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# ON THE UPPER BOUNDS FOR THE KISSING NUMBERS

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ABSTRACT. Let us denote by $\tau_n$ the maximum number (kissing number) of nonoverlapping unit spheres that can simultaneously touch a unit sphere in $\mathbb{R}^n$. Only the following values of $\tau_n$ are known (Levenshtein (1979) and Odlyzko & Sloane (1979)): $\tau_1 = 2$, $\tau_2 = 6$, $\tau_3 = 12$, $\tau_8 = 240$ and $\tau_{24} = 196560$. Estimates for $\tau_n$ have been made by different authors ([1,7,8,9,10,11]). For example, we have $24 \le \tau_4 \le 25$, $40 \le \tau_5 \le 46$, $72 \le \tau_6 \le 82$, etc. The best known upper bounds for $\tau_n$ ($4 \le n \le 24$) were obtained by Odlyzko & Sloane (1979) using a computer to find suitable polynomials satisfying the conditions of the following theorem ([3, 8]):

**Theorem.** *Let $n \ge 3$ and $P(t)$ be a real polynomial such that:*
(C1) $P(t) \le 0$ *for* $-1 \le t \le 1/2$;
(C2) *the coefficients in the expansion of $P(t)$ in terms of Jacobi polynomials*

$$P(t) = \sum_{i=0}^{k} l_i P_i^{((n-3)/2,(n-3/2)}(t)$$

*satisfy* $l_0 > 0$, $l_i \ge 0$ *for* $i = 1, 2, \ldots, k$.
*Then* $\tau_n \le P(1)/l_0$.

In this paper we obtain some restrictions in the form of the polynomials $P(t)$ of degree $k \ge 9$ giving good upper bounds for $\tau_4$ and $\tau_5$. We also propose a new method for finding good polynomials for the above theorem.

**1. Conditions for extremality.** Instead of $P_i^{((n-3)/2,(n-3/2)}(t)$ we shall write $P_i(t)$. Let $A_n = \{P(t) \in \mathbb{R}[t] : \text{(C1) and (C2) are satisfied}\}$ be the set of polynomials giving upper bounds for $\tau_n$. We denote by $l_i(P)$ the coefficients in the expansion of $P(t)$ in terms of Jacobi polynomials.

A polynomial $P(t) \in A_n$ is called extremal for $\tau_n$ if it gives the best upper bound for $\tau_n$ among the polynomials from $A_n$ having the same or lower degree.

We prove that if the polynomial $P(t)$ of degree $k \geq 9$ is extremal for $\tau_4$ or $\tau_5$, then $l_6(P) = l_7(P) = l_8(P) = 0$ under some other conditions. It is not difficult to check that the following Lemma is true:

**Lemma 1.1.** *Let* $P(t) = \sum_{i=0}^{k} l_i(P)P_i(t)$, $Q(t) = \sum_{i=0}^{m} l_i(Q)P_i(t)$ *and* $H(t) = \alpha P(t)+\beta Q(t) = \sum_{i=0}^{deg\,H} l_i(H)P_i(t)$, *where* $\alpha > 0$, $\beta > 0$, $l_0(P) > 0$, $l_0(Q) > 0$, $P(1) > 0$ *and* $Q(1) > 0$. *Then the number* $H(1)/l_0(H)$ *is between the numbers* $P(1)/l_0(P)$ *and* $Q(1)/l_0(Q)$.

In particular, it follows from Lemma 1.1 that if two polynomials give some upper bounds for $\tau_n$, then their linear combinations with positive coefficients give upper bounds for $\tau_n$ in the interval between these two bounds.

