigma)$ elements.

Now we separate from the sum (34) the "fixed point part" by means of

Lemma 6. $|i|$ $(\sigma, \mathfrak{a} + \mathfrak{t}) \in \mathfrak{G}$ *has a fixed point* $\mathfrak{x}_0 \in \mathfrak{V}$ *iff* $\mathfrak{a} + \mathfrak{t} \in \mathfrak{V}^{\perp}(\sigma)$.

$|ii|$ *If* $\mathfrak{t}_i \in \mathrm{rep}(\Gamma - \Gamma^{\perp}(\sigma))$ *with* $\mathfrak{a} + \mathfrak{t}_i \in \mathfrak{V}^{\perp}(\sigma)$ $(i = 1, 2)$, *then* $\mathfrak{t}_2 \in (\mathfrak{t}_1 + \Gamma^{\perp}(\sigma))$ *is representative of the same coset of* Γ *relative to* $\Gamma^{\perp}(s)$, *and analogously for* $\mathfrak{t}_1$.

$|iii|$ *For* $(\sigma, \mathfrak{a}) \in \mathfrak{G}$ *there exists at most one* $\mathfrak{t}_0 \in \mathrm{rep}(\Gamma - \Gamma^{\perp}(\sigma))$ *giving rise to fixed points of* $(\sigma, \mathfrak{a} + \mathfrak{t}_0) \in \varkappa(\sigma)$.

($|i|$ is the Lemma 4 from [1], $|ii|$ is clear almost by itself and $|iii|$ follows from $|ii|$).

With respect to Lemma 6 let

$$(35) \qquad \mathring{\mathfrak{L}} := \{\sigma \in \mathfrak{L} \mid \exists \mathfrak{t}_0 \in \mathrm{rep}(\Gamma - \Gamma^{\perp}(\sigma)) \quad \text{with} \quad \mathfrak{a} + \mathfrak{t}_0 \in \mathfrak{V}^{\perp}(\sigma)\}.$$

Because of $|iii|$ for each $\sigma \in \mathring{\mathfrak{L}}$ there is exactly one such $\mathfrak{t}_0$. So for $\sigma \in \mathfrak{L}$ we can exempt $\mathrm{rep}(\Gamma - \Gamma^{\perp}(\sigma))$ in (34) from the vector $\mathfrak{t}_0$ – if there is any which produces fixed points $\mathfrak{x}_0 \in \mathfrak{V}$ according to $|i|$:

$$(36) \qquad R(\sigma) := \begin{cases} \mathrm{rep}(\Gamma - \Gamma^{\perp}(\sigma)) \setminus \{\mathfrak{t}_0\} & \sigma \in \mathring{\mathfrak{L}} \\ & \text{if} \\ \mathrm{rep}(\Gamma - \Gamma^{\perp}(\sigma)) & \sigma \in \mathfrak{L} \setminus \mathring{\mathfrak{L}} \end{cases}.$$

Now we insert (34) in (32) and sum up a) first over all $\sigma \in \mathring{\mathfrak{L}}$ then b) over $\sigma \in \mathfrak{L} \setminus \mathring{\mathfrak{L}}$. Furthermore the inward sum of a) will be decomposed into that part with $\mathfrak{t} \in R(\sigma)$ and that with $\mathfrak{t} = \mathfrak{t}_0$.

Then Corollary 2 can be read as

Corollary 3.

$$(J_0) \qquad N_\varepsilon(\lambda) = \frac{1}{r}\Big\{\sum_{\sigma\in\overset{\circ}{\mathfrak{L}}} \int_{\mathfrak{V}^\perp(\sigma)} f(\mathfrak{y}+\mathfrak{a}+\mathfrak{t}_0)\,d\mu_\sigma^\perp(\mathfrak{y})$$

$$(J) \qquad +\sum_{\sigma\in\mathfrak{L}} \sum_{\mathfrak{t}\in R(\sigma)} \int_{\mathfrak{V}^\perp(\sigma)} f(\mathfrak{y}+\mathfrak{a}+\mathfrak{t})\,d\mu_\sigma^\perp(\mathfrak{y})\Big\}.$$

