Serdica
Bulgariacae mathematicae publicationes

Сердика
Българско математическо списание

SERDICA — Bulgaricae
mathematicae publicationes
19 (1993) 3-7

JOIN DECOMPOSITION OF CERTAIN SPACES

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ABSTRACT. The notion of a join of double coset and orbit spaces (studied as Pasch geometries) is given in [1]. A space which is a join of projective spaces is naturally the generalized projective space of [5] and the join decomposition is equivalent to the concept of direct union decomposition into irreducible spaces. In this paper geometries the decomposition of which give either projective spaces, groups or orbit spaces isomorphic to Q/Q^+ are characterized. The direct union decompositions of [5] is obtained as a corollary.

Introduction. The definitions and the concept of a Pasch geometry (which we simply call a geometry here) can be found in [4] and the notion of a join in [1]. A geometry in which $x^\# = x$ for all x and whenever $(x, x, y) \in \Delta$, $y = x$ or $y = e$ is naturally a projective space (including projective points and lines, see [2]). A join of projective spaces gives a generalized projective space (and hence a complemented modular lattice in case of finite dimension) of [3] or [5]. We show that a geometry whose join decomposition admits projective spaces is characterized by (what we called) Lt-connectedness and the property that $x^\# = x$ for all x. However, if we assume only Lt-connectedness, then such a geometry in its decomposition is a join of geometries each of which is either a projective space, a group or isomorphic to the orbit space Q/Q^+ where Q is the field of rational numbers and Q^+ the multiplicative group of positive rationals. Then additional assumption of $x^\# = x$ for all x excludes Q/Q^+ and a group with this property is itself a projective space (i.e. a vector space over Z_2), thus resulting in a join of projective spaces. This is easily seen to be equivalent to the direct union decomposition of [5], except that the join structure depends on the (linear) ordering of the indexing set of the spaces whereas the direct union of projective spaces in [5] does not.

Now let (A, e, Δ) be a geometry. For $x, y \in A$, let $xy = \{t : (x, y, t^\#) \in \Delta\}$. (Considering A as a hypergroup or multigroup, xy is the set of all elements which appear in the multivalued product of x and y. In case A is sharp (i.e. a group), xy is just the product of the elements x and y and in case A is a projective space, xy is the set of all other points colinear with x and y.) Since $t \in xx^\# \iff t^\# \in xx^\#$, so $xx^\# = \{t : (x, x^\#, t) \in \Delta\}$. For $x \in A$, we have $A : x = \{y : (x, t, y) \in \Delta \Rightarrow t = x^\#\}$ (see [1]). Thus $A : x = \{y : yx = \{x\}\}$ and so $A : x$ is the set of all elements which fix x (see [7]). We note that $A : x \subseteq xx^\#$. Clearly $x^\# \notin A : x$ for $x \neq e$ but $x^\#$ may well be an element of $xx^\#$.

For $x, y \in A^* = A - \{e\}$, define a relation $\sim$ by $x \sim y$ if $y^\# \notin A : x$ and $x^\# \notin A : y$. Since $x^\# \notin A : x, x \sim x$. Clearly $x \sim y \Longrightarrow y \sim x$. However, in general $\sim$ may not be transitive.

Definition 1. *We say that a geometry is* Lt*-connected (locally transitively connected) if*

i) $xx^\# - \{x^\#\} = A : x$ *(i.e. every element of $xx^\#$ except possibly $x^\#$, fixes x),*

ii) $\sim$ *defined above is transitive.*

Remark 1. Supposing i), the equivalence relation $\sim$ compares with the relation of conjoint points of [3] or the relation of not begin on a degenerate line of [5]. In fact, suppose $x^\# = x$ for all x (which as shown below gives the join of projective spaces). If $x \sim y, x \neq y$ (i.e. x and y are conjoint), then for any $t \in A$ with $(x, y, t) \in \Delta, t \notin A : y$ since $y^\# = y \neq x$ and $y \notin A : t$, since if $x = t^\# = t$, then $y \in xx^\# - \{x\} = A : x$ contradicting $x \sim y$. So t is a third point conjoint to x and y. In other words, the line determined by x and y is not degenerate.

