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A MODEL OF TIME WITH WALKER'S DEFINITION OF INSTANTS BY EVENTS

ANDREANA S. MADGUEROVA

This article constructs a model of Time, using Walker's definition of instants by events. It follows from the proposed system of axioms on the events, that the instants, constructed by events after Walker's definition, compose an open-ended linear continuum with a "dense" sequence of instants, i. e. Time continuum has the properties, characterizing the real line. Here the exposition is based only on Walker's definition of instants without using Russell's definition of instants. The used here system of axioms is simpler than those preceding it in the literature and it treats only events.

The attemp of mathematical contructions of the instants of Time by events follow Russell and Whitehead [1,2]. Such constructions of Time are elaborated also by Robbs [3], N. Wiener [4], Walker [5], Whitrow [6], Thomason [7]. Two models of Time, based on Russell's definition of the instants by events were constructed in papers [8-10]. Physiologists, psychologists and philosophers agree that the conception of the events is more primary and fundamental whereas the instants are intuitive-mental constructions. Russell and Whitehead have posed the problem so as to obtain the construction of the instants from the events in a logical-mathematical way ([1,2,6]). The two different models proposed here are based on Walker's definition and have more simple requirements with respect to events (cf. [6,7]). (For instance here only the relations $\prec$ and $\odot$ are required in the set of the events, whereas the paper [7] uses relations $\prec_1$, $\prec_0$, $\prec$, $\odot$. Here the constructions and proofs use only Walker's definition of the instants (without using Russell's definition of the instants also (cf. [7]).

The model of Time constructed with Walker's definition of instants (see [5-7]) is based on a different system of axioms on the events. It follows from this system, that the instants, constructed by events after Walker's definition [5-7], have the discussed in the literature [6,7] properties of the continuum of Time of Mathematical Physics, namely that instants compose an open-ended linear continuum with a "dense" sequence of instants, which are characterizing properties of the real line.

Let us denote by $\mathcal{E}$ the whole complex of all events.

The model of Time constructed with Walker's definition of instants by events. The model of Time here consists of Walker's definition [5-7] of instants by events and by the following axioms on the events.

Axiom $\mathcal{A}$ (B. Russell [2]).

1. $\mathcal{E} \neq \emptyset$. *For any two events either one of them is "before" ("earlier than", $\prec$) the other or in the opposite case they are "simultaneous" (at least partially) (i.e. they "overlap", i.e. are "contemporary", $\odot$). This is, for any two events $a, b \in \mathcal{E}$ one and only one of the following statements is true: either $a \prec b$ or $b \prec a$, or $a \odot b$; We have $a \odot a$ for $\forall a \in \mathcal{E}$.*

2. *If $a \prec b$, $b \odot c$, $c \prec d$, then $a \prec d$ for any events $a, b, c, d \in \mathcal{E}$.*

It follows from Axiom $\mathcal{A}$ that the relation $\prec$ is transitive, i.e. if $a \prec b$ and $b \prec d$, then $a \prec d$, where $a, b, d \in \mathcal{E}$. It also follows that if $a \odot b$, then $b \odot a$. Thus the set $\mathcal{E}$ of all events is partially ordered by the relation $\prec$.

Axiom $\mathcal{B}$. *There exists a sequence K of events from $\mathcal{E}$, such that for any arbitrarily fixed events $a, b \in \mathcal{E}$ with $a \prec b$ there is an event $k \in K$ with $a \prec k \prec b$.*

Axiom $\mathcal{C}$. *For any arbitrarily fixed event $a \in \mathcal{E}$ there are events $b, c \in \mathcal{E}$ such that $b \prec a \prec c$*

Axiom $\mathcal{D}$. *Whenever $c \prec a \prec b \prec d$ $(a, b, c, d \in \mathcal{E})$, then there is an event s, simultaneous with a and b, $a \odot s$, $b \odot s$, for which $c \prec s \prec d$.*

Remark. There exist complexes $\mathcal{E}$, satisfying Axioms $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$. Such is the set of all compacts (i.e. closed and finite) nonempty segments of the real line.

We shall formalize Walker's construction of the instants by events

Definition of the instants (after Walker [5]). *Let (P, Q, R) be a triple of subsets P, Q, R of $\mathcal{E}$, such that*

(i) *P, Q, R are nonempty, $P \neq \emptyset$, $Q \neq \emptyset$, $R \neq \emptyset$.*

(ii) *Each event of P is before any event of R.*

(iii) *Any event of Q is simultaneous with an event of P and with an event of R.*

There exist such triples after Axioms $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.

Let $\mathbf{W}$ be the set of all such triples (P, Q, R). We introduce a partial order in $\mathbf{W}$ by inclusions (in the sense of the Set Theory) of the triples in $\mathbf{W}$: Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}$, $\mathbf{w}_1 = (P_1, Q_1, R_1)$, $\mathbf{w}_2 = (P_2, Q_2, R_2)$. We shall deem that $\mathbf{w}_2$ follows ($\mathbf{w}_1$, $\mathbf{w}_1 \prec w_2$), iff $P_1 \subset P_2$, $Q_1 \subset Q_2$, $R_1 \subset R_2$.

The maximal elements of $\mathbf{W}$ will be called instants (moments) (after Walker) and will be denoted by small Greek letters. The class of all instants will be denoted by $\mathcal{W}$.

Theorem 1. *$\mathcal{W}$ is not empty, i.e. $\mathcal{W}$ has at least one element.*

Remark. If $\alpha = (P, Q, R) \in \mathcal{W}$ and the event a is simultaneous with any event $q \in Q$, $q \odot a$, then we shall say that the instant α belongs to the event a, $\alpha \in a$.