2.4. Estimation of the integral J_0 from (J_0). The transformation $\mathfrak{z} = \mathfrak{y}+\mathfrak{a}+\mathfrak{t}_0$ of J_0 for $\sigma \in\overset{\circ}{\mathfrak{L}}$ and next the substitution $\Phi(\mathfrak{z}+\mathfrak{b}) := f(\mathfrak{z})$ with $\mathfrak{b}$ from $(\sigma,\mathfrak{b}) \in \tau \in \mathcal{T}$ leads by (29) to $J_0 = e(\sigma)I_\tau(\Phi)$. If we take for $I_\tau(\Phi)$ its alternative expression (30) and use (after a substitution) the Fourier transform $\tilde{\Phi}(\mathfrak{v}) = \exp\{-i\langle\mathfrak{v},\mathfrak{b}\rangle\}\tilde{f}(\mathfrak{v})$ we obtain

$$J_0 = (1/2\pi)^{n(\sigma)} \int_{\mathfrak{V}^*(\sigma)} \tilde{f}(\mathfrak{v})\,d\mu_\sigma^*(\mathfrak{v}) = \int_{\mathfrak{V}^*(\sigma)} (\chi_\lambda * \rho_\varepsilon)(\mathfrak{v})\,d\mu_\sigma^*(\mathfrak{v}),$$

using the transformation $\mathfrak{v} = 2\pi\mathfrak{v}'$ and Lemma 3. Writing out the product $\chi_\lambda * \rho_\varepsilon$, using $\rho_\varepsilon(\mathfrak{v}) = \varepsilon^{-n}\rho(\varepsilon^{-1}\cdot\mathfrak{v})$ and the transformation $\mathfrak{w} = \varepsilon\mathfrak{w}'$ of $\mathfrak{V}^*$ we obtain

$$(37) \qquad J_0 = \int_{\mathfrak{V}^*(\sigma)} \int_{\mathfrak{V}^*} \chi_\lambda(\mathfrak{v}-\varepsilon\mathfrak{w}')\rho(\mathfrak{w}')\,d\mu^*(\mathfrak{w}')\,d\mu_\sigma^*(\mathfrak{v}).$$

If we divide into two parts the integrand $\chi_\lambda(\mathfrak{v}-\varepsilon\mathfrak{w})\rho(\mathfrak{w}) = \chi_\lambda(\mathfrak{v})\rho(\mathfrak{w}) + (\chi_\lambda(\mathfrak{v}-\varepsilon\mathfrak{w}) - \chi_\lambda(\mathfrak{v}))\rho(\mathfrak{w})$ and pay attention to the definition of χ_λ, ρ and $n(\sigma) = \dim\mathfrak{V}^*(\sigma)$, then the integration in (37) yields

$$J_0 = \mathrm{vol}_{n(\sigma)}(\mathfrak{V}^*(\sigma)\cap\mathbf{D})\cdot\lambda^{n(\sigma)} \qquad (\text{see}(6))$$

$$(J_1) \qquad +\int_{\mathfrak{V}^*(\sigma)}\int_{\mathbb{D}} (\chi_\lambda(\mathfrak{v}-\varepsilon\mathfrak{w}) - \chi_\lambda(\mathfrak{v}))\rho(\mathfrak{w})\,d\mu^*(\mathfrak{w})\,d\mu_\sigma^*(\mathfrak{v}).$$

Estimation of the integral J_1 from (J_1):

Lemma 7.

$$(38) \qquad \chi_\lambda(\mathfrak{v}-\varepsilon\mathfrak{w}) - \chi_\lambda(\mathfrak{v}) = 0 \quad \forall\mathfrak{v}\notin\mathbf{U}_\lambda(\varepsilon),\ \forall\mathfrak{w}\in\mathbf{D}.$$

P r o o f. This difference is zero only in two cases: a) $\chi_\lambda(\mathfrak{v}-\varepsilon\mathfrak{w}) - \chi_\lambda(\mathfrak{v}) = 0-0$ iff $\mathfrak{v}-\varepsilon\mathfrak{w}\notin\lambda\cdot\mathbf{D}$ and $\mathfrak{w}\notin\lambda\cdot\mathbf{D}$, b) $= 1-1$ iff $\mathfrak{v}-\varepsilon\mathfrak{w}\in\lambda\cdot\mathbf{D}$ and $\mathfrak{v}\in\lambda\cdot\mathbf{D}$. Now a simple geometric consideration yields the assertion.