In the next we assume that the geometry is Lt-connected.

Proposition 1. *Let $B = A : x - \{x\}$, for $x \in A^*$. Then B is a subgeometry of A.*

P r o o f. $e \in B$, since $e \in A : x$. Let $e \neq y \in B$. Then $(x, x^\#, y) \in \Delta$ implies $(x, x^\#, y^\#) \in \Delta$, so $y^\# \in xx^\#$. But $y^\# \neq x^\#$, so $y^\# \in xx^\# - \{x^\#\} = A : x$. Finally, suppose $(x_1, x_2, y) \in \Delta$, $x_1, x_2 \in B$. To show that $y \in A : x$, let $(x, t, y) \in \Delta$. Then (y, x, t), $(y, x_1, x_2) \in \Delta$ implies $\exists z \in A$ with $(z, x_1^\#, x)$, $(z, x_2, t^\#) \in \Delta$. $x_1^\# \in B \subseteq A : x$, so $z = x^\#$. Substituting this value of z and noting that $x_2^\# \in A : x$, we get $t = x^\#$. Thus $y \in A : x$. But $y = x$ easily gives a contradiction, so $y \in A : x - \{x\} = B$. So B is a subgeometry of A.

A routine check on the cases $x \sim x^\#$ and $x \not\sim x^\#$ results in:

Lemma 1. $\forall x \in A^*, y \in A : x, y \neq x \Rightarrow y \in A : x^\#$.

Proposition 2. *Suppose $\mathcal{Z}_A = \{\{e\}, A\}$. Then $A : x \subseteq \{e, x\} \forall x \in A^*$ and hence $xx^\# \subseteq \{e, x, x^\#\}$.*

P r o o f. We have $\mathcal{Z}_A = \{B : B$ is a subgeometry of A and $A - B^*$ is a weak subgeometry$\}$ (see [1]). We let $B = A : x - \{x\}$ and show that $B \in \mathcal{Z}_A$. By Proposition 1, B is a subgeometry. Consider $A - B^*$. Let $(y_1, y_2, y) \in \Delta$, $y_1, y_2 \in A - B^*$, $y_2 \neq y_1^\#$. Clearly $y \in A - B^*$ if y is y_1 or $y_1^\#$ or y_2 or $y_2^\#$ or x. So suppose y is different from these.

Since in $(y_1, y_2, y) \in \Delta, y_2 \neq y_1^{\#}$, it is easily seen that $y_1^{\#} \notin A : y$ and $y^{\#} \notin A : y_1$ and so $y \sim y_1$. Now, if $y_1 \sim x$, then $x \sim y$ so $y^{\#} \notin A : x$ giving $y^{\#} \in A - A : x \subseteq A - B^*$. So suppose $y_1 \neq x$. Then either $y_1^{\#} \in A : x$ or $x^{\#} \in A : y_1$. But $y_1^{\#} \notin A : x$ unless $y_1 = x$ which it is not. Thus $y_1^{\#} \notin A : x$ and so $x^{\#} \in A : y_1$. Now just suppose $y \in A : x$. Then since $y \neq x$, by Lemma 1, $y \in A : x^{\#}$. Thus $y \in A : x, x^{\#} \in A : y_1$ and it is routine to check that this implies $y \in y_1 y_1^{\#}$. But $y \neq y_1$, so $y \in A : y_1$ giving $y_2 = y_1^{\#}$ a contradiction. So $y \notin A : x$ and hence $y \in A - B^*$. Thus $A - B^*$ is a weak subgeometry and we have $B \in Z_A = \{\{e\}, A\}$. Since $x \notin B$, $B \neq A$, so $B = \{e\}$, i.e. $A : x - \{x\} = \{e\}$ giving $A : x \subseteq \{e, x\}$. □