**Lemma 1.2.** *If the polynomial* $P(t)$ *of degree* $k \geq 9$ *is extremal for* $\tau_4$, *then* $l_6(P) = 0$.

Proof. We consider the polynomial

$$Q_1(t) = S_4(t) - \alpha\left(t^3 + 3t^2/8 - 3t/8 - 1/16\right)^2$$

where

$$S_4(t) = (t^3 + at^2 + bt + c)^2 \left[(t-e)^2 + f(1/2-t)(1+t)\right](t - 1/2) \in A_4$$

for $a = 1.368502$, $b = 0.5015963831$, $c = 0.0446307$, $e = 0.0994729872$, $f = 0.744631$ is the polynomial obtained in [2,8] for $\tau_4$; $\alpha = 0.1812926$. Setting $\left(t^3 + \frac{3t^2}{8} - \frac{3t}{8} - \frac{1}{16}\right)^2 = \sum_{i=0}^{6} l_i^* P_i(t)$ we have $l_i(Q_1) = l_i(S_4) - \alpha l_i^*$ for $i = 0, 1, 2, \ldots, 6$ and $l_i(Q_1) = l_i(S_4)$ for $i = 7, 8, 9$. When $\alpha = 0.1812926 = l_5(S_4)/l_5^*$ we have $l_9(Q_1) > 0$, $l_8(Q_1) = l_7(Q_1) = l_5(Q_1) = 0$, $l_6(Q_1) < 0$, $l_i(Q_1) > 0$, for $i = 0, 1, 2, 3, 4$ and $Q_1(1)/l_0(Q_1) = 21.427$.

Assume that $P(t)$ is extremal for $\tau_4$, $deg(P) \geq 9$ and $l_6(P) > 0$. Then there exists $\beta > 0$ such that $l_6(H) = l_6(P)+\beta l_6(Q_1) \geq 0$ where $H(t) = P(t)+\beta Q_1(t)$. Obviously $H(t) \in A_n$ and $deg(H) = deg(P)$. Using Lemma 1.1 we obtain $21.427 = Q_1(1)/l_0(Q_1) < H(1)/l_0(H) < P(1)/l_0(P)$ – a contradiction ($24 \leq \tau_4 \leq 25$ by [6,chap.13]). □

**Lemma 1.3.** *If the polynomial* $P(t)$ *of degree* $k \geq 9$ *is extremal for* $\tau_4$, *then* $l_7(P) = 0$.

Proof. The proof is similar to the proof of Lemma 1.2 using the polynomial

$$Q_2(t) = t^2(t+1)^2(t+1/2)^2(t^2 - 5t/2 + 9/5)(t - 1/2).$$

instead of $Q_1(t)$. We have now $Q_2(1)/l_0(Q_2) = 24$ and $l_i(Q_2) \geq 0$ for $i \neq 7$ but $l_7(Q_2) < 0$. □

**Lemma 1.4.** *If the polynomial* $P(t)$ *of degree* $k \geq 9$ *is extremal for* $\tau_4$, *and* $P(1)/l_0(P) > 25.4901$, *then* $l_8(P) = 0$.

Proof. One can use the polynomial

$$Q_3(t) = (t^3 + at^2 + bt + c)^2[(t-e)^2 + f(1/2-t)(1+t)](t-1/2)$$

with $a = 1.1268$, $b = 0.259261614$, $c = 0.0039482945$, $e = 0.04526608$, $f = 0.8244$ in the proof of this Lemma. Now we have $l_i(Q_3) \geq 0$ for $i \neq 8$, $l_8(Q_3) < 0$ and $Q_3(1)/l_0(Q_3) = 25.4901$. □

**Theorem 1.5.** *If the polynomial $S_4(t)$ gives the best upper bound for $\tau_4$ among the polynomials of degree 9 in $A_4$, then $S_4(t)$ is extremal for $\tau_4$.*

Proof. In paper [5] it was proven that the polynomial $E(t) = (t^2+t+1/6)^2(t-1/2)$ (used in [1]) is extremal for $\tau_4$. Since $E(1)/l_0(E) = 26 > S_4(1)/l_0(S_4) = 25.5585$ (the best known upper bound for $\tau_4$), it is enough to prove that polynomials of degrees 6,7 or 8 giving better upper bounds for $\tau_4$ than $S_4(t)$ do not exist. Assume, for example, $f(t) \in A_4$, $deg(f) = 6$, and $f(1)/l_0(f) < S_4(1)/l_0(S_4)$. Then $l_6(f) > 0$ and using Lemmas 1.1 and 1.2 we can obtain a polynomial $H(t) = f(t) + \beta Q_1(t) \in A_4$, $\beta > 0$ with $deg(H) = deg(S_4) = 9$ and $H(1)/l_0(H) < f(1)/l_0(f) < S_4(1)/l_0(S_4)$ - a contradiction. Similarly one can prove that no polynomials of degrees 7 or 8 give better upper bounds for $\tau_4$ than $S_4(t)$. □