(38) and the definition of ρ make it possible to estimate J_1:

$$J_1 \leq \int_{\mathfrak{V}^*(\sigma)\cap\mathbb{U}_\lambda(\varepsilon)} \int_{\mathbb{D}} 1\cdot\rho(\mathfrak{w}d\mu^*(\mathfrak{w})d\mu^*_\sigma(\mathfrak{v})$$

$$= \int_{\mathfrak{V}^*(\sigma)\cap\mathbb{U}_\lambda(\varepsilon)} d\mu^*_\sigma(\mathfrak{v}) = \mathrm{vol}_{n(\sigma)}(\mathfrak{V}^*(\sigma)\cap\mathbf{U}_\lambda(\varepsilon))$$

$$= \left((\lambda+\varepsilon)^{n(\sigma)} - (\lambda-\varepsilon)^{n(\sigma)}\right)\cdot\mathrm{vol}_{n(\sigma)}(\mathfrak{V}^*(\sigma)\cap\mathbf{D}), \tag{39}$$

the latter by (25). Now choose $\varepsilon = \lambda^{-1+2/(n+1)}$. Then in (39) the factor before $\mathrm{vol}_{n(\sigma)}$ is of order $O(\lambda^{n-2+2/(n+1)})$ because $n(\sigma)\leq n\ \forall\sigma\in\mathfrak{L}$ (see the notation in (28)). Thus

$$J_0 = \mathrm{vol}_{n(\sigma)}(\mathfrak{V}^*(\sigma)\cap\mathbf{D})\cdot\lambda^{n(\sigma)} + O(\lambda^{n-2+2/(n+1)}). \tag{40}$$

2.5. Estimation of the integral J in (J). Let $\mathfrak{t}\in\mathrm{rep}(\Gamma-\Gamma^\perp(\sigma))$ with $\mathfrak{a}+\mathfrak{t}\notin\mathfrak{V}^\perp(\sigma)$, i.e. $(\sigma,\mathfrak{a}+\mathfrak{t})\in\varkappa(\sigma)$ has no fixed points (Lemma 6, $|i|$). In J set $\mathfrak{z}=\mathfrak{y}+\mathfrak{a}+\mathfrak{t}$ and instead of $f(\mathfrak{z})$ consider its Fourier transform:

$$J = (2\pi)^{-n}\int_{\mathfrak{V}^\perp(\sigma)}\int_{\mathfrak{V}^*}\exp\{i\langle\mathfrak{v},\mathfrak{z}\rangle\}\tilde{f}(\sigma)d\mu^*(\mathfrak{v})d\mu^\perp_\sigma(\mathfrak{y}).$$

Now we set $\mathfrak{v}=2\pi\mathfrak{v}'$, use $\tilde{f}(2\pi\mathfrak{v}')=(\chi_\lambda*\rho_\varepsilon)(\mathfrak{v}')$ (Lemma 3) and write $\chi_\lambda*\rho_\varepsilon$ from the definition of $*$. The so obtained 3-fold integral can be transferred to

$$J = (2\pi)^{2n}\cdot\int_{\mathfrak{V}^\perp(\sigma)}\hat{\rho}_\varepsilon(2\pi\mathfrak{z})\hat{\chi}_\lambda(2\pi\mathfrak{z})d\mu^\perp_\sigma(\mathfrak{y}) \tag{41}$$

where "$\wedge$" means the inverse Fourier transformation. If we use $\hat{\chi}_\lambda(\mathfrak{x})=\lambda^n\cdot\hat{\chi}(\lambda\cdot\mathfrak{x})$ and $\hat{\rho}_\varepsilon(\mathfrak{x})=\hat{\rho}(\varepsilon\cdot\mathfrak{x})$ we obtain from (41)

$$J = (2\pi)^{2n}\cdot\lambda^n\cdot\int_{\mathfrak{V}^\perp(\sigma)}\hat{\rho}(2\pi\varepsilon\cdot\mathfrak{z})\hat{\chi}(2\pi\lambda\cdot\mathfrak{z})d\mu^\perp_\sigma(\mathfrak{y}). \tag{42}$$