Theorem 2. *For any fixed event $a \in \mathcal{E}$ there exists an instant α, belonging to a, $\alpha \in a$.*

Theorem 3. *Let a and b be arbitrarily fixed simultaneous events, $a, b \in \mathcal{E}$. Then there exists at least one instant γ with $\gamma \in a$, $\gamma \in b$.*

Theorem 1-3 are formulated and proved separately because these results have been widely discussed in the literature (see [2,3,7]). In some papers these results are axioms (cf. [2,3,6]).

The order in $\mathcal{W}$. We shall say that the instant α is before (earlier than), the instant β, ($\alpha \prec \beta$) if there are events $q_\alpha \in Q_\alpha$, $q_\beta \in Q_\beta$, with $q_\alpha \prec q_\beta$, where $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$. If any two events $q_\alpha \in Q_\alpha$ and $q_\beta \in Q_\beta$ are simultaneous, $q_\alpha \odot q_\beta$, then we shall say that $\alpha = \beta$. (It is not necessary to have $P_\alpha = P_\beta$, $Q_\alpha = Q_\beta$, $R_\alpha = R_\beta$).

Proposition 4. *The relation "=" among instants is transitive, i.e. if $\alpha = \beta$, $\beta = \gamma$ with $\alpha, \beta, \gamma \in \mathcal{W}$, then $\alpha = \gamma$. Moreover, $\alpha = \alpha$ for any instant $\alpha \in \mathcal{W}$, this is any two events $q'_\alpha, q''_\alpha \in Q_\alpha$ are simultaneous, $q'_\alpha \odot q''_\alpha$ where $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$ is an arbitrarily fixed instant of $\mathcal{W}$.*

Proposition 5. *Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$ is an arbitrarily fixed instant of $\mathcal{W}$. Then we have*

a) $P_\alpha \cup Q_\alpha \cup R_\alpha = \mathcal{E}$;

b) $P_\alpha \cap Q_\alpha = \emptyset$; $Q_\alpha \cap R_\alpha = \emptyset$; $P_\alpha \cap R_\alpha = \emptyset$;

c) Any two events q'_α and q''_α of Q_α are simultaneous, $q'_\alpha \odot q''_\alpha$.

Proposition 6. *If the instant α is before the instant β, $\alpha \prec \beta$, then it is not true that $\beta \prec \alpha$.*

Proposition 7. *Time order of the instants of $\mathcal{W}$ is a transitive relation.*

Proposition 8. *Time order of the instants of $\mathcal{W}$ is a linear order.*

Moreover, the axioms $\mathcal{A}$ - $\mathcal{D}$ ensure all desired [2,6,7] properties of Time continuum for the class $\mathcal{W}$ of all instants. This is, $\mathcal{W}$ is an open-ended linear continuum with a "dense" sequence of instants, which are characterizing properties of the real line. Thus, the properties of Time continuum T, used in Mathematical Physics, are the following after [6,7]:

1. T is linearly ordered;

2. T is a "dense" set, i.e. if the instant π is earlier than the instant κ, then there exists at least one instant ρ between π and κ, $\pi \neq \rho$, $\kappa \neq \rho$.

3. T satisfies Dedekind's postulate, this is: If T_1 and T_2 are two non-empty disjoint parts of T, such that each instant of T belongs either to T_1 or T_2 and each

instant of T_1 is before any instant of T_2, then there exists at least one instant $\tau \in T$, such that every instant before τ belongs to T_1 and every instant after τ belongs to T_2.

4. T contains a countable subset G, such that for any two different instants π and κ of T there exists at least one instant ρ of G, which is between π and κ, $\pi \neq \rho$, $\kappa \neq \rho$.

The property 4 immediately implies the property 2 of T. These four properties of T are satisfied also by a model of Time, which has the earliest and the last moments, i.e. by a model of Time with a beginning and an end. Thereofore one more property should be added [7, 8-10]:

5. For any arbitrarily fixed moment τ of T there exist instants τ_1 and τ_2, such that τ is between τ_1 and τ_2, $\tau \neq \tau_1$, $\tau \neq \tau_2$.

Axioms $\mathcal{A}$-$\mathcal{D}$ ensures the following theorems.

Theorem 9. *The complex $\mathcal{W}$ of all instants is an open-ended linear continuum with a dense sequence of instants, i.e. $\mathcal{W}$ has the properties 1-5 of Time continuum T of Mathematical Physics, which are properties typical of the real line.*

The author has constructed another model of Time, based on Walker's definition of the instants, in which model not all the elements are bounded, as in the exposed here model.

Proofs for the model of time (with Walker's definition of instants). We shall use Zorn's Lemma in the proofs of Theorems 1-3. Let us remind it:

Lemma of Zorn (Zorn [9]) *Let X be a partially ordered nonempty set. If any linearly ordered subset A of X is upper bounded in X, then X contains at least one maximal element.*

Proof of Theorem 1. Evidently, $\mathbf{W}$ is a partially ordered complex. We shall prove that $\mathbf{W}$ is not empty. There exists at least one event a after Axiom $\mathcal{A}$. There is an event b with $a \prec b$ after Axiom $\mathcal{C}$. Applying Axiom $\mathcal{D}$, we get an event s with $a \odot s$, $b \odot s$. We have that the triple $(\{a\}, \{s\}, \{b\}) \in \mathbf{W}$.