The next three lemmas can be proven in the same way as Lemmas 1.2, 1.3, 1.4 ([2]). We use the polynomials $R_1(t)$, $R_2(t)$, $R_3(t)$ respectively, where

$$R_1(t) = S_5(t) - -\alpha(t^2 + 6t/7 + 5/49)^2(t-1/2)^2$$

where

$$S_5(t) = (t^3 + at^2 + bt + c)^2[(t-e)^2 + f(1/2-t)(1+t)](t-1/2)$$

with $a = 1.33342$, $b = 0.474050127$, $c = 0.033080914$, $e = 0.107556573$, $f = 0.73185$ ($S_5(t)$ was used in [2] to estimate $\tau_5$) and $\alpha = 0.1910323$;

$$R_2(t) = Q_2(t)$$

(now we have $R_2(1)/l_0(R_2) = 42$);

$$R_3(t) = (t^3 + at^2 + bt + c)^2[(t-e)^2 + f(1/2-t)(1+t)](t-1/2)$$

where $a = 0.914$, $b = 0.064680836$, $c = -0.030523192$, $e = -0.0349966$, $f = 0.9669$.

**Lemma 1.6.** *If the polynomial $P(t)$ of degree $k \geq 9$ is extremal for $\tau_5$, then $l_6(P) = 0$.*

**Lemma 1.7.** *If the polynomial $P(t)$ of degree $k \geq 9$ is extremal for $\tau_5$ and $P(1)/l_0(P) > 42$, then $l_7(P) = 0$.*

**Lemma 1.8.** *If the polynomial $P(t)$ of degree $k \geq 9$ is extremal for $\tau_5$ and $P(1)/l_0(P) > 42.356$, then $l_8(P) = 0$.*

The following theorem can be proven in the same way as Theorem 1.5

**Theorem 1.9.** *If the polynomial $S_5(t)$ gives the best upper bound for $\tau_5$ among the polynomials of degree 9 in $A_5$, then $S_5(t)$ is extremal for $\tau_5$.*

It is not difficult to see with the help of Lemma 1.1 that the following theorem is true:

**Theorem 1.10.** *Let $P(t) \in A_n$, $deg(P) = k$, $m > k$ be an integer and let $\varepsilon > 0$ be an arbitrary real number. Then there exists a polynomial $Q(t) \in A_n$ of degree $m$ such that*

$$Q(1)/l_0(Q) < P(1)/l_0(p) + \varepsilon.$$

With the help of the polynomial $Q_2(t)$ from Lemmas 1.3 and 1.7 one can also prove the following lemmas:

**Lemma 1.11.** *If the polynomial $P(t)$ of degree $k \geq 9$ is extremal for $\tau_6$, then $l_7(P) = 0$.*

**Lemma 1.12.** *If the polynomial $P(t)$ of degree $k \geq 9$ is extremal for $\tau_7$, then $l_7(P) = 0$.*

Similar lemmas can be proven in higher dimensions for certain coefficients of the extremal polynomials.

**2. A method for obtaining upper bounds for $\tau_n$.** We propose a new method diving good upper bounds for $\tau_n$ using polynomials of the form $P(t) = G(t)(t - 1/2) \in A_n$. Obviously $G(t) \geq 0$ for $-1 \leq t \leq 1/2$ and by Karlin's theorem [4, p.75] we have the representation

$$G(t) = \begin{cases} C_m^2(t) + (1/2 - t)(1 + t)B_{m-1}^2(t) & \text{if } deg(G) = 2m \\ (t + 1)C_m^2(t) + (1/2 - t)B_m^2(t) & \text{if } deg(G) = 2m + 1 \end{cases}$$

where $C_m(t) = t^m + c_1 t^{m-1} + \cdots + c_m$, $B_j(t) = b_0 t^j + \cdots + b_j$, $j = m$ or $m - 1$.