If a set $A = \{a, a^{\#}, e\}$ is made into a geometry by taking (e, e, e), $(a, a^{\#}, e)$, $(a, a, a^{\#})$, $(a^{\#}, a^{\#}, a)$ and any permutation of these to be in Δ, the geometry is easily seen to be isomorphic to the orbit space Q/Q^+ where Q is the field of rational numbers and Q^+ the multiplicative group of positive rationals (in fact, to F/F^+ where F is any ordered field and F^+ is the group of positive elements).

Theorem 1. *Let A be* Lt*-connected geometry with $\mathcal{Z}_A = \{\{e\}, A\}$,*

i) *If $x^{\#} = x \forall x \in A$, then A is projective,*

ii) *If $\exists x \in A$ with $x \neq x^{\#}$, then*

(a) *A is isomorphic to Q/Q^+ if $x \not\sim x^{\#}$*

(b) *A is sharp (i.e. a group) if $x \sim x^{\#}$.*

P r o o f. i) We have $x^{\#} = x \forall x \in A$. Let $(x, x, y) \in \Delta$. Then by Proposition 2, $y \in x x^{\#} \subseteq \{e, x\}$, so $y = x$ or $y = e$. Hence A is projective.

ii) $\exists x \in A$ with $x^{\#} \neq x$.

(a) Suppose $x \not\sim x^{\#}$. Then either $x \in A : x$ or $x^{\#} \in A : x^{\#}$. In either case $(x, x^{\#}, x) \in \Delta$ and so both $x \in A : x$ and $x^{\#} \in A : x^{\#}$. We claim $A = \{e, x, x^{\#}\}$. Suppose $y \in A, y$ different from $e, x, x^{\#}$. Then $y^{\#} \notin A : x \subseteq \{e, x\}$ and $x^{\#} \notin A : y \subseteq \{e, y\}$. So $y \sim x$. Similarly $y \sim x^{\#}$ giving $x \sim x^{\#}$ a contradiction. So $A = \{e, x, x^{\#}\}$. Since $x \in A : x$, $(x, x, x) \notin \Delta$. Now it is easy to see that A is isomorphic to Q/Q^+.

(b) $x^{\#} \sim x$. Let $(z, z^{\#}, y) \in \Delta$, $zy \in A$. For sharpness, we need to show that $y = e$. First consider $z = x$. Then $y \in x x^{\#} \subseteq \{e, x, x^{\#}\}$. If $y = x$, then since $x \neq x^{\#}, x \in A : x$ which contradicts $x \sim x^{\#}$. So $y \neq x$ and similarly $y \neq x^{\#}$ giving $y = e$. Now if z is any other element with $z \neq z^{\#}$, then $z \sim z^{\#}$, since otherwise $z \not\sim z^{\#}$ would reduce to case (a) which it is not. So $z \neq z^{\#}$ and $z \sim z^{\#}$ gives $y = e$ as in the case of $z = x$. So it remains to consider the case when $z = z^{\#} \neq e$. Thus $(z, z, y) \in \Delta$ and $y \in \{e, z\}$. Suppose $y = z$ giving $(z, z, z) \in \Delta$. $\exists t \in A$ with $(z, x, t) \in \Delta$. Hence $t \neq x^{\#}$ since otherwise $z = e$ from above. So $(z, x, t), (z, z, z) \in \Delta \Rightarrow \exists s \in A$ with $(s, z, x), (s, z, t^{\#}) \in \Delta$. Again $(z, x, t), (z, x, s) \in \Delta \Rightarrow \exists \omega \in A$ with $(\omega, x^{\#}, x), (\omega, s, t^{\#}) \in \Delta$. From $(x^{\#}, x, \omega) \in \Delta$ we get $\omega = e$. So $s = t$ and $(t, z, t^{\#}) \in \Delta$ gives $z \in \{e, t, t^{\#}\}$. Since $z \neq e$, we have $z = t = t^{\#}$. Also $(z, x, z) \in \Delta$ gives $x \in \{e, z\}$ and so $x^{\#} = x$ a contradiction. Thus $y = e$ in all cases proving that A is sharp. □