Now, let $\mathbf{A}$ be a linearly ordered subset of $\mathbf{W}$, $\mathbf{A} = \{(P_i, Q_i, R_i),\ i \in I\}$, where I is a complex of indices. Let us examine the triple

$$\mathbf{w}^* = (P, Q, R),\ P = \bigcup_{i \in I} P_i,\ Q = \bigcup_{i \in I} Q_i,\ R = \bigcup_{i \in I} R_i.$$

We shall prove that $\mathbf{w}^* \in \mathbf{W}$, i.e. the requirements (i)-(iii) are satisfied:

(i) If $\mathbf{A} \neq \emptyset$, then $P \neq \emptyset$, $Q \neq \emptyset$, $R \neq \emptyset$;

(ii) Let $p \in P$, $r \in R$. We shall prove that $p \prec r$. Evidently, $p = p_{i'}$ and $r = r_{i''}$ for some i', $i'' \in I$. Since $\mathbf{A}$ is linearly ordered, then one of the triple $\mathbf{w}_{i'} = (P_{i'}, Q_{i'}, R_{i'})$, $\mathbf{w}_{i''} = (P_{i''}, Q_{i''}, R_{i''})$ follows the other. If $\mathbf{w}_{i''} \preceq \mathbf{w}_{i'}$, then we have $p_{i'} \in P_{i'}$, $r_{i''} \in R_{i'}$, which implies $p \prec r$. If $\mathbf{w}_{i'} \preceq \mathbf{w}_{i''}$, then we have $p_{i'} \in P_{i''}$, $r_{i''} \in R_{i''}$ which implies $p \prec r$ again.

(iii) Now, let $q \in Q$. Thus $q = q_{i^0}$ for some $i^0 \in I$. Since $\mathbf{w}^0 = (P_{i^0}, Q_{i^0}, R_{i^0}) \in \mathbf{W}$, then there exist events $p_{i^0} \in P_{i^0}$, $r_{i^0} \in R_{i^0}$ with $p_{i^0} \odot q$, $r_{i^0} \odot q$. But we have also $p_{i^0} \in P$, $r_{i^0} \in R$ due to the construction of $\mathbf{w}^*$. Therefore the triple $\mathbf{w}^* \in \mathbf{W}$.

Evidently, this triple $\mathbf{w}^*$ upper bounds $\mathbf{A}$. Therefore there exists at least one maximal element α of $\mathbf{W}$ after Zorn's Lemma. We have $\alpha \in \mathcal{W}$ by the definition of $\mathcal{W}$. $\square$

Proof of Theorem 2. Let us fix an event $b \succ a$. Such an event b exists after Axiom $\mathcal{A}$. Let s be an event, simultaneous with a and b. Such an event exists after Axioms $\mathcal{C}$ and $\mathcal{D}$. Then the triple $\mathbf{w}^0 = (\{a\}, \{s\}, \{b\}) \in \mathbf{W}$. Let $\mathbf{V}$ be the subset of $\mathbf{W}$ which contains all triples $\mathbf{w} \in \mathbf{W}$ with $\mathbf{w} = (P, Q, R) \succ \mathbf{w}^0 = (\{a\}, \{s\}, \{b\})$ and such that any event of P is not after the event a, i.e. either $p \odot a$ or $p \prec a$ for any event $p \in P$.

Evidently, $\mathbf{V}$ is not an empty partially ordered subset of $\mathbf{W}$ with the same relation of order $\prec$. We shall see that $\mathbf{V}$ satisfies the requirement of Zorn's Lemma. Let $\mathbf{A} = \{(P_i, Q_i, R_i), i \in I\}$ be nonempty linearly ordered subset of $\mathbf{V}$, where I is a complex of indices. Let us examine the triple $\mathbf{w}^* = (P, Q, R)$ with

$$P = \bigcup_{i \in I} P_i, \ \ Q = \bigcup_{i \in I} Q_i, \ \ R = \bigcup_{i \in I} R_i.$$

We have $\mathbf{w}^* \in \mathbf{W}$ since the requirements (i)-(iii) are satisfied: We have

(i) $P \neq \emptyset$, $Q \neq \emptyset$, $R \neq \emptyset$ as $\mathbf{A}$ is not empty;

(ii) If $p \in P$, $r \in R$, then $p \in P_{i'}$, $r \in R_{i''}$ for some $i', i'' \in I$. $\mathbf{A}$ is linearly ordered. This is, we have either $\mathbf{w}_{i'} \preceq \mathbf{w}_{i''}$ and $p \in P_{i''}$, $r \in R_{i''}$, so $p \prec r$, or $\mathbf{w}_{i''} \preceq \mathbf{w}_{i'}$ and $p \in P_{i'}$, $r \in R_{i'}$, thus $p \prec r$ also.

(iii) Let $q \in Q$ be arbitrarily fixed. Then q belongs to some Q_{i^0}, $i^0 \in I$, $\mathbf{w}_{i^0} = (P_{i^0}, Q_{i^0}, R_{i^0}) \in \mathbf{W}$. That is why there exist events $p \in P_{i^0}$, $r \in R_{i^0}$, simultaneous with q, $p \odot q$, $r \odot q$. Thus the proof of $\mathbf{w}^* \in \mathbf{W}$ id completed.

Moreover, $\mathbf{w}^* \in \mathbf{V}$ since we have: 1) It is true that $\mathbf{w}^* \succeq \mathbf{w}^0$, since $\mathbf{w}_i \succeq \mathbf{w}^0$ for $\forall i \in I$; 2) Let the event $p \in P$ be arbitraly fixed. Then $p \in P_{i^0}$ for some $i^0 \in I$. Since $(P_{i^0}, Q_{i^0}, R_{i^0}) \in \mathbf{V}$, hence the event p cannot be after the event a, i.e. either $p \odot a$ or $p \prec a$.

Evidently, the triple $\mathbf{w}^*$ upper bounds $\mathbf{A}$. Then $\mathbf{V}$ contains at least one maximal element α after Zorn's Lemma. Obviously, α is a maximal element of $\mathbf{W}$ also. This is, α is an instant of $\mathcal{W}$. Moreover, we have $\alpha \in a$ after the construction of $\mathbf{V}$. $\square$

Proof of Theorem 3. Let us denote by D_a (resp. by D_b) the set of all events d_a (resp. d_b) which are after the event a (resp. b). There exist the following three cases only: I. We have $a \prec d_b$, $b \prec d_a$ for any $d_a \in D_a$, $\forall d_b \in D_b$. Let $a = c$, $d_a = d$ for an arbitrarily fixed event $d_a \in D_a$, and let s be an event simultaneous with a and d. We have $c \prec d$, $c \odot s$, $d \odot s$.