The polynomials $C_m(t)$, $B_{m-1}(t)$ and $B_m(t)$ have zeros only in the interval [-1,1/2] (see [4, p.75]). We prove that $C_m(t)$ and $B_{m-1}(t)$ (respectively $B_m(t)$) have at least three common zeros if the polynomial $P(t)$ is extremal for $\tau_n$ and $n \geq 4$.

**Lemma 2.1.** *If the polynomial $P(t) = G(t)(t - 1/2)$ of degree $k \geq 4$ is extremal for $\tau_n$, $n \geq 2$ and $l_1(P) > 0$, then polynomials $C_m(t)$ and $B_{m-1}(t)$ (or $B_m(t)$) have at least one common zero.*

Proof. Assume that $C_m(t)$ and $B_{m-1}(t)$ (or $B_m(t)$) have no common zero. Then from Karlin's theorem [4, p.75] we obtain $G(t) \geq \varepsilon > 0$ for some $\varepsilon$ and $-1 \leq t \leq 1/2$. Consider the polynomial

$$P_\varepsilon = (G(t) - \varepsilon)(t - 1/2) \leq 0$$

for $-1 \leq t \leq 1/2$. Since $P_\varepsilon(t) = P(t) - \varepsilon(t - 1/2)$, we have $l_i(P_\varepsilon) = l_i(P)$ for $i \geq 2$, $l_1(P_\varepsilon) = l_1(P) - \frac{2\varepsilon}{n-1}$, $l_0(P_\varepsilon) = l_0(P) + \varepsilon/2$, $P_\varepsilon(1) + P(1) - \varepsilon/2$, $deg(P_\varepsilon) = deg(P)$. For small enough $\varepsilon > 0$ we can obtain $l_1(P_\varepsilon) > 0$ and hence $P_\varepsilon(t) \in A_n$. But one can check directly that

$$\frac{P_\varepsilon(1)}{l_0(P_\varepsilon)} = \frac{P(1) - \varepsilon/2}{l_0(P) + \varepsilon/2} < \frac{P(1)}{l_0(P)}$$

– a contradiction since $P(t)$ is extremal. □

**Lemma 2.2.** *If the polynomial $P(t) = G(t)(t - 1/2)$ of degree $k \geq 6$ is extremal for $\tau_n$, $n \geq 4$ and $l_i(P) > 0$ for $i = 1, 2, 3$, then the polynomials $C_m(t)$ and $B_{m-1}(t)$ (or $B_m(t)$) have at least two common zeros.*

Proof. By Lemma 2.1 $C_m(t)$ and $B_{m-1}(t)$ (or $B_m(t)$) have a common zero $\alpha \in [-1, 1/2]$. Assume that they have no other common zero. Then we obtain the representation

$$G_1(t) = \begin{cases} C_{m-1}^2(t) + (1/2 - t)(1 + t)B_{m-2}^2(t) & \text{if } deg(G) = 2m \\ (t+1)C_{m-1}^2(t) + (1/2 - t)B_{m-1}^2(t) & \text{if } deg(G) = 2m + 1 \end{cases}$$

where $G(t) = (t - \alpha)^2 G_1(t)$ and $G_1(t) \geq \varepsilon > 0$ for some $\varepsilon$ and $-1 \leq t \leq 1/2$ again by [4, p.75].

Now consider the polynomial

$$P_\varepsilon(t) = (t - \alpha)^2 (G_1(t) - \varepsilon)(t - 1/2) = P(t) - \varepsilon(t - \alpha)^2 (t - 1/2) \leq 0$$

for $-1 \leq t \leq 1/2$. As in Lemma 2.1 we have $l_i(P_\varepsilon) = l_i(P)$ for $i \geq 4$ and $l_i(P_\varepsilon) > 0$ for $i = 0, 1, 2, 3$ and small enough $\varepsilon$, i.e. $P_\varepsilon(t) \in A_n$. Since $l_0(P_\varepsilon) = l_0(P) + \frac{\varepsilon}{2n}(n\alpha^2 + 4\alpha + 1) \geq l_0(P)$ for $n \geq 4$, $P_\varepsilon(1) = P(1) - \varepsilon(1 - \alpha)^2 < P(1)$ and $deg(P_\varepsilon) = deg(P)$ we obtain again $P_\varepsilon(1)/l_0(P_\varepsilon) < P(1)/l_0(P)$ – a contradiction. □