For $\mathfrak{v}\in\mathfrak{V}^*$, $\mathfrak{x}\in\mathfrak{v}$ let $\|\mathfrak{v}\|$ be the norm (10) in $\mathfrak{V}^*$, $\|\mathfrak{x}\|$ the belonging infimum norm in $\mathfrak{V}$ which is equivalent to the euclidean norm $|\mathfrak{x}|$: $c_1|\mathfrak{x}|\leq\|\mathfrak{x}\|\leq c_2|\mathfrak{x}|$. Since $\rho\in\mathcal{S}(\mathfrak{V}^*)$ also $\hat{\rho}\imath\mathcal{S}(\mathfrak{V})$ is a Schwartz function,

$$|\hat{\rho}(\mathfrak{x})|\leq M(p)(1+\|\mathfrak{x}\|^2)^{-p}\leq M(p)(1+\varepsilon^2\|\mathfrak{z}\|^2)^{-p},\quad p=0,1,\ldots \tag{43}$$

where $\mathfrak{x}=2\pi\varepsilon\cdot\mathfrak{z}$ so as in (42) and $\mathfrak{z}=\mathfrak{y}+\mathfrak{a}+\mathfrak{t}$. To estimate $\hat{\chi}(2\pi\lambda\cdot\mathfrak{z})$ in (42) we can write because $\mathfrak{z}\neq\mathfrak{o}$

$$|\hat{\chi}(2\pi\lambda\cdot\mathfrak{z})| = |\hat{\chi}(2\pi\|\lambda\cdot\mathfrak{z}\|\cdot(\lambda\cdot\mathfrak{z})/\|\lambda\cdot\mathfrak{z}\|)|. \tag{44}$$

If $\mathfrak{z} \neq \mathfrak{o}$ runs through $\mathfrak{V}$, then $\mathfrak{w} := 2\pi(\lambda \cdot \mathfrak{z})/\|\lambda \cdot \mathfrak{z}\|$ varies in the compact $\mathcal{K} = \{\mathfrak{w} \in \mathfrak{V} \mid \|\mathfrak{w}\| = 2\pi\}$.

Lemma 8. *For every compact* $\mathcal{K} \subset\subset \mathfrak{v} \setminus \{\mathfrak{o}\}$ *and* $t \to \infty$ *one has*

$$|\hat{\chi}(t \cdot \mathfrak{w})| \leq M_{\mathcal{K}} \cdot |t|^{-(n+1)/2} \quad \textit{uniformly for} \quad \mathfrak{w} \in \mathcal{K}.$$

This assertion from [10], 1., (1.7) is clearly applicable to (44) because $\mathfrak{w} := \lambda \cdot \mathfrak{z}/\|\lambda \cdot \mathfrak{z}\|$ varies on the unit sphere $\mathcal{K}$ around of $\mathfrak{o}$ and $t := 2\pi\|\lambda \cdot \mathfrak{z}\|$. Thus from (44) it follows

$$|\hat{\chi}(2\pi\lambda \cdot \mathfrak{z})| \leq M_{\mathcal{K}} \cdot \lambda^{-(n+1)/2} \cdot \|\mathfrak{z}\|^{-(n+1)/2}. \tag{45}$$

Now (43), (45) make possible the following estimation of J from (42):

$$\begin{aligned} J &\leq M \cdot \lambda^{(n-2)/2} \cdot \int_{\mathfrak{V}^{\perp}(\sigma)} (1 + \varepsilon^2\|\mathfrak{z}\|^2)^{-p} \cdot \|\mathfrak{z}\|^{-(n+1)/2} d\mu_{\sigma}^{\perp}(\mathfrak{y}) \\ &= M \cdot \lambda^{(n-1)/2} \cdot \varepsilon^{-(n-1)/2} \times \\ &\times \int_{\mathfrak{V}^{\perp}(\sigma)} \varepsilon^{n(\sigma)} \cdot (1 + \|\mathfrak{x} + \varepsilon(\mathfrak{a} + \mathfrak{t})\|^2)^{-p} \cdot \|\mathfrak{x} + \varepsilon(\mathfrak{a} + \mathfrak{t})\|^{-(n+1)/2} d\mu_{\sigma}^{\perp}(\mathfrak{x}), \end{aligned} \tag{46}$$

for $\mathfrak{x} = \varepsilon \cdot \mathfrak{y}$ and the constant M depending on p. Now we pass on to (J):

$$\sum_{\mathfrak{t} \in R(\sigma)} J \leq M \cdot (\lambda/\varepsilon)^{\frac{n-1}{2}} \cdot \mathcal{J}_{\varepsilon} \tag{47}$$