II. There is an event $d'_a \in D_a$ with $b \odot d'_a$. Then we have $a \prec d'_a$, $d'_a \odot b$, $a \prec d_b$.

It follows applying Axiom $\mathcal{A}$, point 2, that $a \prec d_b$ for $\forall d_b \in D_b$. Let us denote a by c, b by s and $d'_a = d$. We have $c \prec d$, $c \odot s$, $d \odot s$.

III. There is an event $d'_b \in D_b$ with $a \odot d'_b$. Then we have $b \prec d'_b$, $d'_b \odot a$, $a \prec d_a$. Thus we obtain $b \prec d_a$ for $\forall d_a \in D_a$, applying Axiom $\mathcal{A}$, point 2. Let us denote b by c, a by s and d'_b by d. We have $c \prec d$, $c \odot s$, $d \odot s$. Thus we conclude in each of the cases I-III that the triple $\mathbf{w}^0 = (P^0, Q^0, R^0)$ with $P^0 = \{c\}$, $Q^0 = \{s\}$, $R^0 = \{d\}$ belongs to $\mathbf{W}$.

Let us study the partially ordered complex $\mathbf{V} = \{\mathbf{w} = (P, Q, R) \in \mathbf{W}, \mathbf{w} \succeq \mathbf{w}^0$ and any event of P is not after the event c $\}$ with the order, induced by the order of $\mathbf{W}$.

$\mathbf{V}$ satisfies the requirement of Zorn's Lemma. Indeed, let $\mathbf{A} = \{\mathbf{w}_i\}_{i \in I}$ be a nonempty linearly ordered subset of $\mathbf{V}$, where $\mathbf{w}_i = (P_i, Q_i, R_i)$ and let I be a complex of indices.

Let us denote by $\mathbf{w}^*$ the triple $\mathbf{w}^* = (P, Q, R)$ with

$$P = \bigcup_{i \in I} P_i;\ Q = \bigcup_{i \in I} Q_i;\ R = \bigcup_{i \in I} R_i.$$

We shall prove that the triple $\mathbf{w}^* \in \mathbf{W}$, since the requirements (i)-(iii) hold for $\mathbf{w}^*$:

(i) Since $\mathbf{A}$ is not empty we have $P \neq \emptyset$, $Q \neq \emptyset$, $R \neq \emptyset$.

(ii) If $p \in P$, $r \in R$, then $p \in P_{i'}$, $r \in R_{i''}$ for some $i', i'' \in I$. In the case $\mathbf{w}_{i'} \preceq \mathbf{w}_{i''}$, we get $p \in P_{i''}$, $r \in R_{i''}$, and therefore $p \prec r$. In the case $\mathbf{w}_{i''} \prec \mathbf{w}_{i'}$, we obtain $p \in P_{i'}$, $r \in R_{i'}$, and thus $p \prec r$.

(iii) Let $q \in Q$, then $q \in Q_{i^0}$ for some $i^0 \in I$. That is why there exist events $p \in P_{i^0}$, $r \in R_{i^0}$, such $q \odot p$, $q \odot r$. It follows by the construction of $\mathbf{w}^*$ that $p \in P$, $r \in R$. Thus we obtain $\mathbf{w}^* \in \mathbf{W}$.

Moreover, $\mathbf{w}^* \in \mathbf{V}$ since the following statements hold:

1. We have $c \in P$, $s \in Q$,$d \in R$ as $c \in P_i$, $s \in Q_i$, $d \in R_i$ for $\forall i \in I$. Thus we get $\mathbf{w}^* \succeq \mathbf{w}^0$.

2. Any event $p \in P$ cannot be after the event c since $p \in P_{i^0}$ for some $i^0 \in I$ and $\mathbf{w}_{i^0} \in \mathbf{V}$. Thus we obtain $\mathbf{w}^* \in \mathbf{V}$.

Applying Zorn's Lemma to $\mathbf{V}$, we conclude that there exists a maximal element γ of $\mathbf{V}$. Evidently γ is a maximal element of $\mathbf{W}$ also. Thus γ is an instant, for which $\gamma \in a$, $\gamma \in b$, after the contruction of $\mathbf{V}$. $\square$

P r o o f o f P r o p o s i t i o n 5. P o i n t a) Let a be an arbitrarily fixed event of $\mathcal{E}$. At first we shall discuss the following cases I-III: There exists an event $p_0 \in P_\alpha$, such that $a \prec p_0$. Then $a \in P_\alpha$ in this case, after the maximality of the instant α in $\mathbf{W}$.

II. There exists an event $r_0 \in R_\alpha$ with $r_0 \prec a$. Then $a \in R_\alpha$, after the maximality of the instant α of $\mathbf{W}$.

III. There exist events $p_1 \in P_\alpha$, $r_1 \in R_\alpha$ with $p_1 \odot a$, $r_1 \odot a$. In this case $a \in Q_\alpha$, again after the maximality of the instant $\alpha \in \mathbf{W}$.

Let a be an event, for which no requirement of any of the cases I-III is satisfied. Then we have either $p^0 \odot a$ or $p^0 \prec a$ for each event $p^0 \in P_\alpha$. If $a \odot p^0$ for some $p^0 \in P_\alpha$, then $a \prec r$ for all $r \in R_\alpha$, since the event a does not satisfy the requirements of the cases II and III. Therefore $a \in P_\alpha$, after the construction of α.