**Lemma 2.3.** *For any $n \geq 10$ and arbitrary real $\alpha$ and $\beta$ we have*

(1) $L(\alpha, \beta, n) = -\beta^2/2 - \alpha^2/2n + 2\alpha\beta/n - \beta/n + 6\alpha/n(n+2) - 3/2n(n+2) \leq 0$.

Proof. The discriminant of $L(\alpha, \beta, n)$ with respect to $\alpha$ is

(2) $D(\beta, n) = \frac{1}{n^2}[(4 - n)\beta^2/4 + (10 - n)\beta/2(n+2) + 3(10 - n)/4(n+2)^2]$.

The discriminant of $n^2 D(\beta, n)$ with respect to $\beta$ is $\frac{1}{(n+2)^2} D$ where

(3) $D = (10 - n)(n - 1)/2 \leq 0$

for $n \geq 10$ which proves the lemma. □

**Lemma 2.4.** *If $l_0(f) > 0$ for some $n$, $4 \leq n \leq 9$ where $f(t) = (t^2 + \alpha t + \beta)^2(t - 1/2)$, then $f(t) \in A_n$.*

Proof. Note that $l_0(f) = L(\alpha, \beta, n)$ (see(1)). It is enough to prove that $l_i(f) \geq 0$ for $i = 1, 2, 3, 4, 5$. These coefficients can be easily expressed by $\alpha$, $\beta$ and $n$. So, we have $l_5(f) = 3840/(n+3)(n+4)\dots(n+7) > 0$. □

If $l_4(f) = 768(\alpha - 1/4)/(n+2)(n+3)(n+4)(n+5) \leq 0$, then we obtain $\alpha \leq 1/4$. But the discriminant of $l_0(f)$ with respect to $\beta$ is $\frac{1}{n^2}[(4-n)\alpha^2 + 2(n-1)(4\alpha-1)/(n+2)] \leq 0$ for $n \geq 4$ and $\alpha \leq 1/4$ – a contradiction. Therefore $l_4(f) > 0$.

Since $l_0(f) = -\beta^2/2 + (n+1)l_2(f)/8 - (n+3)(n+5)l_4(f)/128 > 0$ and $l_4(f) > 0$ we obtain that $l_2(f) > 0$.

Because of $l_0(f) = L(\alpha, \beta, n) > 0$ we must have $D(\beta, n) > 0$( see (2)). Thus $\beta > -1/4$ for $n = 4$ and $\beta \in \left(\frac{n - 10 + 2\sqrt{D}}{(n+2)(4-n)}, \frac{n - 10 - 2\sqrt{D}}{(n+2)(4-n)}\right)$ ($D$ is determined in (3)) for $5 \leq n \leq 9$. One can check directly that for such $\beta$ we have $K(\beta, n) = 2\beta - (n - 34)/4(n+6) > 0$. Hence $l_3(f) = 48[(\alpha - 1/2)^2 + K(\beta, n)]/(n+1)(n+2)(n+3) > 0$

For arbitrary real $x$ and $n > 0$ we have $M(x, n) = x^2 + 6x/(n+2) + 15(2n+7)/(n+2) + 15(2n+7)/(n+2)(n+4)(n+6) > 0$. Thus $l_1(f) = \frac{2}{n-1}[M(\beta - \alpha/2, n) + (10-n)\alpha^2/4(n+2)] > 0$. The Lemma is proven.

We are now in a position to state the main theorem concerning common zeros.