where $\mathcal{J}_{\varepsilon}$ is the sum over $\mathfrak{t} \in R(\sigma)$ of the $\mathfrak{t}$-belonging integrals in (46). $\mathcal{J}_{\varepsilon}$ can be interpreted as the intermediate sum to the integral

$$\mathcal{J} = \int_{\mathfrak{V} - \mathfrak{V}^{\perp}(\sigma)} \int_{\mathfrak{V}^{\perp}(\sigma)} (1 + \|\mathfrak{x} + \mathfrak{w}\|^2)^{-p} \cdot \|\mathfrak{x} + \mathfrak{w}\|^{-(n+1)/2} d\mu_{\sigma}^{\perp}(\mathfrak{x}) d\mu_{\sigma}^{\perp}(\mathfrak{w});$$

$\mu_{|\sigma}^{\perp}$ is the Lebesgue measure on the difference module $\mathfrak{V} - \mathfrak{V}^{\perp}(\sigma)$, normed by $\mu_{|\sigma}^{\perp}(\mathcal{F}(\Gamma - \Gamma^{\perp}(\sigma))) = 1$; the difference – $\mathbb{Z}$-module $\Gamma - \Gamma^{\perp}(\sigma)$ is to be understood as a lattice in the difference – $\mathbb{R}$-module $\mathfrak{V} - \mathfrak{V}^{\perp}(\sigma)$; $n(\sigma) = \dim(\mathfrak{V} - \mathfrak{V}^{\perp}(\sigma)) = \dim(\Gamma - \Gamma^{\perp}(\sigma))$. For simplification of $\mathcal{J}$ Lemma 3.1, (3.12) from [7] reads: If $\varphi \in L_1(\mathfrak{V})$, $\bar{\mathfrak{w}} := \mathfrak{w} + \mathfrak{V}^{\perp}(\sigma)$ is a coset of $\mathfrak{V} - \mathfrak{V}^{\perp}(\sigma)$, so

$$\int_{\mathfrak{V} - \mathfrak{V}^{\perp}(\sigma)} \int_{\mathfrak{V}^{\perp}(\sigma)} \varphi(\mathfrak{x} + \mathfrak{w}) d\mu_{\sigma}^{\perp}(\mathfrak{x}) d\mu_{|\sigma}^{\perp}(\bar{\mathfrak{w}}) = \int_{\mathfrak{V}} \varphi(\mathfrak{y}) d\mu(\mathfrak{y}).$$

It follows that

$$\mathcal{J} = \int_{\mathfrak{V}} (1 + \|\mathfrak{y}\|^2)^{-p} \cdot \|\mathfrak{y}\|^{-(n+1)/2} d\mu(\mathfrak{y}).$$

Because of $\lim_{\varepsilon \to 0} \mathcal{J}_\varepsilon = \mathcal{J}$, respectively by $\mathcal{J}_\varepsilon = \mathcal{J} + o(\varepsilon)$, and according to the choice $\varepsilon = \lambda^{-1+2/(n+1)}$ (established in (39)) we obtain from (J), (47)

$$\sum_{\mathfrak{t} \in R(\sigma)} \int_{\mathfrak{V}^\perp(\sigma)} f(\mathfrak{y} + \mathfrak{a} + \mathfrak{t}) d\mu_\sigma^\perp(\mathfrak{y}) = O(\lambda^{n-2+2/(n+1)}). \tag{48}$$

2.6. Estimation of $N_\varepsilon(\lambda)$ and $N(\lambda)$. If we consider the estimations (40), (48) and Corollary 3, we obtain

$$N_\varepsilon(\lambda) = \frac{1}{r} \sum_{\sigma \in \mathring{\mathfrak{L}}} \mathrm{vol}_{n(\sigma)}(\mathfrak{V}^*(\sigma) \cap \mathbf{D}) \lambda^{n(\sigma)} + O(\lambda^{n-2+2/(n+1)}). \tag{49}$$