Now let us eliminate the previous case too. So, the event a is not simultaneous with any of the events of P_α. Since the case I is eliminated for the event a as well, then $a \succ p$ for $\forall p \in P_\alpha$. Therefore $a \in R_\alpha$ in this case, after the construction of α and after the maximality of α in $\mathbf{W}$. Thus we obtain $P_\alpha \cup Q_\alpha \cup R_\alpha = \mathcal{E}$.

The assertion of Point b) is evident after the construction of $\mathbf{W}$.

Point c). Let events $q'_\alpha, q''_\alpha \in Q_\alpha$. Then there exist events $p', p'' \in P_\alpha$, $r', r'' \in R_\alpha$ with $p' \odot q'_\alpha$, $r' \odot q'_\alpha$, $p'' \odot q''_\alpha$, $r'' \odot q''_\alpha$, after the construction of $\mathbf{W}$.

Let us assume that $q'_\alpha \prec q''_\alpha$. Then we have $q'_\alpha \prec q''_\alpha$, $q''_\alpha \odot p''$, $p'' \prec r$, for all $r \in R_\alpha$. Applying Axiom $\mathcal{A}$, Point 2, we obtain that $q'_\alpha \prec r$ for all $r \in R_\alpha$, which contradicts $\alpha \in \mathbf{W}$. Thus the assumption $q'_\alpha \prec q''_\alpha$ is not true.

Now, let us assume $q''_\alpha \prec q'_\alpha$. Since we have $q''_\alpha \prec q'_\alpha$, $q'_\alpha \odot p'$, $p' \prec r$, $\forall r \in R_\alpha$, hence $q''_\alpha \prec r$ for all $r \in R_\alpha$. This contradicts $\alpha \in \mathbf{W}$ again.

Thus only the relation $q'_\alpha \odot q''_\alpha$ is possible.

P r o o f o f P r o p o s i t i o n 4. We have $\alpha = \alpha$ after the definition of the relation "=" and after the already proved item c) of Proposition 5.

Also, if $\alpha = \beta$, then $\beta = \alpha$, after the definition.

Now let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$ $\alpha = \beta$, $\beta = \gamma$. Let us assume $\alpha \neq \gamma$. Then there exist events $q^*_\alpha \in Q_\alpha$, $q^*_\gamma \in Q_\gamma$ which are not simultaneous. Then we have either $q^*_\alpha \prec q^*_\gamma$ or $q^*_\gamma \prec q^*_\alpha$. Since $\mu = \nu$ for some instants μ, ν, implies also $\nu = \mu$, it is sufficient to reject the possibility $q^*_\alpha \prec q^*_\gamma$. If $q^*_\alpha \prec q^*_\gamma$, then there exists an event d with $q^*_\alpha \prec d \prec q^*_\gamma$, after Axiom $\mathcal{B}$.

We shall prove that $d \in Q_\beta$. Let us assume $p_\beta \prec d$ for all $p_\beta \in P_\beta$. Then we have $d \notin P_\beta$, $d \notin Q_\beta$ and $d \in R_\beta$, after the maximality of the instant β in $\mathbf{W}$. Therefore there exists an event s with $s \odot d$, $s \odot p^*_\beta$ for some fixed $p^*_\beta \in P_\beta$ and $s \prec q^*_\gamma$, after Axiom $\mathcal{D}$. The maximality of β in $\mathbf{W}$ implies $s \in Q_\beta$. But $\beta = \gamma$, which yields $s \odot q^*_\gamma$. The contradiction obtained proves that $d \notin R_\beta$ and $d \odot p^*_\beta$ for some event $p^*_\beta \in P_\beta$.

Now, let us assume that $d \prec r_\beta$ for all $r_\beta \in R_\beta$. Then $d \in P_\beta$, after the maximality of β in $\mathbf{W}$, and there exists an event v with $q^*_\alpha \prec v$, $v \odot d$, $v \odot r^*_\beta$ for some arbitrarily fixed event r^*_β of R_β. The last two relations involve $v \in Q_\beta$, after the maximality of β in $\mathbf{W}$. Therefore we must have $v \odot q^*_\alpha$, since $\alpha = \beta$, whereas it holds $q^*_\alpha \prec v$. The contradiction thus obtained proves that $d \notin P_\beta$ and $d \odot r^*_\beta$ for some $r^*_\beta \in R_\beta$.

So we have $d \odot p^*_\beta$, $d \odot r^*_\beta$, $p^*_\beta \in P_\beta$, $r^*_\beta \in P_\beta$. Therefore $d \in Q_\beta$, after the maximality of β in $\mathbf{W}$.

Then the relations $d \in Q_\beta$ and $\alpha = \beta$ prove that $d \odot q^*_\alpha$, while we have $q^*_\alpha \prec d$. The contradiction obtained implies the impossibility of $q^*_\alpha \prec q^*_\gamma$. After the symmetry of "=", this is sufficient to assert that $\alpha = \gamma$. $\Box$

Proof of Proposition 6. Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, be instants, $\alpha \prec \beta$, with $q'_\alpha \prec q'_\beta$, $q'_\alpha \in Q_\alpha$, $q'_\beta \in Q_\beta$. Let us assume that simultaneously we have $\beta \prec \alpha$ with $q''_\beta \prec q''_\alpha$, $q''_\alpha \in Q_\alpha$, $q''_\beta \in Q_\beta$. Therefore we have $q'_\alpha \prec q'_\beta$, $q'_\beta \odot q''_\beta$, $q''_\beta \prec q''_\alpha$. Axiom $\mathcal{A}$, Point 2, involves $q'_\alpha \prec q''_\alpha$, whereas we have $q'_\alpha \odot q''_\alpha$, after Proposition 5, Point c). The contradiction obtained proves that the assumption $\beta \prec \alpha$ is not true. $\Box$