**Theorem 2.5.** *If the polynomial $P(t) = G(t)(t - 1/2)$ of degree $k \geq 8$ is extremal for $\tau_n$, $n \geq 4$ and $l_i(P) > 0$ for $i = 1, 2, 3, 4, 5$, then the polynomials $C_m(t)$ and $B_{m-1}(t)$ (or $B_m(t)$) have at least three common zeros.*

Proof. It follows from Lemma 2.2 that $C_m(t)$ and $B_{m-1}(t)$ (or $B_m(t)$) have at least two common zeros. Hence we have

$$P(t) = (t^2 + \alpha t + \beta)^2 G_2(t)(t - 1/2) = f(t)G_2(t)$$

where

$$G_2(t) = \begin{cases} C_{m-2}^2(t) + (1/2 - t)(1+t)B_{m-3}^2(t) & \text{if } deg(G) = 2m \\ (t+1)C_{m-2}^2(t) + (1/2 - t)B_{m-2}^2(t) & \text{if } deg(G) = 2m+1 \end{cases}$$

If $C_{m-2}(t)$ and $B_{m-3}(t)$ (or $B_{m-2}(t)$) have no common zero, then by [4, p.75] we obtain $G_2(t) \geq \varepsilon > 0$ for $-1 \leq t \leq 1/2$ and some $\varepsilon$.

Now we consider the polynomial $P_\varepsilon(t) = f(t)(G_2(t) - \varepsilon) = P(t) - \varepsilon f(t) \leq 0$ for $-1 \leq t \leq 1/2$. We have $l_i(P_\varepsilon) = l_i(P)$ for $i \geq 6$ and $l_i(P_\varepsilon) = l_i(P) - \varepsilon l_i(f) > 0$ for $i = 0, 1, \dots, 5$ and small enough $\varepsilon$, i.e. $P_\varepsilon(t) \in A_n$.

If $l_0(f) \leq 0$ (this is a fact for $n \geq 10$ by Lemma 2.3), then we easily obtain that

$$\frac{P_\varepsilon(1)}{l_0(P_\varepsilon)} = \frac{P(1) - \varepsilon f(1)}{l_0(P) - \varepsilon l_0(f)} < \frac{P(1)}{l_0(P)}$$

– a contradiction.

If $l_0(f) > 0$, then we have $4 \leq n \leq 9$ by Lemma 2.3 and hence $f(t) \in A_n$ by Lemma 2.4. Therefore we have $f(1)/l_0(f) > P(1)/l_0(P)$. But the last inequality is equivalent to $P_\epsilon(1)/l_0(P_\epsilon) < P(1)/l_0(P)$ – a contradiction. This completes the proof. □

We remark that the requirements $l_i(P) > 0$ for $i = 1,2,3,4,5$ in Theorem 2.5 are not essential, since if we have found an extremal polynomial satisfying these conditions, then each extremal polynomial having the same degree must satisfy them. Therefore without loss of generality we can search for extremal polynomials having at least three double zeros in the interval [-1,1/2] (see also the remark at the end of the next paragraph).

We use Theorem 2.5 and Lemmas 1.2, 1.3, 1.4, 1.6, 1.7 and 1.8 to propose a new method for finding good upper bounds for $\tau_4$ and $\tau_5$. Let $P(t) = G(t)(t - 1/2)$ where $G(t) \geq 0$ for $-1 \leq t \leq 1/2$. Using Theorem 2.5 one can see that the polynomial $G(t)$ depends on $k - 4$ parameters, where $k = deg(P) = deg(G) + 1$. By the conditions $l_6(P) = l_7(P) = l_8(P) = 0$ we express 3 of these parameters in terms of the remaining $k - 7$ parameters. For small $k$ we determine these $k - 7$ parameters so that the ratio $P(1)/l_0(P)$ is minimal and $P(t) \in A_n$. We do this by a Monte Carlo method using a PC as a calculator. A similar approach can be used in higher dimensions and degrees (we have shown in a very long technical proof that if the polynomial $P(t)$ of degree $k \geq 10$ is extremal for $\tau_n$, $n \geq 9$, then it must have at least four double zeros in the interval [-1,1/2]).

Using this method for $k = 9$ we have obtained in [2] the polynomials $S_4(t)$ for $n = 4$ and $S_5(t)$ for $n = 5$. They give the estimates $\tau_4 \leq 25.558429$ ([2,8]) and $\tau_5 \leq 46.345916$([2]) ( $\tau_5 \leq 46$ was obtained in [8] with a polynomial of degree 10).

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