Finally let $\varepsilon = \lambda^{-1+2/(n+1)}$ (so as chosen for (48)). Because of $(\lambda + c\varepsilon)^{n(\sigma)} = \lambda^{n(\sigma)} + O(\lambda^{n(\sigma)-2+2/(n+1)})$ and the fact that in (49) all $\sigma \in \mathring{\mathfrak{L}}$ with $n(\sigma) \leq n - 2$ yield summands belonging to $O(\ldots)$ we obtain from (26)

$$N(\lambda) = \frac{1}{r} \sum_{\substack{\sigma \in \mathring{\mathfrak{L}} \\ n(\sigma) \geq n-1}} \mathrm{vol}_{n(\sigma)}(\mathfrak{V}^*(\sigma) \cap \mathbf{D}) \lambda^{n(\sigma)} + O(\lambda^{n-2+2/(n+1)}). \tag{50}$$

a) The main term in (5) comes from (50) for $n(\sigma) = n$. Then $\sigma = e$, $\mathfrak{V}^*(e) = \mathfrak{V}^* \supset \mathbf{D}$ and $\delta_e = 1$ (see (35) and Lemma 6 $|i|$).

b) The second term in (5) arises for $n(\sigma) = n - 1$, $\sigma \in \mathring{\mathfrak{L}}$.

Thus the proof of the theorem is finished. □

Remark 5. Theorem (5) is true also for operators $P(D)$ from (1) with $|\alpha| \leq q$.

Example. For the euclidean spaces $\mathfrak{V}$, $\mathfrak{V}^*$, for a properly discontinuous group $\mathfrak{G}$ of isometries acting on $\mathfrak{V}$ we consider the operator $P(D) = \Delta^k$, $k \in \mathbf{N}$, Δ: Laplacian. Δ^k is $\mathfrak{G}$-invariant because Δ is isometry-invariant; $q = 2k$. The characteristic polynomial $P(\mathfrak{v}) = -(v_1^2 + \ldots + v_n^2)^k$ ($\mathfrak{v} = v_i \mathfrak{b}^i \in \mathfrak{V}^*$, $\{\mathfrak{b}^i\}$ orthonormal base) leads to the norm $\|\mathfrak{v}\| = (v_1^2 + \ldots + v_n^2)^{1/2}$ and defines the gauge domain $\mathbf{D} : \|\mathfrak{v}\| \leq 1/(2\pi)$, which is the n-dimensional ball in $\mathfrak{V}^*$ with $\mathrm{vol}_n(\mathbf{D}) = 1/(2\pi)^n \cdot \mathrm{vol}_n(\mathbf{K}_n)$, $\mathbf{K}_n = 2\pi \cdot \mathbf{D}$ is the n-dimensional unit ball. If $k = n - 1$ we obtian for the $(n-1)$-dimensional hyper-planes $\mathfrak{V}^*(\sigma) \subset \mathfrak{V}^*$ going throughout $\mathfrak{o} \in \mathfrak{V}^*$: $\mathrm{vol}_{n-1}(\mathfrak{V}^*(\sigma) \cap \mathbf{D}) = 1/(2\pi)^{n-1} \cdot$

$\mathrm{vol}_{n-1}(\mathfrak{V}^*(\sigma) \cap \mathbf{K}_n) = 1/(2\pi)^{n-1} \cdot \mathrm{vol}_{n-1}(\mathbf{K}_{n-1})$. Denote by $|\mathcal{F}(\Gamma^*)|$, $|\mathcal{F}(\Gamma^*(\sigma))|$ the volumes of the fundamental domains of the lattices Γ^*, $\Gamma^*(\sigma)$. Then

$$N(\lambda) = \frac{1}{r}\Big(\frac{1}{(2\sqrt{\pi})^n|\mathcal{F}(\Gamma^*)|\Gamma(\frac{n}{2}+1)} \cdot \lambda^n + \frac{1}{(2\sqrt{\pi})^{n-1}\Gamma(\frac{n-1}{2}+1)} \sum_{\sigma \in \mathfrak{L}_{n-1}}{}' \frac{\delta_\sigma}{|\mathcal{F}(\Gamma^*(\sigma))|} \cdot \lambda^{n-1}\Big) + O(\lambda^{n-2+2/(n+1)}). \tag{51}$$

To understand how to calculate $\mathfrak{L}_{n-1}$, δ_σ let $\mathfrak{G}$ be a crystallographic group, e.g. $\mathfrak{G} = \Delta^2_{p31m}$ for $n = 2$ (see [1], 4., Example). Then we obtain $N(\lambda) = \sqrt{3}(48\pi)^{-1} \cdot \lambda^2 + (4\pi)^{-1} \cdot \lambda + O(\lambda^{2/3})$.

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