Proof of Proposition 7. Let us have $\alpha \prec \beta$, $\beta \prec \gamma$ for some instants $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$, belonging to $\mathcal{W}$, with $q_\alpha \prec q'_\beta$, $q''_\beta \prec q_\gamma$ for some events $q_\alpha \in Q_\alpha$, $q'_\beta, q''_\beta \in Q_\beta$, $q_\gamma \in Q_\gamma$. We have $q'_\beta \odot q''_\beta$ after Proposition 5, Point c). Therefore Axiom $\mathcal{A}$, Point 2 implies $q_\alpha \prec q_\gamma$. Thus $\alpha \prec \gamma$. $\Box$

Proof of Theorem 8. It is sufficient to show after proving Propositions 6 and 7, that if μ and ν are arbitrarily fixed instants with $\mu \neq \nu$, then we have either $\mu \prec \nu$ or $\nu \prec \mu$. Let $\mu = (P_\mu, Q_\mu, R_\mu)$, $\nu = (P_\nu, Q_\nu, R_\nu)$. Since $\mu \neq \nu$, then there exists at least one pair of events $q^*_\mu \in Q_\mu$, $q^*_\nu \in Q_\nu$, which are not simultaneous. Then we have only two possibilities: 1. Either $q^*_\mu \prec q^*_\nu$, or 2. $q^*_\nu \prec q^*_\mu$. In the first case we have $\mu \prec \nu$ from the definition of the order of $\mathcal{W}$. In the second case we have $\nu \prec \mu$. $\Box$

Lemma 10. *Let instant $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$ belongs to the event q. If the event b is before q, $b \prec q$, then $b \in P_\alpha$. If c is an event after q, $q \prec c$, then $c \in R_\alpha$.*

Proof. Since $\alpha \in q$, hence $b \notin Q_\alpha$, $c \notin Q_\alpha$, as q should be simultaneous with any event of Q_α. We shall prove that $b \notin R_\alpha$. Let us assume the contrary. Then $p \prec b$ for all $p \in P_\alpha$. Let us fix one event $p' \in P_\alpha$. Since we have $p' \prec b \prec q$, hence there is an event s with $s \odot p'$, $s \odot b$, $s \prec q$. But α is an instant, i.e. α is a maximal element of $\mathbf{W}$. Hence $s \in Q_\alpha$, as $s \odot p'$, $s \odot b$, $p' \in P_\alpha$, $b \in R_\alpha$. Moreover, $\alpha \in q$, thus q should be simultaneous with any element of Q_α. So $q \odot s$ and $s \prec q$. The contradiction obtained proves that $b \notin R_\alpha$. Since we have $b \notin Q_\alpha$ too, then $b \in P_\alpha$, according to the maximality of α in $\mathbf{W}$.

Now we shall prove, that $c \notin P_\alpha$. Let us assume the contrary. Thus $c \prec r$ for all $r \in R_\alpha$. Let us fix one event $r' \in R_\alpha$. There exists an event s' with $s' \odot r'$, $s' \odot c$, $s' \succ q$, since $q \prec c \prec r'$ and Axiom $\mathcal{D}$ holds. We get $s' \in Q_\alpha$ according to the maximality of α in $\mathbf{W}$ and $s' \odot c$, $c \in P_\alpha$, $s' \odot r'$, $r' \in R_\alpha$. Therefore we obtain $q \odot s'$ and $q \prec s'$. The contradiction thus obtained proves that $c \notin P_\alpha$. But $c \notin Q_\alpha$. Then $c \in R_\alpha$ after the maximality of α in $\mathbf{W}$. $\Box$

Lemma 11. *Let the instant α belongs to the event q. If β is an instant, belonging to the event b before q, $b \prec q$, then $\beta \prec \alpha$.*

If γ is an instant, belonging to the event c after q, $q \prec c$, then $\alpha \prec \gamma$.

Proof. Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$. Applying the Axiom $\mathcal{B}$, we get the existence of events a', a'', b', b'', c', c'', k, l with $b' \prec b \prec b'' \prec k \prec a' \prec q \prec a'' \prec l \prec c' \prec c \prec c''$. We have $b' \in P_\beta$, $b'' \in R_\beta$, $a' \in P_\alpha$, $a'' \in R_\alpha$, $c' \in P_\gamma$, $c'' \in R_\gamma$ after Lemma 10. Therefore there exist events s',s'',s''' with

$$\begin{array}{ccc} s' \odot b' & s'' \odot a' & s''' \odot c' \\ s' \odot b'' & s'' \odot a'' & s''' \odot c'' \\ s' \prec k & k \prec s'' \prec l & l \prec s''' \end{array}$$

Thus we obtain $s' \in Q_\beta$, $s'' \in Q_\alpha$, $s''' \in Q_\gamma$, $s' \prec s'' \prec s'''$. Hence $\beta \prec \alpha \prec \gamma$. □

Lemma 12. *Let α, β, γ be instants with $\alpha \prec \beta \prec \gamma$. Then there exist events a, b', b'', c with $\alpha \in a$, $\beta \in b'$, $\beta \in b''$, $\gamma \in c$ and $a \prec b'$, $b'' \prec c$. We have for any such events a and c that $a \prec c$.*

Proof. Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$. The existence of events $a \in Q_\alpha$, $b', b'' \in Q_\beta$, $c \in Q_\gamma$ with $a \prec b'$, $b'' \prec c$ follows by the definition of the relation "$\prec$". Now, let a, b', b'', c be arbitrarily fixed events, satisfying the requirements of Lemma 12. We must prove $a \prec c$. It is sufficient to prove the impossibility of the cases $c \prec a$ and $a \odot c$. Let us assume that $c \prec a$. Then we get $\gamma \prec \alpha$, according to Lemma 11. But $\gamma \prec \alpha$ contradicts the conditions of Lemma 12. Thus the assumption $c \prec a$ is not true. Now, let us assume that $a \odot c$. Then, Theorem 3 involves the existence of an instant $\xi \in a$ and $\xi \in c$. Since $a \prec b'$, $\xi \in a$, $\beta \in b'$, hence $\xi \prec \beta$ after Lemma 11. As $b'' \prec c$, $\beta \in b''$, $\xi \in c$, then $\beta \prec \xi$. Therefore $\xi \prec \beta \prec \xi$. The contradiction obtained proves that the case $a \odot c$ is not possible. Thus it remains only $a \prec c$.