Theorem 2. *Let A be an* Lt-*connected geometry. Then A in its join decomposition is a join of geometries each of which is either projective, sharp or isomorphic to Q/Q^+.*

Proof. $\forall i = B \in Z_A^*, Z_{A_i} = \{\{e\}, A_i\}$ (see [1]). Each weak subgeometry $A_i, i \in \Gamma = \mathcal{Z}_A^*$ is Lt-connected (which can be easily seen by noting that for $x \in A_i, A_i : x = A_i \cap (A : x)$). So by Theorem 1, each A_i is as stated in the theorem. □

Corollary 1. *Any* Lt-*connected geometry with $x^\# = x$ for all x is a join of projective spaces.*

It should also be noted that a join of projective spaces is a (multi) group in the terminology of [6].

Remark 2.

(i) If a geometry is a join of geometries which are either projective, sharp or isomorphic to Q/Q^+, then it is easily seen that such a geometry is Lt-connected. Hence the condition in Theorems 1 and 2 is both necessary and sufficient.

(ii) If A is a probability group, then there is a natural induced geometry structure on A (see [1]). However, the converse is not true. In fact the geometry $Q/Q^+ = \{[0], [1], [-1]\}$ is not induced by any probability structure. Given two elements $[1], [1]$, we have a unique element $[-1]$ such that $([1], [1], [-1]) \in \Delta$. Hence if this geometry were induced by a probability structure p on A, then $p_{[1]}([1], [1]) = 1$ which contradicts Proposition 3.2 (2) of [1]. Now, since the decomposition of a probability group as a geometry gives it as a join of probability groups (see [1]), we call a probability group Lt-connected if the induced geometry is Lt-connected. Therefore, from the above theorem and the observation made here, we get that every probability group which is Lt-connected is a join of probability groups each of which is either a group or a projective space.

(iii) A geometry is called semi-sharp if $(a, b, c), (a, b, d) \in \Delta$, $b \neq a^\#$ implies $c = d$ (i.e. ab gives a unique element for $b \neq a^\#$). It is a routine check to see that every semi-sharp geometry is Lt-connected. It is also easily seen that a semi-sharp geometry which is projective but not sharp (i.e. not a vector space over Z_2) must consist of two elements (i.e. is a projective point). Hence we get that every semi-sharp geometry is a join of geometries each of which is either a group, a projective point or isomorphic to Q/Q^+. For an example, the geometry of conjugacy classes of the group of quaternion units is semi-sharp and is a join of groups isomorphic to Z_2 (the group of order two) and $Z_2 \times Z_2$ (the non-cyclic group of order 4).

Acknowledgments. The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for their hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

[1] H. N. Bhattarai and J. W. Fernandez, Joins of double coset spaces, *Pacific J. Math.* **98** (1982), 271-281.

[2] H. N. Bhattarai, An orbit space representation of a geometry *J. Algebra* **84** (1983), 142-151.

[3] G. Birkhoff, Combinatorial relations in projective geometries, *Ann. Math.* **36** (1935).

[4] D. K. Harrison, Double coset and orbit spaces, *Pacific J. Math.* **80** (1979), 451-491.

[5] O. Frink, Complemented modular lattices and projective spaces of infinite dimension, *Trans. Amer. Math. Soc.* **60** (1946).

[6] W. Prenowitz, Projective geometries as multigroups, *Amer. J. Math.* **65** (1943), 235-256.

[7] R. Roth, Character and conjugacy class hypergroups, *Ann. Mat. Pura Appl.* **55** (1975), 295-311.

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Received 25.08.1987