Proof of Theorem 9.

I. $\mathcal{W}$ is linearly ordered after Theorem 8.

II. $\mathcal{W}$ has the property 4 of T: Let K be a fixed sequence of events from Axiom $\mathcal{B}$. Each arbitrarily fixed event $k \in K$ defines at least one instant $\kappa \in k$, according to Theorem 2. Let us fix arbitrarily such an instant $\kappa \in k$. Let $\mathcal{K}$ be the sequence of these instants κ, $(k \to \kappa)$, when k ranges K.

Let α and β be instants of $\mathcal{W}$ with $\alpha \prec \beta$, $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$ and let $q_\alpha \prec q_\beta$ for some events $q_\alpha \in Q_\alpha$, $q_\beta \in Q_\beta$. Axiom $\mathcal{B}$ implies the existence of an event $k^0 \in K$ with $q_\alpha \prec k^0 \prec q_\beta$. Let $\kappa^0 \in k^0$ be the chosen instant of $\mathcal{K}$, corresponding to k^0. Further on, we have $\alpha \in q_\alpha$, $\beta \in q_\beta$, $\kappa^0 \in k^0$. Lemma 11 implies that $\alpha \prec \kappa^0 \prec \beta$. Thus the sequence $\mathcal{K}$ is a dense sequence of instants.

III. $\mathcal{W}$ has the property 5 of T, i.e. $\mathcal{W}$ is open-ended: Let α be an arbitrarily fixed instant of $\mathcal{W}$, $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$. Let q be an arbitrarily fixed event of Q_α. There exist events m and n with $m \prec q \prec n$, after Axom $\mathcal{C}$. Theorem 2 implies the existence of instants $\mu \in m$, $\nu \in n$. Then we have $\mu \prec \alpha \prec \nu$ after Lemma 11 as $\alpha \in q$. Therefore $\mathcal{W}$ has the property 5 of T.

IV. We shall prove that $\mathcal{W}$ is a continuum: Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two disjoint $(\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset)$ nonempty parts of $\mathcal{W}$, whose union is $\mathcal{W}$, $(\mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{W})$, and each instant of $\mathcal{W}_1$ is before any instant of $\mathcal{W}_2$. We must prove the existence of an instant γ, such that each instant before γ belongs to $\mathcal{W}_1$ and each instant after γ belongs to $\mathcal{W}_2$.

Let $\varepsilon \in \mathcal{W}_1$, $\kappa \in \mathcal{W}_2$. Since $\mathcal{W}_1$ and $\mathcal{W}_2$ are nonempty, we can choose and fix such instants. Let $\rho \prec \varepsilon$ ans $\kappa \prec \delta$. Such instants ρ, δ exist after the proved property 5 of $\mathcal{W}$. Let r, e, m, k, n, d be events with $\rho \in r$, $\varepsilon \in e$, $m \in K$, $n \in K$, $\kappa \in k$, $\delta \in d$ and $r \prec m \prec e$, $k \prec n \prec d$. Here K is the sequence from Axiom $\mathcal{B}$. Let μ, ν be arbitrarily fixed instants of m and n, respectively, $\mu \in m$, $\nu \in n$.

Further on, the events m and n have the following properties:

1^0. $m \prec n$ (after Lemma 12).

2^0. Each instant of m belongs to $\mathcal{W}_1$ (since we have $m \prec e$, $\varepsilon \in e$, $\varepsilon \in \mathcal{W}_1$).

3^0. Each instant of n belongs to $\mathcal{W}_2$ (since we have $k \prec n$, $\kappa \in k$, $\kappa \in \mathcal{W}_2$).

Any pair of events m and n, having the properties 1^0-3^0, will be denoted by $m\&n$. Let us fix such a pair $a\&b$. Then there exists at least one event s, simultaneous with a and b, $a \odot s$, $b \odot s$, according to Axiom $\mathcal{D}$.

Let us construct the following class of events

$$\mathbf{Q} = \{s : \ \exists a, b \in \mathcal{E}, \ a\&b, s \odot a, s \odot b\}.$$

Let $\mathbf{P}$ be the set of all events a, corresponding to events s of $\mathbf{Q}$; let $\mathbf{R}$ be the set of all events b, corresponding to events s of $\mathbf{Q}$. We want to prove that the triple $\mathbf{\Gamma} = (\mathbf{P}, \mathbf{Q}, \mathbf{R}) \in \mathbf{W}$, i.e. has the properties (i)-(iii):

(i) We have shown the existence of events m, n, s with $m\&n$, $s \odot m$, $s \odot n$. Therefore $\mathbf{R} \neq \emptyset$, $\mathbf{Q} \neq \emptyset$, $\mathbf{P} \neq \emptyset$.;

(ii) Let the events $p \in \mathbf{P}$, $r \in \mathbf{R}$ be arbitrarily fixed. We must prove that $p \prec r$. There exist events $m^0, \hat{n}$ with $p\&\hat{n}$, $m^0\&r$, by the construction of $\mathbf{P}, \mathbf{Q}, \mathbf{R}$. Now we shall prove that the cases $r \prec p$ and $p \odot r$ are impossible. Let us assume that $r \prec p$. Let ξ, η be instants with $\xi \in p$, $\eta \in r$ (cf. Theorem 2). Since we have assumed that $r \prec p$, then $\eta \prec \xi$ after Lemma 11. But $p\&\hat{n}$ implies $\xi \in \mathcal{W}_1$ after the property 2^0 of $p\&\hat{n}$. The relation $m^0\&r$ involves $\eta \in \mathcal{W}_2$, according to the property 3^0 of $m^0\&r$. Therefore $\xi \prec \eta$ after the choice of $\mathcal{W}_1$ and $\mathcal{W}_2$. The contradiction thus obtained proves that the relation $r \prec p$ is not possible. Now, let us assume $p \odot r$. Let the instant $\zeta \in p$, $\zeta \in r$, according to Theorem 3. Since we have $p\&\hat{n}$, then $\zeta \in \mathcal{W}_1$. But since $m^0\&r$ and $\zeta \in r$, then $\zeta \in \mathcal{W}_2$ by the property 3^0 of "&". This contradicts $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$. Thus the assumption $p \odot r$ is not true. Then it remains only $p \prec r$.

(iii). Evidently this requirement is satisfied by the construction of $\mathbf{P}, \mathbf{Q}, \mathbf{R}$. Thus we obtain $\mathbf{\Gamma} \in \mathbf{W}$.

Let $\mathbf{V}$ be the subset of $\mathbf{W}$ of all triples $\mathbf{w} \in \mathbf{W}$ with $\mathbf{w} \succeq \mathbf{\Gamma}$. $\mathbf{V}$ satisfies the requirement of Zorn's Lemma: Let $\mathbf{A} = \{\mathbf{w}_i = (P_i, Q_i, R_i), i \in I\}$ be a linearly ordered nonempty subset of $\mathbf{V}$. Then the triple $\mathbf{w}^* = (P, Q, R)$ with

$$P = \bigcup_{i \in I} P_i, \ Q = \bigcup_{i \in I} Q_i, \ R = \bigcup_{i \in I} R_i,$$

belongs to $\mathbf{V}$. Moreover, $\mathbf{w}^*$ upper bounds $\mathbf{A}$. After Zorn's Lemma $\mathbf{V}$ has at least one maximal element γ. It is evident that γ is a maximal element of $\mathbf{W}$ too. Thus γ is an instant, $\gamma \in \mathcal{W}$.

We shall prove that any instant ξ with $\xi \prec \gamma$ belongs to $\mathcal{W}_1$ and that any instant η with $\gamma \prec \eta$ belongs to $\mathcal{W}_2$. Now, let $\xi \prec \gamma$. Let us assume the contrary, i.e., that $\xi \in \mathcal{W}_2$. Then we must have $\gamma \in \mathcal{W}_2$ by the choice of $\mathcal{W}_1$ and $\mathcal{W}_2$. Since $\xi \prec \gamma$, there are events $r, \xi \in r$, $n \in K$, g, $\gamma \in g$, for which $r \prec n \prec g$. Let ν be some fixed instant of n. Let m be an event of $\mathcal{W}_1$, this is, if $\mu \in m$, then $\mu \in \mathcal{W}_1$. We have shown that such events exists. We shall prove that the pair of events m and n has the properties 1^0-3^0, i.e., $m\&n$:

Let us examine the order between m and n. If we assume that $n \prec m$, since $\mu \in m$, $\nu \in n$, then $\nu \prec \mu$. But this is impossible because $\mu \in \mathcal{W}_1$, $\nu \in \mathcal{W}_2$. Thus the assumption $n \prec m$ is not true. Now, let us assume that $m \odot n$. Then, there exists an instant $\mu^* \in m$, $\mu^* \in n$, after Theorem 3. $\mu^* \in m$ involves $\mu^* \in \mathcal{W}_1$; $\mu^* \in n$ implies $\mathcal{W}_2 \ni \mu^*$. But we have $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$. The contradiction proves that the assumption $m \odot n$ is not true. Thus, it remains $m \prec n$.

Moreover, since any instant of m belongs to $\mathcal{W}_1$ and any instant of n belongs to $\mathcal{W}_2$, hence we have $m\&n$.

Let s be an event, simultaneous with m and n, $s \odot m$, $s \odot n$, and let $s \prec g$. Such an event s exists after Axiom $\mathcal{D}$. Since $s \in \mathbf{Q}$, then $\gamma \in s$. On the other hand we have $s \prec g$ and $\gamma \in g$. This is, we must have $\gamma \prec \gamma$ after Lemma 11, since $\gamma \in s$ and $\gamma \in g$. This contradiction proves that the assumption $\xi \in \mathcal{W}_2$ is not true. Therefore $\xi \in \mathcal{W}_1$ for any instant ξ with $\xi \prec \gamma$.

The statement that $\eta \in \mathcal{W}_2$, whenever $\gamma \prec \eta$ can be proved in a similar way.

Thus Theorem 9 is true. □

Proposition 13. *The instant γ, separating $\mathcal{W}_1$ and $\mathcal{W}_2$, determined by $\mathbf{\Gamma} = (\mathbf{P}, \mathbf{Q}, \mathbf{R})$, is unique.*

Proof. Let us assume that there are at least two such different instants. We shall denote the first one by γ_1 and the other one by γ_2, i.e. $\gamma_1 \prec \gamma_2$. Then there exists an instant τ with $\gamma_1 \prec \tau \prec \gamma_2$, after the properties of $\mathcal{W}$, which have already been proved. It follows from $\gamma_1 \prec \tau$ that $\tau \in \mathcal{W}_2$. But since $\tau \prec \gamma_2$, hence we have $\tau \in \mathcal{W}_1$. This contradicts $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$. The contradiction obtained is a proof for the uniqueness of γ. □

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Institute of Mathematics
Bulgarian Academy of Sciences
P.O.Box 373
1090 Sofia
BULGARIA

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